Symplectic field theory
Problem set 9

Gerard Bargalló i Gómez and Chris Wendl

To be discussed on the 9th of July

The goal of this sheet is to continue to reap the fruits of our hard work as we start to use the basic SFT package to prove a beautiful theorem. The exercises in Part II requires a small amount of Riemannian geometry[1] but those in Part I and III do not. We should highlight that, unlike the older sheets where the emphasis was placed more towards understanding through details, here we expect to get a feel and idea on how to use the machinery. A general convention for now: all manifolds here are assumed to be orientable.

PART I: The question and curves in $\mathbb{CP}^n$. Recall that a Lagrangian in a symplectic manifold (W, ω) is a real half-dimensional closed submanifold L on which $\omega|_L = 0$ (so L satisfies a PDE) and that, by Weinstein's theorem, they have standard neighbourhoods symplectomorphic to a neighbourhood of the zero section in T^*L. It is a central problem in symplectic topology to understand the Lagrangians in a given symplectic manifold, as they hold a lot of algebraic, dynamical and symplectic information.

1. In $\mathbb{CP}^2$ a Lagrangian is a closed surface, show that there are no closed Lagrangians of genus $g > 1$.

   Hint: Consider the pairings $[\omega_{\mathrm{FS}}] \cdot [L]$ and $[L] \cdot [L]$ to derive a contradiction.

To investigate this question in $\mathbb{CP}^n$, let's generalize the genus condition. Observe that the surfaces of genus $g > 1$ are precisely those that admit an everywhere negatively curved Riemannian metric.[2]

Question. Are there Lagrangians in $\mathbb{CP}^n$ that admit a Riemannian metric of negative sectional curvature?

Observe that the condition on the Lagrangians is not just geometric: we have not asked if negatively curved Lagrangians embedded in $\mathbb{CP}^n$ isometrically with that given metric, but if they embedd as soon as they admit such a metric.[3] Note, as well, that while the sectional curvature of $\mathbb{CP}^n$ is positive (in fact, bounded between 1 and 4) there is no reason to expect it does not have negatively curved submanifolds. So, by imposing the Lagrangian condition, we achieve the following striking result due to Viterbo (see remark 1):

Theorem. *There are no Lagrangians that admit Riemannain metrics of negative sectional curvature in complex projective space $\mathbb{CP}^n$.*

The reason this is true is that near a negatively curved Lagrangian holomorphic curves must be sparse, but $\mathbb{CP}^n$ has too many of them. We make this rigorous as follows: in Part I we show that

[1] Chapters 34 and 25 of Prof. Wendl' differential geometry notes cover more than enough for what we need, which is the definition of geodesic, the Riemann curvature tensor and sectional curvature.

[2] You may have realized that an even simpler argument also rules out genus Lagrangians 0 in $\mathbb{CP}^2$. The analogous naive generalization does not work, as $\mathbb{RP}^3$ is spherical and a Lagrangian on $\mathbb{CP}^3$.

[3] Note that admitting such a metric imposes heavy restrictions on the topology of the manifold, especially on the structure of the fundamental group, which becomes very rich.

$\mathbb{CP}^n$ has genus 0 curves through every point; in Part II we study the SFT of the cotangent bundle of a negatively curved manifold and heavily constraint the moduli space of holomorphic curves that can arise; in Part III we introduce and study the compactness properties of "neck stretching", a way to relate the holomorphic curves of a manifold with the curves in the cotangent bundle of a Lagrangian (and a lot more, see that section). In Part IV we put everything together.

Let us turn to J-curves in $\mathbb{CP}^n$. In the last sheet we showed that there is a J-line through every point in $\mathbb{CP}^2$ no matter the J. We now generalize this: we establish the analogue for $J = i$ and higher n and then use the continuation method to understand them for generic J.

2. Consider the moduli space $\mathcal{M}_0([H], i)$ of rational curves in the class of the line in $(\mathbb{CP}^n, i)$,[4] sketch an explanation of why it is compact, why all curves in it are Fredholm regular (deduce this from the analogous statement in $\mathbb{CP}^2$) and why the evaluation map $ev : \mathcal{M}_{0,1}([H], i) \to \mathbb{CP}^n$ is surjective.

