Symplectic field theory

Problem set 2

Gerard Bargalló i Gómez and Chris Wendl

To be discussed on the 30th of April

The goal of this exercise sheet is to explore the ellipticity condition further (by popular request) in a general setting and in the more concrete setting of the $\overline{\partial}$ operator. This is the model version of the holomorphic curve equation. Real-linear Cauchy-Riemann operators are the infinitesimal counterpart to the non-linear holomorphic curve equation. Locally, as we shall see, all these look like a perturbation of $\overline{\partial}$. So we begin the analytic underpinnings of holomorphic curve theory.

I have tried to follow the notation of the lecture and, just in case, have made a cheat sheet at the end to establish some prerequisits.

Problem 1. General definition of elliptic operators and elliptic regularity. Let M^n be a smooth manifold and E and F vector bundles over it. A *differential operator of order k* is a linear map $D : \Gamma(M, E) \longrightarrow \Gamma(M, F)$ such that for any choice of local coordinates on $U \subseteq M$ and corresponding trivializations of E and F it can be written as

$$Ds = \Sigma_{|\alpha|\leq k} a_\alpha \partial^a s,$$

for some $a_\alpha : U \longrightarrow \mathrm{Mat}_{\mathbb{R}}(\mathrm{rk}\, F \times \mathrm{rk}\, E)$ and $s \in \Gamma(U, E) \cong \mathcal{C}^\infty(U, \mathbb{R}^{\mathrm{rk}\, E})$. Note the multi-index notation, see fact 1 in the cheat sheet. We have stated the definition for real vector bundles but obvious analog works for complex ones.

a. Convince yourself that the following are differential operators of order 1: the exterior derivative d (with $E = \Lambda^m T^*M$ and $F = \Lambda^{m+1}T^*M$); any connection ∇ on a vector bundle E (with $F = T^*M \otimes E$); a Cauchy-Riemann type operator as in the lectures (E is a complex vector bundle and $F = \overline{\mathrm{Hom}}_{\mathbb{C}}(TM, E)$). Note that part *b.* may help.

b. Given a real-linear Cauchy-Riemann operator D, show that any other such operator D' takes the form $D' = D + A$ with $A : E \to F = \overline{\mathrm{Hom}}_{\mathbb{C}}(TM, E)$ a smooth linear bundle map.[1]

We now define the principal symbol of a differential operator. It is meant to be an algebraico-topological object associated to a differential operator that captures its terms of leading order. Let $\pi : T^*M \to M$ be the natural projection, the *principal symbol* $\sigma_D \in \Gamma(T^*M, \mathrm{Hom}(\pi^*E, \pi^*F))$ of D is defined as follows: given a covector $\xi \in T^*_x M$ we define the linear mapping

$$\begin{aligned} \sigma_D(\xi) : E_x &\longrightarrow F_x \\ e &\longmapsto \frac{1}{k!} D(f^k \bar{e})|x, \end{aligned}$$

for any smooth function f on M such that $f(x) = 0$, $df(x) = \xi$ and extension $\bar{e} \in \Gamma(M, E)$ of e. The following exercise should explain the definition both in an abstract way (leading order behaviour) and concrete (this definition is very well suited to operators with nice Leibniz rules).

[1] Essentially, the exercise shows that the space of real-linear Cauchy-Riemann operators is affine $\Gamma(M, \mathrm{Hom}(E, \overline{\mathrm{Hom}}_{\mathbb{C}}(TM, E)))$. The local version of this exercise essentially shows part *a* as it shows that locally all real-linear Cauchy-Riemann operators are a 0-th order deformation of $\overline{\partial}$. For connections the analogue of this zero-th order terms are Christoffel symbols.

c. In local coordinates as in the definition, show that

$$\sigma_D(\xi)\cdot e = \sum_{|\alpha|=k} a_\alpha(x)\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}\cdot e.$$

Given a differential operator on $\mathbb{R}^n$, what is the Fourier transform of its leading order terms?

d. Deduce from this that the symbol is well-defined and has the property that two differential operators D and D' (on the same bundles) of order k have the same symbol if and only if $D - D'$ has order $k-1$.

e. Compute the symbol for the examples above d, ∇ and D.

