

Symplectic field theory

Problem set 4

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This problem set has two parts. The first is meant to help digest the Fredholm property, its relation to the spectral flow and the index theorem. The second focuses on some aspects of the Conley-Zehnder index that we will need in applications.

PART I: The Fredholm property and the index theorem.

For these problems, we can make use of lectures 1-4 freely, unless otherwise stated.

Problem 1. Finite dimensional “Fredholm index = spectral flow”. The goal of this exercise is to prove the following theorem, which is the Morse analogue of Lectures 3 and 4, by mirroring the strategies developed in class:

Theorem. Given a smooth path of matrices $A : \mathbb{R} \rightarrow \text{End}(\mathbb{R}^n)$ with limit matrices $A_{\pm} = \lim_{s \rightarrow \pm\infty} A(s)$ (that exist) and are both invertible and symmetric,¹ then the operator

$$D_A : H^1(\mathbb{R}, \mathbb{R}^n) \longrightarrow L^2(\mathbb{R}, \mathbb{R}^n) \\ \eta \longmapsto \frac{d}{ds}\eta + A\eta$$

is Fredholm and the index is given by

$$\text{ind } D_A = \#\{\text{negative eigenvalues of } A_-\} - \#\{\text{negative eigenvalues of } A_+\}.$$

Notice that the above count is with multiplicities and that the finite-dimensional spectral theorem says that the index is the same as the spectral flow. The first three items breakdown the proof of Fredholmness and the other 3 the index computation.

a. We want to show that D_A is Fredholm. Deduce this assuming the following estimate holds

$$\|\eta\|_{H^1(\mathbb{R})} \leq C(\|\eta\|_{L^2(I)} + \|D_A\eta\|_{L^2(\mathbb{R})}), \quad (1)$$

for $I = [-T, T]$ a sufficiently large interval. Verify this estimate for $I = \mathbb{R}$ and explain why it is not sufficient for Fredholmness.

b. If D_A is translation-invariant, i.e. $A(s) = A_0$ for all s , show that it is an isomorphism. From this, it follows that $\|\eta\|_{H^1} \leq C\|D_A\eta\|_{L^2}$.

Hint: Fourier transform.

c. No longer assuming D is translation-invariant, prove (1).

Hint: From the previous item we have that $\|\eta\|_{H^1} \leq C\|D_A\eta\|_{L^2}$ for an η that vanishes on $[1 - T, T - 1]$ for T large enough.

¹It suffices to ask that no eigenvalues lie in the imaginary axis. In that case we replace the numbers in the index count by the dimensions of the sum of the generalized eigenspaces of positive real parts.

d. Given a matrix $B \in \text{End}(\mathbb{R}^n)$, show that the following subspaces

$$E^s(B) = \{v \in \mathbb{R}^n : \lim_{t \rightarrow \infty} e^{tB}v = 0\},$$

$$E^u(B) = \{v \in \mathbb{R}^n : \lim_{t \rightarrow -\infty} e^{tB}v = 0\}$$

respectively correspond to the sum of generalized eigenspaces of eigenvalues of B of negative and positive real parts (it's okay to prove it in the symmetric case only).

e. Show that $\dim \ker D = \dim E^u(A_+) \cap E^s(A_-)$.

Hint: Consider the fundamental solution of the ODE defined by the kernel along with its own stable and unstable subspaces.

f. Conclude that $\text{ind } D_A = \dim E^s(A_-) - \dim E^s(A_+)$.

Hint: Compute the adjoint.

Remark. I think it is really worth it to reflect on the two main points of this exercise: D_A being an isomorphism when A is translation-invariant being the key to Fredholmness; and how the solutions to D_A are related to the eigenvalues/vectors of its asymptotic limits. This is the finite-dimensional intuition as to why elliptic operators in Floer theory have the Fredholm property, crucial to moduli theory, and how the Fredholm index is related to the spectral flow. In other problems we work on cultivating the SFT version of this.

Problem 2. Automatic transversality and index computations on the sphere. Consider a closed Riemann surface (Σ, j) along with a complex line bundle $(E, J) \rightarrow (\Sigma, j)$. Let D be any linear Cauchy-Riemann type operator on $(E, J) \rightarrow (\Sigma, j)$ (hence Fredholm).

a. Show that as soon as $c_1(E) < 0$, then D must be injective.

Hint: Similarity principle.

b. Show that as soon as $c_1(E) > -\chi(\Sigma)$, then D must be surjective.

Hint: Use *a.* appropriately.

c. As a corollary, compute that if (\mathbb{C}, i) is the trivial complex line bundle over \mathbb{S}^2 and D a linear Cauchy-Riemann type operator on it, then $\text{ind } D = 2$.

Remark. This problem is the basis to automatic transversality, which shows the unobstructedness of certain (complex) codimension 1 holomorphic phenomena. Later in the course we will see how useful this is.