3. Sketch a proof that for a generic $J \in \mathcal{J}(\omega_{\text{FS}})$ ω_{FS}-compatible almost complex structure, $\mathcal{M}_{0,1}([H], J)$ is a compact manifold and $ev : \mathcal{M}_{0,1}([H], i) \to \mathbb{CP}^n$ is surjective.

Upshot: *For generic J, there exists a rational J-curve in the class of a line through any given point in $\mathbb{CP}^n$.*

PART II: The SFT of cotangent bundles of Riemannian manifolds Let (L, g) be a closed Riemannian n-manifold. Recall that g defines the musical isomorphisms

$$T^*L \underset{\flat}{\overset{\sharp}{\rightleftarrows}} TL$$

by $Y_\flat \mapsto g(Y, -)$ and $Y^\sharp$ is defined as the inverse of $-_\flat$. This defines the dual metric on T^*L which we continue to denote by g or $\langle , \rangle$. Writing $\widehat{W} = T^*L$ for the total space and (q, p) for the base-fiber coordinates of T^*L, we can write the unit cotangent as follows:

$$M := \mathbb{S}^*L = \{(q, p) \in T^*M : \langle p, p\rangle = 1\}.$$

The (dual of the) Levi-Civita connection defines a horizontal/vertical splitting $T(q, p)(T^*L) = T_qL \oplus Tq^*L$, for which we write $X = (X^h, X^v) \in T(T^*L)$. Like this, the tangent space of M in $\widehat{W}$ is simply

$$T_{q,p}M = T_{q,p}\mathbb{S}^*M = \{(X^h, X^v) : \langle p, X^v\rangle = 0\} \subseteq T_{(q,p)}\widehat{W}.$$

We define the following 1-form, almost complex structure and vector field on $\widehat{W}$:

$$\lambda_{(q,p)}(X) = p(X^h), \;\; J = \begin{pmatrix} 0 & \sharp \\ -\flat & 0 \end{pmatrix} : TW = H \oplus V \to H \oplus V = TW, \;\; V(q, p) = (0, p).$$

4. Show that $d\lambda = \langle J-, -\rangle$ on $\widehat{W} = T^*L$ and verify that λ is a contact form[5] on M defining the contact structure

$$\xi_{(q,p)} = \{(X^h, X^v) : \;\langle p, X^v\rangle = 0, \; p(X^h) = 0\}$$

on which $d\lambda = \langle J-, -\rangle$ is non-degenerate, J is $d\lambda$-compatible and V is the Liouville vector field on M in $\widehat{W}$ dual to λ. The h/v splitting gives us a splitting $\xi = \xi^h \oplus \xi^v$, check that J maps ξ^h isomorphically to ξ^v.

[4] Recall that by a rational curve we mean a holomorphic curve of genus 0, by class of the line $1 = [H] \in \mathbb{Z} = H_2(\mathbb{CP}^n)$ and i is the standard complex structure.

[5] One can verify that in local coordinates this is the contact form $\sum p_i dq_i$, the canonical contact form on T^*L, which is a primitive of the canonical symplectic structure.

5. Verify that the Reeb vector field R is given by $R(q,p) = (p^\sharp, 0)$ (the components are taken with respect to the h/v splitting). Conclude that each period $T > 0$ closed Reeb orbit $\widetilde{\gamma}$ of $(M, \xi = \ker \alpha)$ projects to a constant speed T closed geodesic[6] γ of (L,g) under $\pi : M = \mathbb{S}^*L \to L$ and that conversely each constant speed closed geodesic lifts to a closed Reeb orbit whose period is that speed.

