Symplectic field theory

Problem set 6

Gerard Bargalló i Gómez and Chris Wendl

To be discussed on the 11th of June

In this sheet we mull over the compactness lectures. First we study bubbling, how it happens explicitly and some ways to use it. Then we turn to genus 0 Deligne-Mumford space and stability (which is meant to be approached with the same amount of rigor as in the lectures). We conclude with a key problem meant to work on the "combinatorics" of the SFT compactness theorem stated at the end of the lecture. There are a couple of fun bonus problems.

Problem 1. An application of bubbling. Let Σ be a closed connected oriented surface and $P \subseteq \Sigma$ a finite set of points such that $\chi(\Sigma \setminus P) < 0$ (i.e. (Σ, P) is a stable marked Riemann surface). Let j be an almost complex structure on Σ and $\mathrm{Aut}(\Sigma, j, P)$ the group of holomorphic diffeomorphisms $(\Sigma, j) \to (\Sigma, j)$ that fix P. We will show that:

Theorem. *If $\chi(\Sigma \setminus P) < 0$, the group $\mathrm{Aut}(\Sigma, j, P)$ is finite.*

This will follow from the following important compactness property of biholomorphisms of Riemann surfaces:

Proposition. *Let $j_k \to j$ and $j'_k \to j'$ $\mathcal{C}^\infty$-convergent sequences of complex structres on Σ and $(\Sigma, j'_k) \xrightarrow{\varphi_k} (\Sigma, j_k)$ a sequence of holomorphic diffeomorphisms that fix the points in P. Then, as long as $\chi(\Sigma \setminus P) < 0$, the sequence $(\varphi_k)_k$ has a $\mathcal{C}^\infty$-convergent subsequence.*

a. * Prove the proposition in the case that Σ has positive genus.

   Hint: Observe that there is no bubbling by topological reasons.

b. * Prove the proposition in the case that $\Sigma = \mathbb{S}^2$.

   Hint: How much energy would a bubble take?

c. If $\chi(\Sigma \setminus P) < 0$, deduce from the proposition that $\mathrm{Aut}(\Sigma, j, P)$ is compact. Deduce the theorem by showing this group must be discrete.

   Hint: Use the Lefschetz fixed point theorem.[1]

Remark. With the second bonus problem this shows that the stable marked Riemann surfaces are exactly those with finite automorphisms. It is a classical theorem of Hurwitz that an unmarked stable Riemann surface (i.e. $g > 1$) has at most $84(g-1)$ automorphisms and that this bound is attained by some Riemann surfaces (the proof consits of a case by case study after considering the projection $(\Sigma, j) \to (\Sigma, j)/\mathrm{Aut}(\Sigma, j)$ and using orbifold Riemann-Hurwitz). Some of the ideas in this problem to rule out bubbling are common in a symplectic geometer's practice

Problem 2. Bubbling experiments. In this problem we go through increasingly elaborate instances bubbling, from a single bubble to a tree. There is a more guided version of this problem at the end that you can check, though starting here may still be a good idea.

[1] We need a weak version of it, so you should in fact prove it yourself: we can algebraically count fixed points of φ by counting intersections of the graph of $\varphi : \Sigma \to \Sigma$ with the diagonal Δ in $\Sigma \times \Sigma$. Show that this count is the count of self-intersections of the zero section of $T\Sigma$, which is $\chi(\Sigma)$. In conclusion, said map must be the identity and so the group is discrete. Discrete and compact implies finite.

a. * Consider the sequence of holomorphic curves $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2, i)$ given by $u_n(z) = nz$ where $z \in \mathbb{C} \cup \infty = \mathbb{S}^2$. Compute the point-wise limit of u_n and exhibit the map biholomorphic map $v : (\mathbb{S}^2, i) \to (\mathbb{S}^2, i)$ defined by $v(z) = z + 1$ as a Gromov limit of the sequence.