Problem 3. “Fredholm index = spectral flow” theorem I. Consider D to be a translation invariant real linear Cauchy-Riemann type operator on the trivial (complex) line bundle over the cylinder with non-degenerate asymptotics. In this case, there is a non-degenerate asymptotic operator $A = -i\partial_t - S(t)$ on the trivial complex line bundle over the circle such that $D = \partial_s - A$:

$$D : H^1(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2) \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2)$$

$$\eta \longmapsto D(\eta) = \bar{\partial} \eta(s, t) + S(t) \cdot \eta(s, t) = \partial_s \eta - A(\eta).$$

In the lectures we have seen that these operators are Fredholm and isomorphisms.

a. If we restrict the domain to $Z_+ = [0, \infty) \times \mathbb{S}^1$, show that the operator $D : H^1(\mathring{Z}_+, \mathbb{R}^2) \rightarrow L^2(\mathring{Z}_+, \mathbb{R}^2)$ is not Fredholm.

Hint: Build zeros of D from eigenfunctions of A .

Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with values $\delta_{\pm}s$ for $\pm s \gg 1$, where δ_{\pm} are two real numbers. Define the following Banach spaces:

$$H_{\delta}^1(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2) = \{u : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}^2 \mid e^{-f}u \in H^1(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2)\}, \quad \|u\|_{H^1\delta} := \|e^{-f}u\|_{H^1},$$

$$L_{\delta}^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2) = \{u : \mathbb{R} \times \mathbb{S}^1 \rightarrow \mathbb{R}^2 \mid e^{-f}u \in L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^2)\}, \quad \|u\|_{L^2\delta} := \|e^{-f}u\|_{L^2}.$$

Consider D defined on these spaces, we write $D_{\delta} : H_{\delta}^1 \rightarrow L_{\delta}^2$ and $D_{\delta}\eta = \partial_s\eta - A(\eta)$.

c. Determine for which parameters of $\delta = \{\delta_+, \delta_-\}$ the operator D_{δ} is Fredholm.

Hint: The map e^{-f} conjugates D_{δ} to $D_f = D + f'$ on the non-weighted spaces.

d. Show that the index is the number of eigenvalues of A in (δ_1, δ_2) when $\delta_1 < \delta_2$ and the number of eigenvalues of $-A$ in (δ_2, δ_1) if $\delta_2 < \delta_1$.

Hint: Explicitly compute the kernel of D_{δ} and D_{δ}^* .

e. Deduce from this the Riemann-Roch theorem with punctures from the Lecture in the case of the trivial line bundle over the cylinder with asymptotics of arbitrary Conley-Zehnder index.

Remark. Compare this problem with the first. We have proven that the Fredholm index of an operator on the cylinder is computed by the spectral flow. Unlike in the ODE problem, here we used *translation-invariant* operators only and we got away with it because we use the spectral theorem. In the next problem we reprove this just using the definition, showing that the Fredholm index behaves like the spectral flow in a precise way. We also have seen how solutions to the equation $D = 0$ on the ends are regulated by eigendirections of the asymptotes and their eigenvalues. In the non-translation invariant case this is true with correction terms but the take-away should be clear: the eigendirections of the asymptotic operator of an orbit are the allowed directions of holomorphic approach.

Bonus Problem. “Fredholm index = spectral flow” theorem II. Our goal now is to prove that the spectral flow computes the Fredholm index of on a cylinder using only the definition of spectral flow (i.e. intersection of a path and the locus of non-invertible operators), so without appealing to the spectral flow theorem or consequences thereof. From the lectures we only use the definition and the results on Fredholm operators (Lecture 4). We will also use Problem 1.

Consider $(\mathbb{C}^n, \omega_{\text{std}}, J_0) \rightarrow \mathbb{S}^1$ the trivial Hermitian vector bundle over \mathbb{S}^1 and $\mathcal{A}(n)$ the space of all asymptotic operators on it. These are exactly the operators of the form

$$H^1(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n})$$

$$\eta \longmapsto A(\eta) = -J_0\partial_t\eta(t) - S(t) \cdot \eta(t),$$

where S ranges over the bounded loops of symmetric matrices, more precisely,² functions $L^{\infty}(\mathbb{S}^1, \text{End}^{\text{sym}}(\mathbb{R}^{2n}))$. Paths $(A_s)_{s \in \mathbb{R}} \in \mathcal{A}(n)$ of these operators define Cauchy-Riemann type operators on the trivial complex vector bundle $(\mathbb{C}^n, J_0) \rightarrow \mathbb{R} \times \mathbb{S}^1$ over the cylinder $\mathbb{R} \times \mathbb{S}^1$, with coordinates (s, t) like so