Observe that the contact formalism for dealing with geodesics geometrizes the usual method of turning a second order ODE into a system of first order ODE's. Now that we understand how Reeb orbits model geodesics, we need to understand how the contact and Riemannian invariants of the orbits are related to each other. Namely, we want to relate the Morse index of the geodesic with the Conley-Zehnder index of its lift. These are indices of the stability operators of both notions: for the CZ index we have the asymptotic operator of the orbit (hessian of the contact action functional) and for the Morse index we have the hessian of the energy functional. Indeed, recall the energy functional

$$E : \mathcal{C}^\infty(\mathbb{S}^1, L) \longrightarrow \mathbb{R}, \;\; \gamma \longmapsto \frac{1}{2}\int_{\mathbb{S}^1} \langle \gamma(t), \gamma(t) \rangle dt.$$

Its critical points are the closed geodesics and it's a standard exercise to verify that the hessian of E at a critical point is a second order linear L^2-symmetric differential operator given by

$$H_\gamma : \Gamma(N_\gamma) \longrightarrow \Gamma(N_\gamma), \;\; X \longmapsto -\nabla_t^2 - R(X, \dot\gamma)\dot\gamma,$$

where N_γ is the normal bundle of the orbit γ in L and R the Riemann curvature tensor.[7] We say that a closed geodesic γ is **non-degenerate** if the kernel of H_γ is trivial and the **Morse index** of γ is the number of negative eigenvalues[8] of H_γ. Cominng from the difficult definition of the Conley-Zehnder index, this should feel surprising, but indeed:

6. Show that the Morse index is well defined by checking that H_γ only has has only finitely many (counting with multiplicities) negative eigenvalues. Deduce as well that all closed geodesics in spaces of negative sectional curvature have Morse index 0.

   Hint: Consider $H_\gamma^\lambda = H_\gamma - \lambda$ for $\lambda < 0$ and compute/bound $\langle X, H_\gamma^\lambda X \rangle$. The curvature term $R_1(X) = R(X, \dot\gamma)\dot\gamma$ is an L^2-bounded operator. When all sectional curvatures are negative, $-R_1$ will have to be a positive operator.

Now we are almost ready to state the main technical theorem of this section. The h/v splitting gives us $\widetilde{\gamma}^*\xi = \widetilde{\gamma}^*\xi^h \oplus \widetilde{\gamma}^*\xi^v$, a trivialization of the normal bundle of the (oriented) geodesic γ determines a trivialization τ of $\widetilde{\gamma}^*\xi$ by taking the horizontal lift of it to $\widetilde{\gamma}^*\xi^h$ (isomorphic to N_γ), to be written as τ^h, and defining $\tau^v = J\tau^h$ and $\tau = \tau^h \oplus \tau^v$.

> Theorem. *Let γ be a closed geodesic of (L,g) and $\widetilde{\gamma}$ a lifted closed Reeb orbit of the associated strict contact manifold (M,α) with the associated trivialization τ as explained above. Then,* $\mathrm{Morse}(\gamma) = \mu_{CZ}^\tau(\widetilde{\gamma})$.

We will prove this theorem (cf. remark 2) when the Riemannian metric has everywhere negative sectional curvature by computing the asymptotic operator of $\widetilde{\gamma}$ and explicitly showing its CZ-index to be 0. The first computation is important but perhaps not that enlightening, so we leave it as a bonus.

[6]Recall that this is a map $\gamma : \mathbb{S}^1 \to L$ such that $\nabla_t\dot\gamma = 0$ and it is of constant speed T if $|\dot\gamma(t)| = T$. A small word of caution: if $\gamma(t)$ is a geodesic $\gamma(-t)$ also is, but the corresponding statement for Reeb orbits is false, each of this lift to different curves in $\mathbb{S}^*L$ (eg. at $\widetilde{\gamma}(0) \neq \widetilde{\gamma}(-0)$ the horizontal part is $\gamma(0) = \gamma(-0)$ but the vertical parts are antipodal).

[7]Technically speaking, we have written down the "normal" Hessian. This is the natural thing to do because the full hessian always is degenerate along the directions tangent to γ, so we split them off. Also, recall that the Riemann curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X - \nabla_{[X,Y]}Z$, i.e. the failure of $\nabla_X\nabla_Y$ to commute made tensorial via $\nabla_{[X,Y]}Z$.