b. * Next, consider the sequence $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2, i \oplus i)$ given by $u_n(z) = (z, nz)$. Exhibit the pair of maps $u, v : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2, i \oplus i)$ defined by $u(z) = (z, \infty)$ and $v(z) = (0, z + 1)$ as a Gromov limit of the sequence.

c. Now, consider $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ given by $u_n(z) = (z, \frac{1}{nz}, \frac{1}{n^2 z})$. Exhibit $u, v_1, v_2 : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ defined by $u(z) = (z, 0, 0)$, $v_1(z) = (0, \frac{1}{z+1}, 0)$ and $v_2(z) = (0, \infty, \frac{1}{z+1})$ as a Gromov limit of the sequence.

d. Finally, consider the following sequence $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ given by $u_n(z) = (z, \frac{1}{n(z-\frac{1}{n})}, \frac{1}{n^2(z+\frac{1}{n})})$. Exhibit $u, v, v_1, v_2 : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ defined by $u(z) = (z, 0, 0)$, $v(z) = (0, 0, 0)$ (yes, this is not a typo, it's a ghost[2]), $v_1(z) = (0, \frac{1}{z-1}, 0)$ and $v_2(z) = (0, 0, \frac{1}{z+1})$ as a Gromov limit of the sequence.

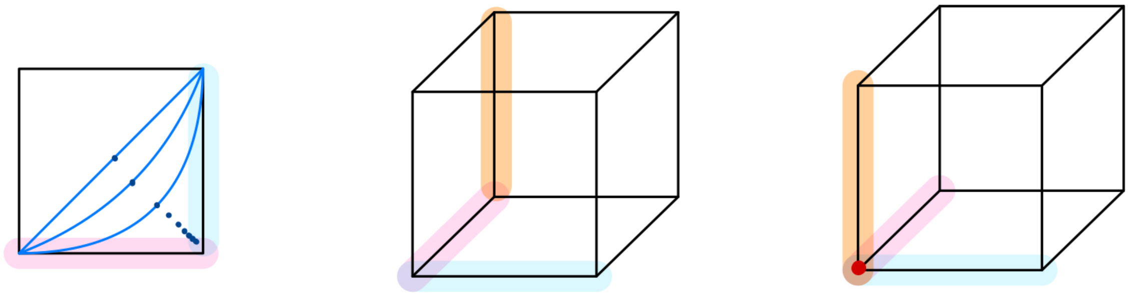


Figure 1: Left: limiting sequence (blue line) in $\mathbb{S}^2 \times \mathbb{S}^2$ drawn as a square for exercise 2.b., the limits are the highlighted sides. Center: the limit in exercise 2.*c.* represented as the highlighted edges. Right: the limit in exercise 2.*d.* represented as the highlighted edges and red vertex.

Problem 3. The genus 0 Deligne-Mumford space. Let's reflect on the genus 0 case of the Deligne-Mumform compactness theorem (aka Knusden's compactness theorem).

a. The non-stable case: show that $\mathcal{M}_{0,m}$ consists of a single point for $m = 0, 1, 2, 3$.[3]

b. * For $m \geq 4$, identify topologically the space $\mathcal{M}_{0,m}$ with $M_m := (\mathbb{CP}^1_*)^{\times(m-3)} \setminus \Delta$.

   Here $\mathbb{CP}^1_*$ is taken to mean $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$, $(\mathbb{CP}^1_*)^{\times(m-3)}$ the product of $\mathbb{CP}^1_*$ with itself $m - 3$ times and Δ the fat diagonal, i.e. the set of points in the product where at least two coordinates agree.

Let's take a moment to reflect on the stability condition and describe the easiest non-trivial example of $\overline{\mathcal{M}}_{0,m}$.

c. Representing nodal genus 0 surfaces with sticks that cross at the nodal points and dots representing the marked points, which of the following are (arithmetic) genus 0 stable nodal Riemann surfaces (as defined in class)?

[2]Ghost bubbles are constant bubbles and they arise naturally. Note that in this case, without the ghost, the limit would not be a tree.