$$H^1(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$$

$$\eta \longmapsto D_A(\eta) = \bar{\partial}\eta(s, t) + S(s, t) \cdot \eta(s, t) = \partial_s\eta - A_s(\eta),$$

and we assume that there is convergence at $\pm\infty$ to some asymptotic operators A_{\pm} . Denote this path space, restricted to the paths whose limits A_{\pm} exist and are non-degenerate ($\ker A_{\pm} = 0$), by $\mathcal{PA}(n)$. Our goal is to show that if $(A_s)_s \in \mathcal{PA}(n)$ and D_A is the Cauchy Riemann operator on the cylinder associated to $(A_s)_s$, then

$$\text{ind } D = \mu^{\text{spec}}((A_s)_s).$$

²This is important as this is the topology we take on $\mathcal{A}(n)$.

- a. Use the index theorem of Lecture 5 in the notes (but nowhere else in this exercise) to verify this formula.

The point of this exercise is to prove this formula using only the spectral flow and Fredholm theory of Lectures 4 and 5. We want to axiomatically characterize the spectral flow so it encompasses the finite dimensional case: denote by $a(n)$ the space of symmetric $(n \times n)$ -matrices and $\mathcal{P}a(n)$ the space of \mathbb{R} -paths whose limits exist, are invertible and symmetric. Denote by \mathcal{P} the union of all the $\mathcal{P}a(n)$ and $\mathcal{P}\mathcal{A}(n)$ for all $n \geq 1$.

- b. Show that the spectral flow, understood as a function

$$\mu^{\text{spec}} : \mathcal{P} \longrightarrow \mathbb{Z}$$

satisfies the following properties:

- I. (constant) if $(A_s)_s \in \mathcal{P}$ is constant, then $\mu^{\text{spec}}((A_s)_s) = 0$,
 - II. (homotopy) it is constant on connected components of \mathcal{P} ,
 - III. (direct sum) If $(A_s)_s, (B_s)_s \in \mathcal{P}$, then $\mu^{\text{spec}}((A_s \oplus B_s)_s) = 0$ for $(A_s \oplus B_s)_s \in \mathcal{P}$,
 - IV. (normalization) For $(A_s)_s \in a(1)$ given by $A_s = \arctan(s)$, $\mu^{\text{spec}}((A_s)_s) = 1$.
- c. * Show that if any other function

$$\mu : \mathcal{P}\mathcal{A} \longrightarrow \mathbb{Z}$$

satisfies these properties (to be called axioms hence) then $\mu = \mu^{\text{spec}}$.

Hint: Given $(A_s)_s \in \mathcal{P}\mathcal{A}(n)$ there is an integer m and a path $(a_s)_s \in \mathcal{P}a(m)$ such that $(A_s \oplus a_s)_s$ is homotopic to a constant path.³ This is not trivial, you can take it as a black box. If you don't want to, then prove it by iteratively removing *simple* crossings by a carefully chosen path in $\mathcal{P}a(1)$.

- d. * Show that if D is a Cauchy-Riemann operator on (\mathbb{C}^n, J_0) as before and $(A_s)_s \in \mathcal{P}\mathcal{A}(n)$ the corresponding path of asymptotic operators, then

$$\text{ind } D = \mu^{\text{spec}}((A_s)_s).$$

Remark. To recap, we have shown that anything that formally behaves like the spectral flow and matches the finite-dimensional count, must be the spectral flow; and that the index behaves this way, partly because our finite-dimensional intuition and the invertibility of the translation-invariant ones. One can use these tools to give an alternative proof of the Spectral theorem, as was done by Robin and Salamon (in a slightly more general setting), who were first to prove it rigorously (I think).

Bonus problem. The cut and past proof of the index theorem. We want to give an alternative proof of the Riemann-Roch theorem with punctures, as stated in Lecture 5. We describe the main ingredient first.

Consider a punctured Riemann surface $\dot{\Sigma}$ with an asymptotically Hermitian vector bundle (E, J) with a linear Cauchy-Riemann type operator D asymptotic to the (nondegenerate) operators $(A_z)_z$ on $(E_z, J_z, \omega_z) \rightarrow \mathbb{S}^1$, summarized by $D = (\dot{\Sigma} = \Sigma \setminus \Gamma, E, D)$. Now, if we have two such examples, $D = (\dot{\Sigma} = \Sigma \setminus \Gamma, E, D)$ and $D' = (\dot{\Sigma}' = \Sigma' \setminus \Gamma', E', D')$, we can perform a gluing operation on a set of positive/negative punctures of $\dot{\Sigma}$ and negative/positive punctures of $\dot{\Sigma}'$ along with the vector bundles E and operators D when their asymptotics match, denoted by $D \#_{\Omega} D' = (\dot{\Sigma} \#_{\Omega} \dot{\Sigma}', E \#_{\Omega} E', D \#_{\Omega} D')$ where Ω represents the choice of punctures and asymptotic information. Then:

³This should be intuitive: such paths should be able to be stabilized by adding enough parameters.