[8]Similarly as with asymptotic operators, $H_\gamma : H^1 \to L^2$ is Fredholm of index 0 and, refarded as an unbounded operator on $L^2(N_\gamma)$ with dense and compactly embedded domain $H^2(N_\gamma) \subseteq L^2(N_\gamma)$, the spectrum of H_γ consists of isolated real eigenvalues, each with finite multiplicity

Bonus 1. Show that, after trivializing $\widetilde{\gamma}^*\xi$ with τ and writing T for the period of $\widetilde{\gamma}$, the asymptotic operator of $\widetilde{\gamma}$ can be written as

$$A_{\widetilde{\gamma}} = -J\nabla_t + \begin{pmatrix} -\frac{1}{T}R(\cdot,\dot{\gamma})\dot{\gamma} & 0 \\ 0 & -T \end{pmatrix}.$$

Deduce that $\ker A_{\widetilde{\gamma}} \cong \ker H_\gamma$ (and so a closed geodesic γ is non-degenerate if and only if $\widetilde{\gamma}$ is non-degenerate).

Hint: Compute the linearization of the gradient of the contact action functional at $\widetilde{\gamma}$ parametrizing it as follows. In $(q,p) \in \mathbb{S}^*L$ coordinates, $\widetilde{\gamma} = (\gamma, p)$ and choose a parametrization so γ is of constant speed $T = p(\dot{\gamma}) > 0$ and $p = \frac{1}{T}\dot{\gamma}_\flat$ (by the geodesic equation).

7. * Show that, when g has everywhere negative sectional curvature, it must hold that $\mu_{\mathrm{CZ}}(A_{\widetilde{\gamma}}) = 0 = \mathrm{Morse}(\gamma)$.

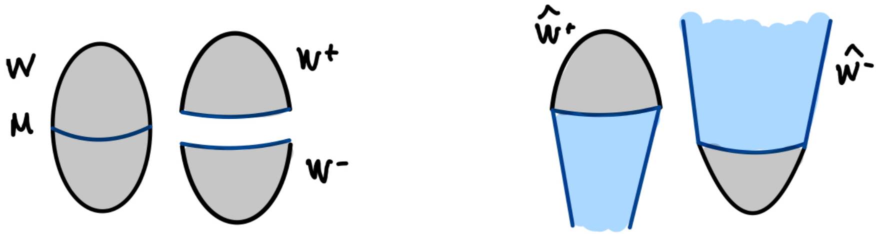
   Hint: Homotope $A_{\widetilde{\gamma}}$ to $A_0 = -J\nabla_t + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ through non-degenerate operators and deduce that $\mu_{\mathrm{CZ}}(A_{\widetilde{\gamma}}) = \mu_{\mathrm{CZ}}(A_0) = 0$. Recall that the operator R_1 defined above is positive...

We are now ready to explain what we meant with there are few holomorphic curves in the cotangent bundle of a negatively curved space. We regard the disc cotangent bundle $W = \mathbb{D}^*L$ as a filling of $\mathbb{S}^*L$ whose completion is identified with $\widehat{W} = T^*L$ (via the Liouville vector field V described above). In this way, we refer to the SFT of (L,g) as the SFT of $\widehat{W} = T^*L$ with the contact structure on the filled M associated to g. In the case g has negative sectional curvature, we have seen the corresponding contact form on M is non-degenerate and so:

8. * Take any list of closed geodesics $\gamma_1,\ldots,\gamma_r$, lifts to closed Reeb orbits $\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_r$ and a class $A \in H_2(\mathbb{D}^*L, \widetilde{\gamma}_1 \cup \cdots \cup \widetilde{\gamma}_r)$ in a negatively curved manifold L of dimension $n > 2$. For a generic choice of SFT-admissible almost complex structure, show that the moduli space $\mathcal{M}_h^*(A, J; \widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_r)$ is either empty or consists of isolated points. Moreover, if $n \geq 3$ and the moduli space is non-empty, then $h = 0$ and $r = 2$.[9]

   Hint: Use the index theorem above and generic transversality for simple curves.