[3]Recall the uniformization theorem and that the group of biholomorphisms of the Rimeann sphere (rational functions of degree 1) acts transitively on thruples of points, see the second bonus exercise.

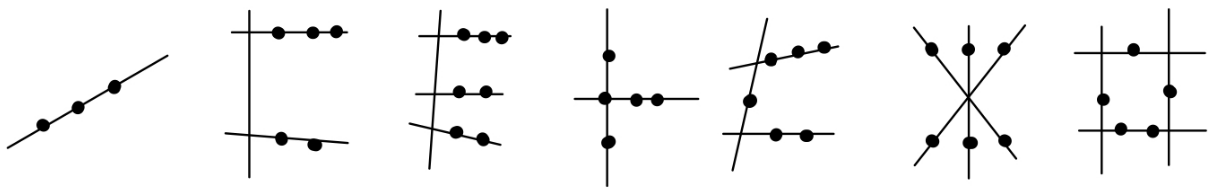


Figure 2: Problem 3.c. (adapted from an "Invitation to Quantum cohomology").

d. * Enumerate the different 4-marked genus 0 nodal Riemann surfaces and convince yourself (via the DM-theorem) that $\overline{\mathcal{M}}_{0,4} = \mathbb{CP}^1$.

Note that $\mathcal{M}_{0,5} = \mathbb{CP}^1_* \times \mathbb{CP}^1_* \setminus \Delta$ (where the fat diagonal Δ is the usual diagonal now). It is tempting to infer from the above that $\overline{\mathcal{M}}_{0,5} = \mathbb{CP}^1 \times \mathbb{CP}^1$, which is not true. The following exercise means to explain why, see figure.

e. Bonus: Take for granted that there is a well-defined continuous map $\overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,m}$ given by forgetting the last marked point (and $m \geq 3$).[4] Considering the case $m = 4$ and using part *b.*, this map is the same as the projection

$$\mathbb{CP}^1_* \times \mathbb{CP}^1_* \setminus \Delta \longrightarrow \mathbb{CP}^1_*, \quad (z_1, z_2) \longmapsto z_1,$$

which should then extend continuously to $\overline{\mathcal{M}}_{0,5} \to \mathbb{CP}^1$. Convince yourself that the naive way to do this by taking $\overline{\mathcal{M}}_{0,5} = \mathbb{CP}^1 \times \mathbb{CP}^1$ does not capture all families of nodal Riemann surfaces.

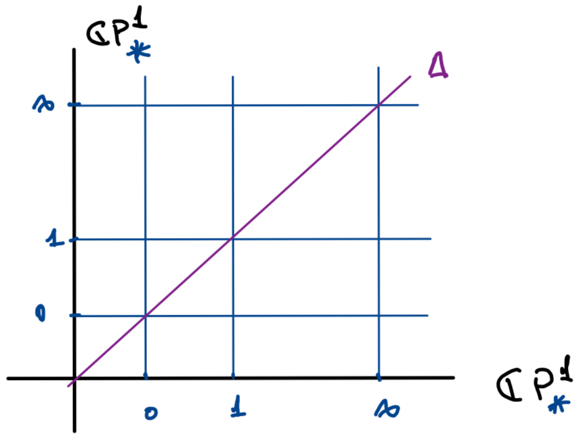


Figure 3: Set up/help for the bonus part in problem 3.