Figure 1: We have two surfaces: one sphere with 3 punctures (one negative, two positive); and another sphere with 4 punctures (three negative, one positive). We glue these at two pairs of punctures; the result is a torus with 2 negative punctures and a single positive one.

Schwarz's theorem. *The glued operator $D\#_{\Omega}D'$ is a linear Cauchy-Riemann operator on an asymptotically Hermitian vector bundle $E\#_{\Omega}E'$ with non-degenerate asymptotics and*

$$\text{ind } D\#_{\Omega}D' = \text{ind } D + \text{ind } D'.$$

The proof of this theorem is analogous to the linear gluing result from the Lectures.

- a. Loosely convince yourself that such a gluing the operators at the punctures is possible, see picture.
- b. Convince yourself that this reduces the proof of the index theorem to proving it in two (classes of) examples, which we have already computed: for Cauchy-Riemann operators on the trivial line bundle over the 2-sphere; and for Cauchy-Riemann operators on the trivial line bundle over the cylinder with asymptotics of arbitrary Conley-Zehnder index.

Remark. The proof of the index theorem in the lectures is beautiful and possible due to a miracle. This proof is more robust, one expects indices of Fredholm operators modeling some geometry to have this cut and paste property. This proof is also beautiful at it justifies the index formula from two simple cases: a closed/topological and a cylindrical/more dynamical. Recasting the strategy of the proof, one can also deduce the theorem from: the usual Riemann-Roch for algebraic curves (which is understood to be topological); the cylinder computations; and knowing the index of a standard plane. Although conceptually less simple, it answers the following question (I had when I started learning the subject): indices of Fredholm operators in geometry are usually algebro-topological in nature, how does the Conley-Zehnder index, a dynamical quantity, appear?

Finally, a word on history. The index theorem for holomorphic curves followed from Riemann-Roch and I think that's how Gromov thought about it. The punctured-Riemann-Roch was first proven by M. Schwarz in his thesis, via the cut and paste construction. Chris' proof follows an idea of Taubes. In whichever case, it is likely that the theorem follows from the much older Atiyah-Patodi-Singer index theorem.

PART II: The Conley-Zehnder index

Problem 4. Asymptotic operators, their parallel transport and multiple covers. For a Reeb orbit of a contact form, its asymptotic operator is equivalent to the linearized Reeb flow. In general, the abstract way to put this is that on a Hermitian vector bundle over the circle a "symplectic parallel transport" is equivalent to a symplectic connection (and hence an asymptotic operator). First, we show this in a trivialization.

- a. * Consider $(\mathbb{C}^n, \omega_{\text{std}}, J_0) \longrightarrow \mathbb{S}^1$ the trivial Hermitian vector bundle over \mathbb{S}^1 , show that there is a 1:1 correspondence between (smooth) asymptotic operators $A = -J_0\partial_t - S(t)$ and smooth paths $(P(t))_{t \in \mathbb{R}}$ of symplectic linear maps $P(t) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ (i.e. $P(t) \in \text{Sp}(2n)$) such that $P(0) = \mathbb{1}$ and $P(t+1) = P(t)P(1)$ for all $t \in \mathbb{R}$.

Hint: Consider the fundamental solution to the (non-autonomous) ODE $A = 0$.

Given an asymptotic operator A as above, we denote $P_A = (P_A(t))_t$ the path of symplectic linear maps and call it its *parallel transport*. The time 1-map $P_A(1)$ is called the *Poincaré return map*. In a contact manifold the return map of an orbit controls much of the C^1 -Reeb dynamics near it.

- b. * Show that $\ker A$ is isomorphic to the eigenspace of eigenvalue 1 of its return map $P_A(1)$. In particular, the asymptotic operator of an orbit is non-degenerate if and only if the return map of the linearized Reeb flow along the orbit does not have 1 as an eigenvalue.
- c. Bonus: Show that if $(\gamma_t)_{t \in (-\varepsilon, \varepsilon)}$ is a smooth family of Reeb orbits of period T_t of a contact manifold (M, α) (in particular, we take this to also mean $(T_t)_t$ is a smooth path of real numbers) then the return map of the linearized flow of γ_0 has an eigenvalue 1.

- d. * For $n = 1$ and a real number $r > 0$, consider ${}_h A_r = -i\partial_t - \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}$. Show that ${}_h A_r$ is non-degenerate and that $\mu_{\text{CZ}}({}_h A_r) = 0$ by definition.