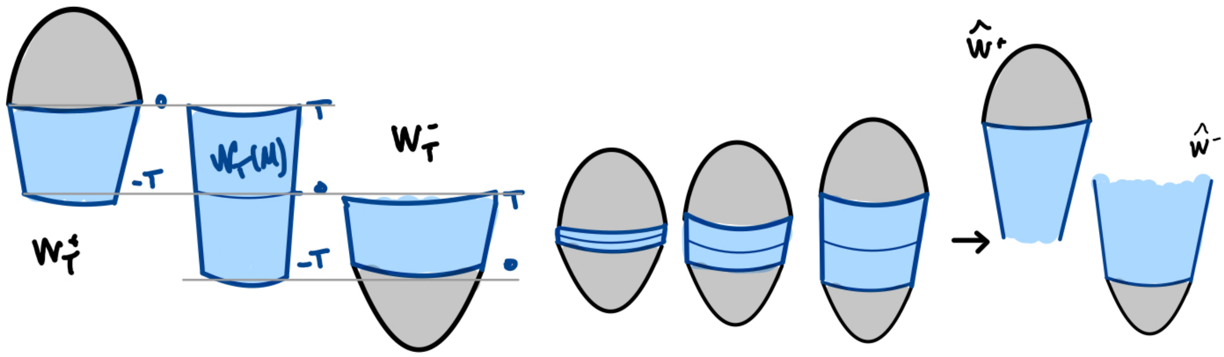
PART III: Neck-stretching and compactness We explain how to neck-stretch along a separating contact hypersurface (M,α) in a closed symplectic manifold (W,ω), see the remark 3 for more general context. Observe that W^+ and W^-, for $W_- \cup W_+ := \overline{W \setminus M}$, are symplectic manifolds with contact type boundary given by (M,α) (where W^+ caps it off and W^- fills it). As usual, we denote the completion of such cobordisms by $\widehat{W}^\pm$. The goal is to relate the closed holomorphic curves in W to asymptotically cylindrical holomorphic curves in $\widehat{W}^-$ and $\widehat{W}^+$.



We have seen in the lectures that M admits a neighbourhood $\mathcal{N}(M) \subseteq W$ symplectomorphic to $(\mathcal{N}_\varepsilon(M), \omega_\varepsilon) := ((-\varepsilon,\varepsilon)_r \times M, d(e^r\alpha))$. We define W^T by the (topologically trivial) replacement of

[9]To rule out the $h = 0$ and $r = 1$ case recall that closed geodesics in negatively curved manifolds are homotopically essential.

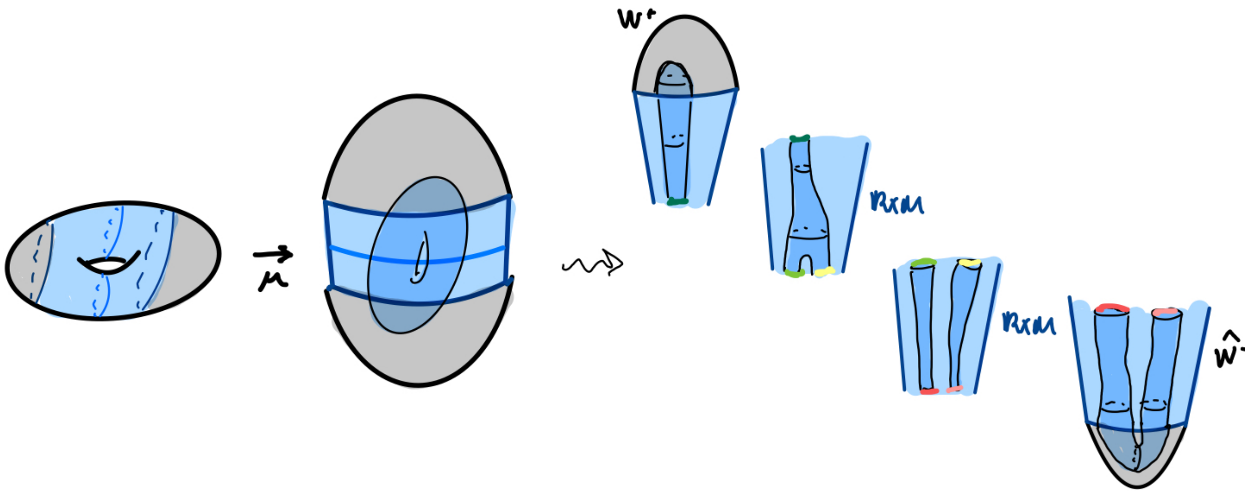
$\mathcal{N}(M)$ by $\mathcal{N}_T(M)$ (stretching this neck). This decomposes $\overline{W^T \setminus M}$ in the truncated completions $W_T^{\pm} = (W^{\pm} \cup ([0, \mp T] \times M, d(e^r\alpha))$. The limit of this procedure makes ∂W^+ and ∂W^- in W^T infinitely far apart, and so we interpret the limit as the **split symplectic manifold** given by $W^\infty = \widehat{W}^+ \sqcup \widehat{W}^-$.[10]