Remark. Deligne and Mumford showed that the space $\overline{\mathcal{M}}_{g,m}$ is, in fact, a projective variety. The genus 0 case was previously done by Knusden. In the bonus exercise you actually realize that $\overline{\mathcal{M}}_{0,5}$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at three points, which is equivalent to the degree 5 del Pezzo

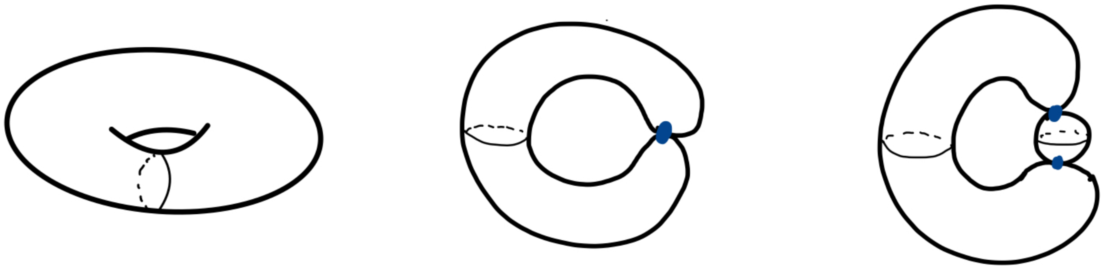
[4] Forgetting the last marked point of a stable marked curve without nodes produces a stable marked curve without nodes. If the curve is nodal, forgetting the last marked point may destabilize the curve. The map is well-defined after fixing this issue: if this happens it means that the corresponding component has no other marked points and two nodes, or a marked point and a node. In the former case we discard the whole component and fuse both of the nodes and in the latter case we discard the component and place the marked point where the node used to be.

surface. Kapranov constructed $\overline{\mathcal{M}}_{0,m}$ as an iterated blow up.[5] Take a moment to notice that the top dimensional boundary components (i.e. with one node; known as the boundary divisors) in $\overline{\mathcal{M}}_{0,m}$ are given by partitioning the set of marked points into two disjoint sets with at least two elements, which describes these boundary components as $\overline{\mathcal{M}}_{0,m-k+1} \times_0 \overline{\mathcal{M}}_{0,k+1}$ for $k \geq 2$, where x_0 indicates the identification of their last marked point. This recursive structure is deeply significant in Gromov-Witten theory (this sort of algebraic structure in the boundary of domain moduli spaces is crucial in Floer theory and SFT as well) where, combined with the fact that the 3 boundary divisors of $\overline{\mathcal{M}}_{0,4}$ are cohomologous (they are points), one can enumerate the number of algebraic curves in $\mathbb{CP}^2$ of a given degree through a given number of points, a huge theorem of Kontsevich in the 90's.

Problem 4. The stability condition. The notion of stability, both for Riemann surfaces and holomorphic curves/buildings, is necessary to ensure Hausdorfness of the respective compact moduli spaces.

a. Explain why if we do not require nodal Riemann surfaces in $\mathcal{M}_{g,m}$ to be stable the space cannot be Hausdorff.

   Hint: Consider the figure and think of the Hausdorff property in terms of limits.



b. Show that, given a punctured holomorphic curve in a symplectic cobordism or symplectization asymptotic to a non-empty set of closed Reeb orbits, one can always break off trivial cylinders at any of the ends. Explain why if we allowed these in the definition of a stable building the SFT-compactification could not be Hausdorff.

Problem 5. Compactness in symplectic cobordisms and contact homology. Suppose that $\mathbf{u}$ is a stable J-holomorphic building in a completed symplectic cobordism $\widehat{W}$ (between two contact manifolds and an SFT-adapted almost complex structure J) with: arithmetic genus 0, exactly one positive puncture at the top level, and every connected component of $\mathbf{u}$ has at least one positive end.

a. * Show that $\mathbf{u}$ has no nodes and all of its connected components have exactly one positive end.

b. * If, in addition, $\mathbf{u}$ also has exactly one negative puncture at the lowest level and no connected component is a plane, show that every level consists of a single cylinder with one positive and negative end.

c. * Describe all possible buildings in *b.* under the assumption that the cobordism is exact or that the target is the symplectization of a contact manifold.