- e. * For $n = 1$, consider ${}_e A_\theta = -i\partial_t - \theta \mathbb{1}$. Show that ${}_e A_\theta$ is non-degenerate if and only if $\theta \notin 2\pi\mathbb{Z}$ and use the following item to compute $\mu_{\text{CZ}}({}_e A_\theta) = 2\lfloor \theta/2\pi \rfloor + 1$, where $\lfloor - \rfloor$ is the integer floor function.

Recall from the lectures that for $n = 1$ each non-trivial eigenfunction of an asymptotic operator A (as above, on the trivialized bundle) has a well defined winding number with the following properties: (non-trivial) eigenfunctions of the same eigenvalue have the same winding, so it defines a function $\text{wind} : \sigma(A) \longrightarrow \mathbb{Z}$; and this function is nondecreasing and each integer value is attained exactly twice (with multiplicity). From this, we can define the *extramal winding numbers* of A , $\alpha_+(A)$ (resp. $\alpha_-(A)$) as the winding of the smallest positive eigenvalue (resp. largest negative eigenvalue), and the parity $p(A) = \alpha_+(A) - \alpha_-(A)$ which is either 0 or 1 if A is non-degenerate.

- f. Bonus: (Re)prove the (magic, right?) formula $\mu_{\text{CZ}}(A) = 2\alpha_{\mp}(A) \pm p(A)$.

We now shift our attention to multiply covered orbits, which unavoidably arise in applications. In a contact manifold (M, α) (now n is not necessarily 1 anymore) given a closed Reeb orbit $\gamma : \mathbb{S}^1 \rightarrow M$ of period T , for each positive integer k , we can associate a Reeb orbit $\gamma^k : \mathbb{S}^1 \rightarrow M$ of period kT by setting $\gamma^k(t) = \gamma(kt)$, i.e going around the same orbit k times. We call γ^k the *k-fold cover* of γ and say it is *multiply covered* if $k > 1$. A *simply covered* Reeb orbit is then an orbit that is not the k -fold cover of any other for $k > 1$. Taking this back to the abstract setting above, given an asymptotic operator A on the trivial bundle with parallel transport P_A , we can define A^k as the unique asymptotic operator (as per the first item) corresponding to the parallel transport maps defined by $P_{A^k}(t) = P_A(kt)$. Given (a section) $\eta : \mathbb{S}^1 \rightarrow \mathbb{C}$ its k -fold cover is $\eta^k(t) := \eta(kt)$.⁴

- g. Show that the k -fold cover of an eigenfunction f_λ of A of eigenvalue λ defines an eigenfunction f_λ^k of A^k of eigenvalue $k\lambda$.

Hint: Show that $A^k = -J_0\partial_t - kS(kt)$.

- h. For $n = 1$ we can say more. Show that an eigenfunction g_λ of A_k of eigenvalue λ is a k -fold cover (i.e. $g_\lambda = f_{\lambda/k}^k$) if and only if its winding number is divisible by k . In fact, we can do better:

$$\text{cov}(g_\lambda) = \max\{m \in \mathbb{Z}^{>0} : g_\lambda \text{ is an } m\text{-fold cover}\} = \text{gcd}(k, \text{wind}(g_\lambda)).$$

⁴Geometrically, if $\varphi_k : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ is the k -cover of the circle given by $t \mapsto kt$, $\gamma^k = \varphi_k^* \gamma = \gamma \circ \varphi_k$. The vector bundle $\gamma_k^* \xi$ then corresponds to $\varphi_k^*(\gamma^* \xi) = (\gamma \circ \varphi_k)^* \xi$. Using the pull-back trivialization you can see that we get exactly what we state for A_{γ^k} . In this language, $\eta^k = \varphi_k^* \eta \in \Gamma(\gamma_k^* \xi)$.

One of the basic questions we can ask about the A_k is what is their Conley-Zehnder index in terms of that of A . This is a difficult question in general (knowing about the eigenvectors that are k -fold covers is not sufficient to understand the spectral flow) as it depends on the Reeb dynamics near the orbit in a subtle way. The 3-dimensional case is treatable, let us start with the following:

- i. * Relate $\mu_{CZ}({}_hA_r^k)$ to $\mu_{CZ}({}_hA_r)$ and $\mu_{CZ}({}_eA_\theta^k)$ to $\mu_{CZ}({}_eA_\theta)$.

These examples happen to be crucial: roughly speaking, all asymptotic operators in 3 dimensions fall in two boxes that qualitatively behave like these two examples. See the remark.