Take note of the tight relationship between part *d.* and part *b.* (after computing part *e.*).

A differential operator is said to be *elliptic* if for every $\xi \in T_xM$ that is non-zero the principal symbol $\sigma_D(\xi) : E_x \to F_x$ is invertible.

f. When are d, ∇ and D elliptic? Hint: it may depend on the dimension of M.

One of the important consequences of ellipticity is *elliptic regularity*, solutions to elliptic PDEs will be "regular" as the functions that define the equation.[2] This allows us to use Sobolev spaces and glorified linear algebra to solve PDE's and then show that the weak solutions found are actually smooth. The goal of the rest of the exercise is to give a very convincing reason to believe this.

Consider differential operator of order k on $M = \mathbb{R}^n$ of constant coefficients:

$$\begin{aligned} D : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell) \\ f &\longmapsto \sum_{|\alpha|\le k} a_\alpha \partial^\alpha f, \end{aligned}$$

where the $a_\alpha \in \mathrm{Mat}_\mathbb{R}(m\times\ell)$ are constant. We assume that D is elliptic (so necessarily $\ell = m$) and show the following theorem: *if $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a smooth function with compact support, then every solution $f \in \mathcal{C}^k(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ to the equation $Df = g$ is smooth.*[3]

g. If $n = 1$, $Df = g$ would be an ODE. Deduce the theorem in that case (not needed for the rest but insightful).

h. We want to work with Fourier transforms (see fact 4 in the cheat sheet), so we complexify D, meaning that now $D : \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m) \to \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C}^m)$ by considering the real entries of a_α as complex entries, $a_\alpha \in \mathrm{Mat}_\mathbb{C}(m\times\ell)$. By applying the Fourier transform to $Df(x)$ one gets a matrix valued polynomial $Q : \mathbb{R}^n \to \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^m)$ defined by $Q(p)\widehat{f}(p:) = \widehat{Df}(p)$. Do this yourself to find the expression of Q and verify that Q_k, the highest order terms of Q, coincide with the symbol up to a constant factor.

i. From the exercise above and ellipticity, we know that $Q_m(p)$ is an invertible matrix for each $p \in \mathbb{R}^n \setminus 0$ (though, in fact, only injectivity will be needed). Show that

$$|Q_m(p)v| \ge C|p|^k|v| \quad \text{for all } p \in \mathbb{R}^n \text{ and } v \in \mathbb{C}^m.$$

j. Deduce from the above the following bound for Q (so Q_m after adding the lower order terms) for a sufficiently large R:

$$|Q(p)v|^2 \ge C'(1+|p|^2)^k|v|^2 \quad \text{for all } p \in \mathbb{R}^n \cap \{|p| > R\} \text{ and } v \in \mathbb{C}^m.$$

[2]This is really not a general phenomenon: $(\partial_1 - \partial_2)f = 0$ admits the solution $f(x_1, x_2) = (x_1 + x_2)^2|x_1 + x_2|$ which is $\mathcal{C}^2$ but not smooth. Can you compute the symbol?

[3]In fact, the proof would work for the weaker conditions that the symbol is injective on covectors, not necessarily bijective, so $m \le \ell$ would be allowed.

k. Consider $f \in L^2(\mathbb{R}^n)$ and $g \in H^2(\mathbb{R}^n)$ for $s \geq 0$ and assume that the equation $Q(p)\widehat{f}(p) = \widehat{g}(p)$ is satisfied for almost all $p \in \mathbb{R}^n$ (this is the Fourier transform of $Df = g$). Show that

$$||f||_{H^{s+k}} \leq C||g||_{H^s} + ||f||_{L^2}.$$

This is the *fundamental elliptic estimate*: in the LHS all derivatives of f appear up to order $s+m$ while only some linear combinations of that order appear in the RHS in the $g = Df$ summand. Ellipticity controls the L^2-norms of the many just by knowing the norms of the few.

l. Deduce the theorem from this (use the Sobolev embedding theorem).