A compatible almost complex structure $J \in \mathcal{J}(W, \omega)$ is said to be adapted to the contact hypersurface (M, α) if $J|_{\mathcal{N}(M)} = J_M|_{\mathcal{N}_\varepsilon(M)}$ for some almost complex structure J_M on the symplectization $\mathcal{N}_\infty(M)$ of M that is SFT-adapted, i.e. $J_M \in \mathcal{J}^{\mathrm{SFT}}(\alpha)$. Now, we can neck-stretch adapted almost complex structures as follows: we define J^T on W^T as J in $W^T \setminus \mathcal{N}_T(M) = W \setminus \mathcal{N}(M)$ and as J_M. This complex structure converges to a J^∞ in $\widehat{W}^{\pm}$ that matches the original on $W^{\pm}$ and is SFT-admissible on the completion parts.

9. * Explain how to start from a Largangian L in (W, ω) and obtain the split symplectic manifold $\widehat{W}^+ = W \setminus L$, $\widehat{W}^- = T^*L$ and the symplectization of $(M, \alpha) = (\mathbb{S}^*L, \alpha)$, where the underlying α is the contact form α associated to *any* Riemannian metric g on L.

We now go on to understand what happens to holomorphic curves under neck-stretching to be able to relate the holomorphic curves on $\mathbb{CP}^n$ to those of T^*L.



10. * Let J be adapted to (M, α), $(T_k)_k$ a sequence of positive real numbers going to ∞ and $(J^{T_k})_k$ the corresponding neck-stretching sequence. If $(j_k)_k$ is a sequence of complex structures on

[10] To do this rigorously we must clarify that this can be done symplectically. The way the cylindrical parts of $W_T^{\pm}$ are fit complimentarily in $\mathcal{N}_T(M)$ is done by translating the $\mathbb{R}$-component, which rescales by an exponential factor the symplectic form). This does not change the symplectic form, as it produces a smooth path of cohomologous symplectic forms, so sympletomorphic ones. The stretching is not so much seen by the sympelctic deformation itself, but how it compares to the almost-complex one which we now explain.

a closed surface Σ converging to j and $u_k : (\Sigma, j_k) \longrightarrow (W^{T_k}, \omega^{J^{T_k}}, J^{T_k})$ a sequence of holomorphic curves with a uniform bound on their gradients, describe the possible limits of these sequence.

Hint: The assumptions rule out the usual lack of compactness, you should think about the distribution of mass/energy of the u_k near, above, or below M along the neckstretching; the figure here should give you a good idea of the "new" compactness phenomenon.

Let us describe the correct notion of holomorphic building for neck-stretching along a contact hypersurface in a closed symplectic manifold. A building will have a main level, just as before, whose components consist of (possibly nodal) holomorphic curves (closed or with punctures) into $\widehat{W}^{\pm}$ and, instead of higher/lower levels, an insert layer consisting of *insert levels*: these are N copies of the symplectization $\mathbb{R} \times M$ where the building has asymptotically cylindrical components (up to $\mathbb{R}$-translation).[11] The other data of a building (nodes, breaking pairs, stability, equivalence) is the same. It is then a theorem that the space of these buildings is compact.