- j. Bonus: Here is a nice geometric way I like to understand this k -fold sections. A trivialization of $\gamma^*\xi$ is essentially an identification of a tubular neighbourhood of γ with a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ where $\gamma = \mathbb{S}^1 \times 0$. A section η of $\gamma^*\xi$ (sufficiently close but disjoint to γ) represents a $(1, q)$ torus knot in $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{D}^2 \setminus \gamma \times 0$. The q is the winding number defined above. The k -fold covers η^k are pullbacks $\varphi_k^*\eta$ (see the footnote), geometrically corresponding to the k -fold cover of the solid torus induced by φ_k . These are often called multi-sections. To visualize them we just note that in $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{D}^2 \setminus \gamma \times 0$ they represent (k, kq) torus-knots. Other sections of the pull-back orbit are of the form (p, q) for $p = k$ and q not necessarily divisible by k .

Remark. The connection between a closed Reeb orbit, its Conley-Zehnder index, the eigenfunctions of its asymptotic operator and holomorphic curves asymptotic to it is deep. Holomorphic curves can only roughly approach in the directions of the eigenfunctions and the “knot/link theory” of those then determines certain “counts at infinite” needed to generalize things like intersection theory from closed holomorphic curves to ones with cylindrical ends. We will quite possibly work on this a little bit later on in the semester.

Also, note that thinking about multiply covered orbits is unavoidable, holomorphic curves naturally have ends asymptotic to such orbits. Even if one insisted on making all of them converge to simple orbits, if two ends approached the same orbit (which does happen), it is possible that they then converged to a multiple cover that doubly covered the orbit. In the next problem we hopefully clear out some of this mystery as we explore this problem more geometrically.

Problem 5. A topological description of the Conley-Zehnder index. In this problem we continue the study of $\text{Sp}(2)$ by focusing on the topology of paths $P : [0, 1] \rightarrow \text{Sp}(2)$ that start at the identity $P(0) = \mathbb{1}$ and end at a non-degenerate matrix, meaning $P(1)$ has no eigenvalue 1. This takes the perspective of studying the linearized Reeb flow along a Reeb orbit (more abstractly, asymptotic operators via their parallel transport) to define invariants that capture something about dynamics.

Denote the degenerate matrices by $\text{Sp}^*(2) = \{A \in \text{Sp}(2) : \det(A - \mathbb{1}) = 0\}$, the non-degenerate ones $\text{Sp}^0(2) = \text{Sp}(2) \setminus \text{Sp}^*(2)$, and the space space of paths described above by $\mathcal{P} = \{P : [0, 1] \rightarrow \text{Sp}(2) \mid P(0) = \mathbb{1}, P(1) \in \text{Sp}^0(2)\}$.

- a. * Show that the eigenvalues of a matrix $A \in \text{Sp}(2)$ are either: $\{r, 1/r\}$ for a non-zero real number r , in which case we say it is (poritive/negative) *hyperbolic* (if $r > 0/r < 0$); $\{\pm 1\}$ (i.e. with multiplicity 2), which we call *degenerate*; or $\{\lambda, \bar{\lambda}\}$ for $\lambda \in \text{U}(1) \subseteq \mathbb{C}$, which we call *elliptic*.
- b. Bonus: Show that the rotation matrix $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$ is conjugate to $R(-\theta)$ in $\text{GL}(2, \mathbb{R})$ but *not* in $\text{Sp}(2)$. Can you classify all conjugacy classes of matrices in $\text{Sp}(2)$? The picture of the next item may help.

The terminology comes from dynamical systems. If you consider the orbits of points under iteration of A , i.e. $A^n(p)$, one sees that: if A is hyperbolic it contracts/repels all points towards/from the axi given by the eigenvectors of eigenvalue with absolute value smaller/greater than 1; the degenerate one is a shear of \mathbb{R}^2 ; and the elliptic is a rotation. A consequence of this exercise will be understanding the dynamics of the Reeb flow much more concretely just from its return map and the Conley Zehnder index.

- c. * Using the symplectic polar decomposition on $\mathrm{Sp}(2)$, show that $\mathrm{Sp}(2)$ is diffeomorphic to $\mathbb{R}_r^+ \times \mathbb{S}_\theta^1 \times \mathbb{R}_z$, i.e. cylindrical $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ (here $\mathbb{R}^+ = (0, \infty)$). For future definitions, we take $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (not $\mathbb{R}/2\pi\mathbb{Z}$).
- d. Describe the set $\mathrm{Sp}^*(2)$ in terms of these coordinates (see the figure) and show that $\mathrm{Sp}^0(2)$ has two connected components, to be called $\mathrm{Sp}_\pm^0(2)$.

