Symplectic field theory
Problem set 7

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In this sheet we work on three different but related things: a finite-dimensional simplification of the moduli space and transversality framework; a first contact with Fredholm geometry by constructing Kuranishi models and proving the Sard-Smale theorem; and another example of a PDE moduli space (from the Kazdan-Warner equation) where a lot of the tools we have develop allow us to study this "analogous" moduli space and prove the uniformization theorem.

Problem 1. The deformation theory of the solution space of a global system of smooth equations with symmetries (in the finite-dimensional setting).[1] A (local) system of (finite-dimensional) smooth equations is simply a smooth function $f : \mathbb{R}^n \to \mathbb{R}^k$ and its solutions space is $\mathcal{M} = f^{-1}(0)$. A global system of smooth equations is a section $s : M \to E$ of a rank k vector bundle E over a closed n-manifold and its solution space is $\mathcal{M} = s^{-1}(0_E) = s \cap 0_E$, where 0_E is the zero-section of $E \to M$.

a. Choose a connection ∇ on E, define the linearization $D_p s : T_pM \to E_p$ of s at a solution $p \in \mathcal{M}$ (note that the target is E_p, not $T_{s(p)}E$) and show that it is independent of the connection. If s is transverse to the zero section at p, $s \pitchfork 0_E$ at p, prove that then $\mathcal{M}$ is a manifold near p of dimension $n - k$ whose tangent space is given by $\ker D_p s$.

Often times these sort of equations come with natural symmetries, which we model as follows: let G be a compact Lie group acting smoothly and properly on M such that E is a G-vector bundle and s is equivariant with respect to this action. Then G acts properly and smoothly on $s^{-1}(0_E)$ and the solution space is defined as $\mathcal{M} = s^{-1}(0)/G$.

c. Consider the orbit map $G \to M$ given by $g \mapsto g \cdot p$, composing it with $s : M \to E$ and linearizing it at point the $p \in M$ that is a zero of s we get:

$$T_{\mathrm{id}}G = \mathfrak{g} \xrightarrow{L^{-1}} T_pM \xrightarrow{L^0 = D_p s} E_p.$$

Show that this is a complex, i.e. $L^0 \circ L^{-1} = 0$.

The complex is denoted $L^\bullet$ and it is called the deformation complex of $[p] \in \mathcal{M}$. The cohomology group $H^{-1}(L^\bullet)$ encodes the infinitesimal automorphisms of $[p]$; $H^0(L^\bullet)$ encodes the infinitesimal deformations of $[p]$; and $H^1(L^\bullet)$ the infinitesimal obstructions. The following justifies this:

d. Show that the stabilizer G_p of the G-action at p is finite if and only if $H^{-1}(L^\bullet) = \ker L^{-1} = 0$.

e. Assuming G_p is finite, show that if $H^1(L^\bullet) = \operatorname{coker} L^0 = 0$, then $\mathcal{M}$ is an orbifold at $[p]$ with isotropy G_p whose tangent space is $H^0(L^\bullet) = \ker L^0 / \operatorname{im} L^{-1}$.

Hint: Check out the slice theorem for smooth proper group actions with finite-stabilizer.

[1]This problem is a syntactical retelling of problem 1 in problem set 3 in terms of "deformation theory". Semantically, the only new thing is the slice theorem (for actions acting with finite-isotropy). The point is to develop intuition for the infinite-dimensional version of this problem, which we work with in class.

We finish this problem with a bonus:[2] when we cannot assume that s is transverse to 0_E we can perturb the equation to achieve transversality, especially because in nature the equation s may naturally depend on some parameters. We model this as follows. Let P be a smooth manifold, denote by E_P the vector bundle over $M \times P$ that trivially extends $E \to M$ (i.e. $E_P = pr_M^* E$, $pr_M : M \times P \to M$) and consider a given section $S : M \times P \to E_P$ (the sections $s_p := s(\cdot, p) : M \to E$ are as before, so S is a P-family of them). Write $\mathcal{M} = S^{-1}(0)$ and $\mathcal{M}_p = s_p^{-1}(0)$.

f. Bonus: Consider the projection $pr_P : \mathcal{M} \subseteq M \times P \to P$. At a point $(x,p) \in \mathcal{M}$ where $D_{(x,p)}S$ is surjective, show that $D_x s_p$ is surjective if and only if π is a submersion at (x,p). Conclude that, if $D_{(x,p)}S$ is surjective at all points $(x,p) \in \mathcal{M}$ there is a comeager set P_{reg} in P such that $\mathcal{M}_p$ is an $n-k$ dimensional manifold for all $p \in P_{\text{reg}}$.