[5] To exemplify this in the $m = 5$ case, recall that there is a unique conic through any 5 points in $\mathbb{CP}^2$. Fixing four of them to be the coordinate points in $\mathbb{CP}^2$, there is a natural map from $\beta : \overline{\mathcal{M}}_{0,5} \to \mathbb{CP}^2$. This realizes $\overline{\mathcal{M}}_{0,5}$ as the blow up of $\mathbb{CP}^2$ at the four coordinate points and β as the blow down map. For general m the construction is similar for $m-1$ points in $\mathbb{CP}^{m-3}$ and blow ups of coordinate points, lines, planes, and so on.

Remark. This "combinatorial" exercise shows that it is reasonable to set-up a Floer-type count of cylinders whenever there are no holomorphic planes. These often exist, but also often do not. When they don't, this exercise tells us that perhaps we can expect to count curves with one positive puncture and multiple negative ones. These two ideas give rise to cylindrical contact homology and contact homology respectively. We will use this in the lectures to distinguish tight contact structures on the 3-torus and more.

Bonus problem. The compactification is not sharp. Let (Σ_0, j_0) be a closed genus g_0 Riemann surface and consider the moduli space $\mathcal{M}_g(j_0, d[\Sigma_0])$ of closed holomorphic curves $(\Sigma, j) \to (\Sigma_0, j_0)$ of degree d (up to reparametrization), where the genus of Σ is g. Show that if $g > g_0$ and $d = 1$ the moduli space $\mathcal{M}_g(j_0, d[\Sigma_0])$ is empty, *but* that $\overline{\mathcal{M}}_g(j_0, d[\Sigma_0])$ is not!

Remark. Determining the closure of a moduli space within its SFT (or Gromov, in the closed case) compactification, i.e. determining which stable buildings (or stable curves) are limits of smooth ones, is a complicated question in general. This exercise shows that there are elements in the compactified moduli space of holomorphic curves that are not limits of smooth ones.

Bonus problem. Genus zero uniformization and automorphisms of Riemann surfaces. We close two gaps in the previous problems: we show all complex structures on $\mathbb{S}^2$ are equivalent and we compute the automorphisms of Riemann surfaces not covered by the theorem above, which are $\Sigma = \mathbb{S}^2$ with $|P| \in \{1, 2, 3\}$ and $\Sigma = \mathbb{T}^2$. Let's begin at the end:

a. Show that $\mathrm{Aut}(\mathbb{S}^2, i) = \mathrm{PSL}(2, \mathbb{C})$, $\mathrm{Aut}(\mathbb{S}^2, i, \{\infty\}) = \mathbb{C} \rtimes \mathbb{C}^*$, $\mathrm{Aut}(\mathbb{S}^2, i, \{0, \infty\}) = \mathbb{C}^*$ and $\mathrm{Aut}(\mathbb{S}^2, i, \{0, 1\infty\}) = \{1\}$. Deduce that $\mathrm{Aut}(\mathbb{S}^2, i)$ acts transitively on triples of points.

   Hint: Deduce first that $\mathrm{Aut}(\mathbb{S}^2, i, \{\infty\})$ is the group $\{az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}$ from the fact that a holomorphic function f on $\mathbb{C}$ such that $\lim_{|z| \to \infty} f(z) = \infty$ is a polynomial.

b. Show that $\mathrm{Aut}(\mathbb{T}^2, i) = \mathbb{T}^2 \rtimes \mathbb{Z}/2$. Here $(\mathbb{T}^2, i)$ is $(\mathbb{C}, i)$ modulo the standard lattice $\mathbb{Z}^2$ and $\mathbb{T}^2$ acts on itself by translation.

   Hint: The $\mathbb{Z}/2$ part corresponds to the automorphisms of $\mathbb{C}$ fixing $\mathbb{Z}^2$ and 0.

c. Bonus: Show that $\mathrm{Aut}(\mathbb{C}/(\mathbb{Z} \oplus \omega\mathbb{Z}), i) = \mathbb{T}^2 \rtimes G_w$, where w is a a complex number with positive imaginary part and G_w is $\mathbb{Z}/2$ unless w is a primitive fourth or sixth root of unity, when then G_w is $\mathbb{Z}/4$ and $\mathbb{Z}/6$ respectively. This completes the computation of the automorphisms of all tori.

d. Show that if j is a complex structure on $\mathbb{S}^2$, then there is a diffeomorphism $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\varphi^* j = i$, the standard complex structure.