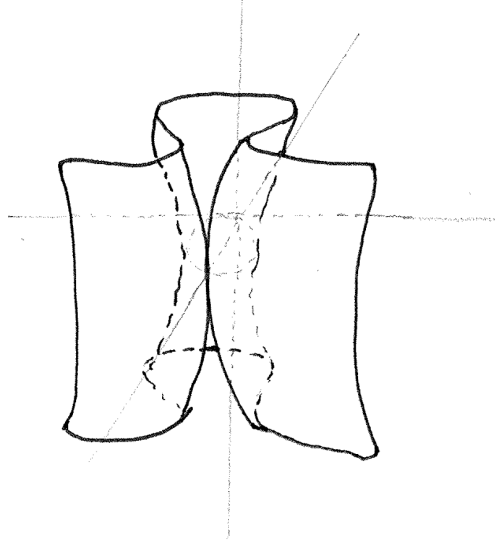


Figure 2: The set $\mathrm{Sp}^*(2)$ in cylindrical coordinates. The axes and a circle of diameter one in the $r - \theta$ plane from $(0, 0, 0)$ to $(1, 0, 0)$ are shaded.

- e. Further describe in cylindrical coordinates (i.e. in the picture) the loci of: positive and negative hyperbolic, degenerate, and elliptic matrices.

We are now ready to define the Conley-Zehnder index in this setting. Consider a path $P \in \mathcal{P}$, the first definition of $\mu_{\mathrm{CZ}}^1(P)$ is the count of how many times P becomes degenerate, more concretely:

Define $\mu_{\mathrm{CZ}}^1(P) = \#(P \cap \mathrm{Sp}^0(2))$ signed count of intersection points of $P(t)$ with the co-orientable hypersurface $\mathrm{Sp}^*(2)$.⁵

For the second definition define the *rotation angle* $\theta(A)$ of A to be θ -component of A in cylindrical coordinates. Define next the *mean index* or *monodromy angle* of a path P to be the net rotation, meaning $\theta(P) := \tilde{\theta}(P(1)) - \tilde{\theta}(P(0))$ where $\tilde{\theta}$ is some lift of $\theta : \mathrm{Sp}(2) \rightarrow \mathbb{S}^1$ to \mathbb{R} (there shouldn't be confusion between the related but different functions $\theta(A)$ and $\theta(P)$). The second definition of Conley-Zehnder index is the count of half-turns:

Define $\mu_{\mathrm{CZ}}^2(P) = 2\theta(R_P * P)$, where R_P is path from $P(1)$ to $R_P(1) := \pm P_0(1) = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm \frac{1}{2} \end{pmatrix}$ contained solely in $\mathrm{Sp}_\pm^0(2)$ if $P(1)$ belongs to either $\mathrm{Sp}_+^0(2)$ or $\mathrm{Sp}_-^0(2)$.

Note that μ_{CZ}^1 is independent of the path-homotopy class by definition, take a second to convince yourself that the same is true for μ_{CZ}^2 .

- f. Show that both definitions are equivalent, i.e. $\mu_{\mathrm{CZ}}^1(P) = \mu_{\mathrm{CZ}}^2(P)$, to be denoted μ_{CZ} from now on.
- g. * Observe that $\theta(P(1))$, which is $\theta(P)$ modulo 1, is 0 if P is positive hyperbolic (or degenerate), $1/2$ if negative hyperbolic, and elliptic otherwise. Show that $\mu_{\mathrm{CZ}}(P) = 2\lfloor \theta(P) \rfloor$ in the positive hyperbolic case and $\mu_{\mathrm{CZ}}(P) = 2\lfloor \theta(P) \rfloor + 1$ in the elliptic or negative hyperbolic one.

⁵To be completely rigorous, note that the path starts at the identity, so this count is not quite well-defined. We resolve this by getting rid of this superfluous intersection: consider $P_0(t) = \begin{pmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix}$ and define $\mu_{\mathrm{CZ}}^1(P) = \#(P * P_0) \cap \mathrm{Sp}^0(2)$, where $P * P_0 : [0, 1] \rightarrow \mathrm{Sp}(2)$ denotes the concatenation.

In light of the first item of the first exercise, given $P \in \mathcal{P}$ we can define its k -iterate recursively by tracing $P^{k-1}(t)$ and then $P(t)P^{k-1}(t)$ and renormalizing time to $[0, 1]$, i.e. $P^2(t) = (P(t)P(1)) * P(t)$ and so on.

- h. * Compute iteration formulas for $\mu_{\text{CZ}}(P^k)$ in terms of $\mu_{\text{CZ}}(P)$ and whether $P(1)$ is positive/negative hyperbolic or elliptic. Identify the limit $\lim_{k \rightarrow \infty} \mu_{\text{CZ}}(P^k)/k$. Careful: that P is non-degenerate does not imply that iterates are. What monodromy angles of a path $P \in \mathcal{P}$ ensure that $P^k \in \mathcal{P}$ for all k ?
- i. Show that if A is a non-degenerate asymptotic operator and P_A the corresponding parallel transport, $[\theta(P_A)] = \alpha_-(A)$; the number of full rotations completed by P_A along the orbit is the winding number of the negative extremal eigenfunction.⁶