   Hint: Recall the first problem on the third problem set.

g. Bonus: By definition of S, we have that $D_{(x,p)} = D_x s_p + R_p$ for a linear map $R_p : T_pP \to E_x$, i.e. the linearization of S in the P direction. Show that $D_{(x,p)}$ is surjective if and only if $pr \circ R_p : T_pP \to \operatorname{coker} D_x s_p$ is surjective, where $pr : E_x \to \operatorname{coker} D_x s_p$ is the natural projection.

The last two items can be interpreted as follows: *as long as the parameters P added to the equation $s = s_{p_0}$ saturate the infinitesimal obstructions of all s_p, then almost all $\mathcal{M}_p$ are transversely cut out.* In particular, notice that it would suffice for $R_p : T_pP \to E_x$ to be surjective to achieve generic transversality.

Remark. This is a finite-dimensional analogue of the deformation theory for an "Elliptic deformation problem", such as is the case with holomorphic curves and other solution spaces to elliptic PDE (like in the last problem of this sheet). While in many cases we can assume the infinitesimal obstructions vanish (we have an orbifold structure), in many other cases we cannot. The right thing to do in that case is the content of a very very very long story. By the right thing to do I mean what the correct framework is to extract (enumerative) invariants from the obstructed moduli space. Later in the course, we hope to touch upon this further, for now note the following: part of a certain line of thought is to work with the deformation complex[3] as the "tangent object" of a weaker notion of space.

Finally, there is one crucial difference (other than this is finite-dimensional) from this nice model to what we do in the lectures. In our setting, the action of G is not quite smooth. It will require some cleverness to get around this in the cases we care about. In general, this is difficult and related to said very very very long story.

Problem 2. The Kuranishi model of a Fredholm map and the Sard-Smale theorem. Let X and Y be Banach spaces and $f : X \to Y$ a smooth map of Banach spaces. A point $y \in Y$ is called a *regular value* of f if every $x \in f^{-1}(y)$ is a *regular point* of f, which means that $T_x f$ is surjective *and* has a bounded right inverse.

a. * (Implicit function theorem) Consider an open set U of X a smooth map $f : U \to Y$ such that $x \in U$ is a regular point of f and $y = f(x)$. Deduce from the inverse function theorem that there is an open neighbourhood $V \subset \ker T_x f$ of 0 and a smooth embedding $\varphi : V \to U$ onto a neighbourhood of x *in* $f^{-1}(y)$.

   Hint: The IFT allows you to locally invert a smooth map of Banach spaces near a point where its derivative is a Banach space isomorphism. The crux of this exercise is realizing how to leverage the existence of the bounded right inverse, which is a necessary condition, to use the IFT.

[2] We forget the symmetry out of simplicity, as achieving transversality for s has to do with H^1 and so the group action is not involved

[3] In fact, note that in the category of complexes of vector spaces the deformation complex $L_\bullet$ is quasi-isomorphic to its cohomology (as a trivial complex). So, the apparent tangent object is a "derived" $\mathbb{R}$-vector space. The non-linear version of these are derived smooth manifolds, which partly arise from adjoining fiber products to the category of smooth manifolds (so non-transverse intersections now exist).

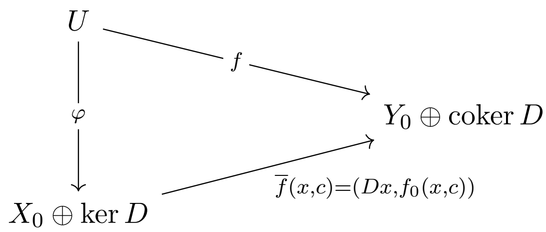
Now assume that f is moreover a *smooth Fredholm map*, which means that its derivative map at x, $T_xf : X \longrightarrow Y$, is a Fredholm operator.

b. The Fredholm index of T_xf does not depend on x, why?

Consequently, given a smooth Fredholm map $f : X \to Y$ it has a well defined index $\operatorname{ind} f$. The IFT implies that *that if y is a regular value of a smooth Fredholm map f, then $f^{-1}(y)$ is a finite-dimensional smooth manifold whose dimension is given by* $\operatorname{ind} f$ *and tangent spaces by* $\ker Tf$.

Our next goal is to describe sets cut out by Fredholm map, not necessarily at regular values. To do this we describe zero sets cut out by smooth Fredholm maps as zero sets cut out by finite-dimensional smooth maps. Consider U an open neighbourhood of 0 in X and assume that the smooth Fredholm map has a zero at zero, i.e. $f(0) = 0$. Denote $D := d_0 f$ and let $X = X_0 \oplus \ker D$ and $Y = Y_0 \oplus \operatorname{coker} D$ be a decomposition of X and Y in virtue of $D : X \to Y$ being a Fredholm operator.

c. Show that after possibly making U smaller, there is a smooth map $\varphi : U \to X_0 \oplus \ker D$ that is a diffeomorphism onto its image such that the diagram



commutes, where $\overline{f}$ is given by $(x, c) \mapsto (Dx, f_0(x, c))$, $D : X_0 \to Y_0$ is a linear isomorphism and $f_0 : X_0 \oplus \ker D \to \operatorname{coker} D$ is a smooth map whose derivative at 0 vanishes.