m. *Bonus:* Show the Sobolev embedding theorem used above, i.e. for $H^s(\mathbb{R}^n)$: *for any real number $s > n/2$ and every integer $k \geq 0$, there exists a continuous linear inclusion $H^{s+k}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^k(\mathbb{R}^n)$.*

Hint: Show first that the linear inclusion $H^{s+k}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^k(\mathbb{R}^n)$ is bounded by showing that if $f \in H^s(\mathbb{R}^n)$ then $||\widehat{f}||_{L^1}$ is bounded, as then $f = \mathcal{F}^*(\widehat{f}) \in \mathcal{C}^0(\mathbb{R}^n)$ as $\mathcal{F}^*$ defines a continuous linear map $L^1(\mathbb{R}^n) \longrightarrow \mathcal{C}^0(\mathbb{R}^n)$.

Remark. In the first part of the exercise we play with 3 differential operators of great importance, some applications of these: d defines deRahm cohomology; ∇ is essential to study geodesics; the CR-operators are the key to the deformation theory of holomorphic curves. All of these are manifestations of ellipticity on closed manifolds (or variants of, like elliptic complexes) through the so called Fredholm property. This essentially allows that the infinite dimensional geometry of the PDE admits a finite-dimensional treatment, and in particular the solution space is also finite-dimensional. We will see this in the course in a few weeks (though for operator on non-closed manifolds, like punctured Riemann surfaces, we will have to work harder). Many moduli problems in geometry are actually Fredholm problems, so the theory we will develop is deep and extrapolable to many other situations, like Gauge theory.

Problem 2. The $\overline{\partial}$-operator: elliptic estimate and fundamental solution for $\overline{\partial}$. The goal of the next two items is to show the fundamental elliptic estimate for $p = 2$.[4] Note that a more involved and general version is shown in Problem 1, k. Fact 2 in the past page may be quite useful for computations.

a. Given $u \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{C})$, show that $||\partial u||_{L^2} = ||\,\overline{\partial}\,u||_{L^2}$.

Hint: Apply Stokes theorem to $d(ud\overline{u}) = du \wedge d\overline{u}$ on a sufficiently large ball.

b. Show that there exists a constant c such that for every $u \in \mathcal{C}_0^\infty(B, \mathbb{C})$,

$$||u||_{W^{1,2}} \leq c||\,\overline{\partial}\,u||_{L^2}.$$

Hint: Poincaré's inequality bounds the L^p-norm of u in terms of du.

Now it may be a good moment to check fact 3 in the cheat sheet. A *fundamental solution* to a PDE $Df = g$ is a function $K \in L^1_{\text{loc}}(\mathbb{C})$ such that $DK = \delta$ in the sense of distributions (D is a differential operator, think of $D = \overline{\partial}$ if you have not done the previous exercise). This is relevant because it implies that for any $g \in \mathcal{C}_0^\infty(\mathbb{C})$, $K * g$ is a smooth solution to the equation $Df = g$.

c. Verify that, indeed, if $DK = \delta$ in the sense of distributions, then $D(K * g) = g$.

The goal of the next two items is to find a fundamental solution to the inhomogenous $\overline{\partial}$-equation that has the property that all of its solutions are the given by the inhomogenous term convoluted with the fundamental solution.

[4]Sadly, this estimate will not be enough for applications: on a two-dimensional domain, $n = 2$, $W^{1,p}$ embeds into $\mathcal{C}^0$ only for $p > 2$. See the remark for how to solve this.

d. Define $K(z) := \frac{1}{2\pi z} \in L^1_{\text{loc}}(\mathbb{C}, \mathbb{C})$. Show that it satisfies $\overline{\partial} K = \delta$ in the sense of distributions.

   Hint (complex analysis recall): If γ is a circle around the origin, $\int_\gamma \frac{dz}{z} = 2\pi i$ by Cauchy's integral formula.

e. Show that for any $f \in \mathcal{C}^\infty_0(\mathbb{C})$ the function $K * f$ satisfies $|(K * f)(z)| \leq \frac{C}{|z|}$ for some constant $C > 0$ and deduce from this that if $u \in \mathcal{C}^\infty_0(\mathbb{C})$ satisfies the equation $\overline{\partial} u = f$, then $u = K * f$.