The punchline of this exercise is this:⁷

- j. * The function $\mu_{\text{CZ}} : \mathcal{P} \rightarrow \mathbb{Z}$ is an isomorphism at the level of connected components.
- k. Bonus: In fact, consider $R(\theta)$ the rotation matrix of angle $2\pi\theta$ and consider the following class of paths, here $r > 1$ and $\theta \in [0, 1]$:

$$H_+ : H_r^{2k}(t) = R(kt) * \begin{pmatrix} 1 + t(r-1) & 0 \\ 0 & \frac{1}{1+t(r-1)} \end{pmatrix},$$

$$H_- : H_r^{2k+1}(t) = R(kt) * \begin{pmatrix} -1 + t(1-r) & 0 \\ 0 & \frac{1}{-1+t(1-r)} \end{pmatrix} * R(\frac{1}{2}t),$$

$$E : E_\theta^{2k+1}(t) = R(kt) * R(\theta t).$$

Then, after a possible conjugation and a homotopy *relative to the endpoints*, every path $P \in \mathcal{P}$ is of the form above, where the $2k$ or $2k + 1$ is its Conley-Zehnder index and the r and θ are determined from the eigenvalues of $P(1)$.⁸ Note that these paths are very easy to draw in the picture of $\text{Sp}(2)$: the topology of the linear dynamics of the Reeb flow is determined and understood from the return map and the Conley-Zehnder index.

Remark. The upshot is that the basic dynamics of the return map of the linearized Reeb flow and the Conley-Zehnder index determine the behavior of the linearized Reeb flow and that the iteration theory, even is non-obvious (specially in the elliptic case), is understood. One can also define the monodromy angle and prove the iteration formulas purely from the perspective of the lectures, which you can find in the lecture notes. This exercise also shows that both approaches describe the same object.

One may wonder how to do this in higher dimensions. The intersection-theoretic definition μ_{CZ}^1 is the same. The second definition goes through as soon as we define the monodromy angle in higher dimensions. This is essentially being able to define the mean index, which is a well-constructed extension of $\det_{\mathbb{C}} : \text{U}(n) \rightarrow \text{U}(1)$. Let me list some of the new features and dangers:

- As we saw in the last problem last sheet, we will also have eigenvalues that come in a 4-tuple $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$. For iteration purposes, these behave like the hyperbolic ones, so they are not a terrible danger.
- If all eigenvalues are simple, then we can define the mean index by essentially adding up the arguments of one of each elliptic pair; and then extend using density. While diagonalizable unitary matrices are unitarily diagonalizable, this is quite false in general: the conjugacy classes in the symplectic group are subtle (described by Y. Long and J. Gutt). A non-trivial consequence of this is that matrices with simple eigenvalues are dense in $\text{Sp}(2n)$.

⁶How curious...

⁷Note that we have proven this in the lecture for asymptotic operators

⁸There is something slightly tricky here: if A is hyperbolic, then the eigenvalues are $r, 1/r$ and we choose the one with absolute value larger than 1; if A is elliptic, we want θ to be determined by the rotation angle of A , which is determined by its eigenvalues are $e^{2\pi i\theta}, e^{-2\pi i\theta}$ in a slightly less trivial way.

- The mean index continues to iterate well and then index iteration theory boils down to bounding the iterated CZ-index by the mean index. The different patterns that arise are interesting and a topic of research, as it is applicable to finding periodic orbits by understanding the grading in Floer homology. Sadly, this kind of qualitative study is the best one can do in $\text{Sp}(2n)$ for $n > 1$.

The material consequence, so to say, of this funny iteration behavior is that multiply-covered holomorphic curves are usually behaved wrong and one needs to struggle to deal with them in applications.

Problem 6. Example: irrational ellipsoids. Recall from the first sheet the irrational ellipsoids:

$$E(a, b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi}{a}|z_1|^2 + \frac{\pi}{b}|z_2|^2 = 1\}, \text{ with } a/b \text{ irrational.}$$

- Compute the monodromy angle of the linearized Reeb flow, and use this to show that the two orbits are non-degenerate and calculate their Conley-Zehnder index.
- Compute the asymptotic operators of the two Reeb orbits, show they are non-degenerate (again) and use calculate their Conley-Zehnder index from this perspective.
- Bonus: what can you say about the Conley-Zehnder indices of the orbits of the induced forms on Lens spaces?

Remark. If the ellipsoids were not irrational, the orbits would come in (Morse-Bott) families and be degenerate. In the future, we may talk about what a good notion of Conley-Zehnder index is in that case or a good way slightly perturb a contact form to obtain non-degenerate orbits from these Morse-Bott families. In the first problem sheet we also saw that Riemannian manifolds endow their unit cotangent with a contact form “compatible” with the metric. For these, when non-degenerate, their Conley Zehnder index is the Morse index. We will come back to this.