11. Show that a component of a limit building must have a component in either $\widehat{W}^{+}$ or $\widehat{W}^{-}$. If there is a component in $\widehat{W}^{-}$ that is not a closed holomorphic curve, then there must be a component in $\widehat{W}^{+}$. Hence, if $\widehat{W}^{-}$ is exact, all limits must have a component in $\widehat{W}^{+}$.[12]

PART IV: Finishing the proof and remarks

12. Put together the upshot of the first part, exercise 8, exercise 9 and the compactness theorem to show the main theorem, that there cannot be a Lagrangian in $\mathbb{CP}^n$ ($n > 2$) that admits a metric of negative sectional curvature.

Remark 1. The theorem and proof we have explained here is due to Eliashberg-Givental-Hofer in the foundational SFT paper, where they attribute the original result to Viterbo. They prove the theorem for uniruled Kähler manifolds, of which $\mathbb{CP}^n$ is an example. The key property, however, is that there should be a class for which any generic J has rational curves through every point (which we prove by hand for $\mathbb{CP}^n$), which is called weakly symplectically uniruled. Uniruled Kähler manifolds have a rational curve through any point for their given complex structure, but it is a difficult theorem of Ruan to show that this implies weak symplectic uniruled.[13] In whichever case, notice that the argument presented here shows that no weakly symplectic uniruled manifold has Lagrangians that would admit a metric of negative curvature.

Remark 2. The second part's "CZ=Morse theorem" is proven in full generality in the lecture notes. In our case, the proof greatly simplifies because the curvature operator is positive so we can homotope it to the identity through positive operators. I want to sketch an alternative point of view on this theorem. For path-geodesics, the Morse index can be defined in the same way (index of the hessian of the energy functional). A classical theorem of Morse states that this is the same as counting conjugate points along the geodesic: these are points on which there is a first order variation of the geodesic (Jacobi fields) with fixed endpoints on the beginning and on the to be called conjugate point. Both of these definitions of the Morse index make sense in so far they measure stability of a geodesic, though it is not obvious that they are equivalent. A similar description can be given for closed geodesics and define *resonant points*, points along which there exists a first order variations of the gedesics that displace (a chosen) origin as much as the

[11] The reason to distinguish this so-called insert layers is that when we neck-stretch in a symplectic cobordism it's useful to tell the different kind of layers appart. Moreover, they are naturally ordered cyclically and not linearly because, contrary to the intuituive figure, both components $W^{\pm}$ should be considered in the main layer (think what would be correct if the hypersurface was not separating).

[12] On the other hand, if there is a non-closed component in $\widehat{W}^{+}$, it is perfectly possible there is no component in $\widehat{W}^{-}$.

[13] Symplectic uniruledness is defined by the non-vanishing of the one-point insertion Gromov-Witten invariants in a given class. This implies the weak version of symplectic uniruledness but it's unclear, perhaps false, if this definitions are equivalent.

to be called resonant point.[14] These variations are exactly the linearized Reeb flow (=linearized cogedesic flow) applied to a given vector on the (chosen) origin. In particular, resonant points model eigenvalues 1 along the orbit of the linearized Reeb flow. But the count of eigenvalues 1 that the linearized flow acquires along the orbit is exactly the Conley-Zehnder index. This should explain CZ=Morse from a more topological perspective.

Remark 3. The other important use of neck-stretching (that I am aware of) is to understand algebraic structures defined using holomorphic curves when we concatenate symplectic cobordisms together. We have in fact seen a light example of this when we showed that cylindrical contact homology is well-defined. For a general exposition of neck-stretching (contact hypersurfaces, not necessarily connected or separating) in a cobordism in the SHS setting you are encouraged to check out the lecture notes.

[14]I have not found this in the literature, so if you know of a reference that does this with another name, please let me know.