   Hint (complex analysis recall): The only holomorphic function on $\mathbb{C}$ that decays at infinity is the zero function.

We can in fact upgrade this to something better:

f. Show that the operator

$$\begin{aligned} T : \mathcal{C}^\infty_0(\mathbb{C}) &\longrightarrow \mathcal{C}^\infty(\mathbb{C}) \\ f &\longmapsto K * f \end{aligned}$$

   that we have defined above extends as a bounded linear operator $T : L^2(B) \longrightarrow W^{1,2}(B)$ which is right inverse to $\overline{\partial} : W^{1,2}(B) \longrightarrow L^2(B)$.

   Hint: Recall that $\mathcal{C}^\infty_0$ is dense in L^2. Do not think about it too hard, all is done by now.

Remark. As explained before, we would really like to have these statements for $p \neq 2$. As we have seen, it all boils down to bounding $||\partial(K * f)||_{L^p}$. This can be done by regarding $\partial(K * -) = (\partial K) * -$ as a singular integral operator ($\partial K = -1/\pi z^2$ is not locally integrable) for which Calderon-Zygmund theory turns L^2 regularity into L^p. The proof of this estimate is the crux of the local elliptic regularity theorem in the lecture. Moreover, the local existence of holomorphic curves, which is foundational to the whole theory, is a direct consequence of T being a right inverse to $\overline{\partial}$ in the correct Sobolev spaces.

Problem 3. Almost-complex structures on surfaces are integrable. The following problem is an excuse to cover a theorem for which there was no time in the lecture. As a (more or less direct) consequence of the existence of a right-inverse of $\overline{\partial}$, in the lectures we have seen for a "*Linear local existence*" result, which can be re-stated as follows (make sure you see why): *If D is a real linear Cauchy-Riemann operator on E, then for any $e \in E$ we can (locally) find a (smooth) section $\eta : \mathbb{D}_\varepsilon \to E$ such that $D\eta = 0$ and $\eta(0) = e$.*

There is a non-linear analogue, which means the analogue for the holomorphic curve equation and not the $D = 0$ equation (which is the linear/infinitessimal analogue). Figure out what the statement of this should be and use it to show the following theorem: Let Σ be a real 2-manifold without boundary with an almost complex structure $j : T\Sigma \longrightarrow T\Sigma$ (ie. $j^2 = -id$), then j is integrable (i.e. comes from a complex altas).

Remark. This is a deep fact, the almost-complex geometry of domains of holomorphic curve is in fact complex geometry, which for curves is exactly algebraic geometry. The Newlander-Nirenberg theorem is a deep theorem that characterizes the integrability of an almost complex structure by a "simple" equation. One can see that for surfaces the equation is verified. The (non-linear) local existence of holomorphic curves is a very non-trivial result that implies the NN-theorem for surfaces. Although we will not prove it in the lecture it will be a consequence of the "non-linear Fredholm set up" and the local existence result (maybe we'll make it an exercise here). In any case, another nice consequence of (a small generalization of) the non-linear local existence is that one can make sense of holomorphic Jet bundles for almost-complex manifolds.

Cheat sheet

Fact 1: *Multi-index notation.* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index and we define $|\alpha| = \sum \alpha_i$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}.$$

Fact 2: *Some notation convinient for computations.* Using the coordinates $z = s + it$ in $\mathbb{C}$, we can define the differential operators ∂_z and $\partial_{\overline{z}}$ (which are the same as ∂ and $\overline{\partial}$ up to a factor of 2) as well as the complex valued 1-forms dz and $d\overline{s}$

$$\partial_z = \frac{1}{2}(\partial_s - i\partial_t), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_s + i\partial_t),$$
$$dz = ds + idt, \qquad d\overline{z} = ds - idt.$$

Consequently, note that $du = \partial_z u dz + \partial_{\overline{z}} u d\overline{z}$ and the usual Lebesgue measure in $\mathbb{C}$ is $dm(z) = \frac{dz \wedge d\overline{z}}{-2i}$.