   Hint: Use Riemann-Roch (which follows from our index theorem and the deRahm-Dolbeault theorem) to show that a genus 0 Riemann surface (Σ, j) admits a meromorphic function with at most a simple pole. This produces a non-constant holomorphic map $(\Sigma, j) \to (\mathbb{CP}^1, i)$ and use that it has as many poles as it has zeros to show that it is injective and hence a biholomorphism.

Problem 2: Bubbling experiments. A guided version. This is a more detailed version of problem 2 so one can check the bubbling explicitly. I have tried to keep notation consistent with the abstract development of bubbling in the notes.

Consider the sequence of holomorphic curves $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2, i)$ given by $u_n(z) = nz$ where $z \in \mathbb{C} \cup \infty = \mathbb{S}^2$.

a. Note that[6] $[u_n] = [\mathbb{S}^2]$ and describe the point-wise limit. Justify the sentence "all mass gets displaced to infinity" by describing the limit of a compact set K that does not contain 0. Taking as a fact that the (Fubini-Study) gradient of u_n at z is given by $\sqrt{2}n\frac{1+|z|^2}{n^2|z|^2+1}$, observe that is blows up along $z_n = 1/n$.

b. Produce the bubble: let $R_n = n$ and $\varepsilon_n = \frac{1}{\sqrt{n}}$ (note that $R_n\varepsilon_n = \sqrt{n} \to \infty$) and consider $b_n : (\mathbb{D}_{R_n\varepsilon_n}(0), i) \to \mathbb{D}_{\varepsilon_n}(0 + 1/R_n)$ given by $b_n(z) = z/n + 1/n$ and compute the limit of $v_n := u_n \circ b_n$.

Consider next the sequence $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2, i \oplus i)$ given by $u_n(z) = (z, nz)$.

c. Find the homology class $[u] \in H_2(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{Z})$, compute the limit u of u_n outside of 0 as a map $u : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2, i \oplus i)$ and describe the homology classes $[u]$ and $[u_n] - [u]$.

d. Consider b_n as before and $v_n := u_n \circ b_n$ and compute the limit.

Now consider the following sequence $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ given by $u_n(z) = (z, \frac{1}{nz}, \frac{1}{n^2 z})$.

e. Find the homology class $[u] \in H_2(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2; \mathbb{Z})$, compute the limit u of u_n outside of 0 as a map $u : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ and describe the homology classes $[u]$ and $[u_n] - [u]$.

f. Consider $b^1_n(z) = z/n + 1/n$ and $v^1_n = u_n \circ b^1_n$, as well as $b^2_n(z) = z/n - 1 + 1/n$ and $v^2_n = v^1_n \circ b^2_n$. Compute the limits appropriately and describe the homology classes and corresponding bubble tree.

Finally, consider the following sequence $u_n : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ given by $u_n(z) = (z, \frac{1}{n(z-\frac{1}{n})}, \frac{1}{n^2(z+\frac{1}{n})})$.

g. Find the homology class $[u] \in H_2(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2; \mathbb{Z})$, compute the limit u of u_n outside of 0 as a map $u : (\mathbb{S}^2, i) \longrightarrow (\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2, i \oplus i \oplus i)$ and describe the homology classes $[u]$ and $[u_n] - [u]$.

h. Consider $b^0_n(z) = z/n$ and $b^\pm_1 = z/n \mp 1/n \pm 1$. Compute the limits (and describe the homology classes) of $v^0_n = u_n \circ b^0_n$, $v^+_n = u_n \circ b^+_n$ and $v^-_n := u_n \circ b^-_n$.

[6] Recall the notation $[u] = u_*[\Sigma] \in H_2(M; \mathbb{Z})$.