Fact 3: *Distributions and convolution.* Let U be an open subset of $\mathbb{R}^n$, $\mathcal{D}(U) := \mathcal{C}^\infty_0(U, \mathbb{C})$ the test functions and $\mathcal{D}'(U)$ the space of continuous complex-linear functionals $\mathcal{D}(U) \to \mathbb{C}$, known as the distributions on U. Denoting (T, φ) the value of a distribution T at a test function φ, we note that any $f \in L^1_{\text{loc}}(U)$ defines a distribution by setting $(f, \varphi) := \int_U f\varphi$ and recall that the Dirac δ-distribution is defined by $(\delta, \varphi) := \varphi(0)$. Recall that we say that a distribution R is the j-th partial derivative of T and write $R = \partial_j T$ if $(R, \varphi) = -(T \partial_j \varphi)$, and this extends uniquely to define higher-order derivatives of T. When a locally integrable function f has distributional derivatives that are also locally integrable we call them weak derivatives.

When $U = \mathbb{R}^n$ we can define the convolution of two test functions by

$$\mathcal{D}(U) \longrightarrow \mathcal{D}(U) : \varphi \longmapsto (f * \varphi)(x) = \int_{\mathbb{R}^n} f(x-y)\varphi(y) dm(y),$$

which extends to a continous linear endomorphism of the space of distributions by the rule $(f * T, \varphi) := (T, f^- * \varphi)$, where $f^-(x) = f(-x)$ and we define $T * f := f * T$. Recall that $\partial_j (f * T) = (\partial_j f) * T = f * (\partial_j T)$ and $f * \delta = f$. Finally, recall that convoluting a distribution with a test function produces a distribution that is representable by a smooth function.

Fact 4: *Fourier transforms.* For $f \in L^1(\mathbb{R}^n)$, we can define the Fourier transform $\mathcal{F}f : \mathbb{R}^n \to \mathbb{C}$ and Fourier inverse transform $\mathcal{F}^* f : \mathbb{R}^n \to \mathbb{C}$ as the function

$$\mathcal{F}f(p) := \widehat{f}(p) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot} dm(x), \ \ \mathcal{F}^* f(x) := \check{f}(x) := \int_{\mathbb{R}^n} f(p) e^{2\pi i x \cdot p} dm(p).$$

They both define bounded linear maps $L^1(\mathbb{R}^n) \longrightarrow \mathcal{C}^0(\mathbb{R}^n)$. These do not preserve the space of test functions but they do preserve the Schwarz space and there they are inverses to each other.

The Schwarz space $\mathcal{S}(\mathbb{R}^n)$ is the space of complex-valued smooth functions that decay, along with all of its derivatives, at least polynomially. Plancharel's theorem states that the L^2-inner product of two Schwarz functions is the same as that of their Fourier transforms. By density of test functions in the Schwarz space and L^2, the operators $\mathcal{F}$ and $\mathcal{F}^*$ have unique extensions to isometries $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The Fourier transform turns differenciation into multiplication by a polynomial: $\widehat{\partial_j f}(p) = 2\pi i p_j \widehat{f}(p)$. Moreover, it turns convolutions into multiplications: $\widehat{f * g}(p) = \widehat{f}(p)\widehat{g}(p)$.

Recall $H^k(\mathbb{R}^n)$ is the space of L^2 functions f such that the norm $||f||_{H^k} := ||(1 + |-|^2)^{k/2} \widehat{f}||_{L^2}$ are finite. A moment's thought shows that this norm is equivalent to $\sum_{|\alpha| \leq m} ||(2\pi i -)^\alpha \widehat{f}||$ and that this one is equivalent to $W^{k,2}$.

A note that will not be needed but is nice: if instead of defining distributions with compactly supported test functions we used Schwarz functions we would obtain the so-called tempered distributions, a class of distributions on which the Fourier transform is well defined and things like $\mathcal{F}(1) = \delta$ and $\mathcal{F}(\delta) = 1$ hold.