From this we can conclude that the zero set of f near 0 is given by the zero set $f_0(0, \cdot) : \ker D \longrightarrow \operatorname{coker} D$, which we call the **Kuranishi model** of f at 0. With this in hand we are ready to prove the main analytic tool that enables transversality arguments:

> Sard-Smale theorem. *Let X and Y be paracompact*[4] *and separable Banach spaces, $U \subseteq X$ an open set and $f : U \longrightarrow Y$ a smooth Fredholm map. Then, the set of regular values of f is comeager in Y (i.e. countable intersection of open and dense subsets).*

d. * Using the (finite-dimensional) Sard's theorem prove the Sard-Smale theorem.

Hint: The separability hypothesis is crucial, with paracompactness it implies that every open cover has a countable subcover. Then, it suffices to show that every point $x \in U$ has an closed neighbourhood V for which the set of regular values of $f|_V$ is open and dense in Y. Keep the Kuranishi model in mind.

Remark. Allow me to elaborate on the Kuranishi model. Firstly, note that if f was equivariant under the action of some compact Lie group on X and Y, then f_0 could be taken equivariant as well (so there is a equivariant finite-dimensional reduction). Secondly, note that this allows us to *locally* present the moduli space of holomorphic curve (irregardless of transversality being achieved or not) as the zero set of a smooth map $\mathbb{R}^k \to \mathbb{R}^c$ equivariant under some finite group action. Locally, this is a good model to extract enumerative information (see previous remark), but it's simply a Herculean task to globalize this and work with "Kuranishi spaces": groupoid like spaces arising from patching up those local Kuranishi models. It is a recent breakthrough that there is a way to obtain global Kuranishi charts that have a tractable algebraic topology and so invariants can be extracted in different settings. This is one of the important plot points in the aforementioned long story.

[4]All Banach spaces are paracompact, I have added this hypothesis for psychological reasons, because it is needed for transporting this statement to the "global" setting, i.e. for Banach manifolds.

Problem 3. The uniformization theorem and the Kazdan-Warner equation. A closed Riemann surface (Σ, j) has a unique conformal class of Riemannian metrics associated to it, of which a compatible metric after choosing an area form compatible with j is a representative.

> Metric uniformization theorem. *In the conformal class induced by j there is a Riemannian metric g of constant scalar curvature $2, 0$ and -2 when $\chi(\Sigma) = 2, 0$ and < 0 respectively. This metric is unique for the first and last cases and unique up to constant for $\chi(\Sigma) = 0$.*

a. Explain why this is equivalent to the uniformization theorem stated in class.

To prove this theorem the first step should be to understand the scalar curvature of different conformal metrics. Let g_0 be a Riemannian metric on Σ and set $g = e^{2f} g_0$ for $f : \Sigma \to \mathbb{R}$. One can compute that

$$R_g = e^{-2f}(R_{g_0} + 2\Delta_{g_0} f),$$

where $\Delta_{g_0} = d^*d$ is the Beltrami Laplacian of g_0 (it depends on g_0 to take the adjoint). Consider g_0 fixed so we can drop the subscripts from the notation and let $\mu : \Sigma \to \mathbb{R}$ a function, the following is the *Kazdan-Warner equation*

$$2\Delta f + \mu e^{2f} = -R.$$

This is understood as a non-linear PDE on a function f given a metric g_0 and a function μ. Another good name is the "prescribed scalar curvature equation", on the grounds that solutions f produce metrics conformal to g_0 with the scalar curvature prescribed by μ.

b. Phrase the metric uniformization theorem in terms of "solving" the Kazdan-Warner equation (for specific μ's).

Before initiating our study of this equation, for reasons that will become clear, we must take care of the $\chi = 2$ and $\chi = 0$ case separately. The former we solved in the previous sheet, for the latter consider:

c. Let $n \in \mathbb{Z}^{>0}$, $p \in \mathbb{Z}$ such that $2p > n$, and (X, g) a closed connected oriented Riemannian n-manifold. There exists a constant $C > 0$ (depending on (X, g) and p) such that for every $h \in L^p(X)$ with $\int_X h = 0$ there exists a unique $f \in W^{2,p}(X)$ solution to

$$\Delta f = h, \quad \int_X f = 0$$

and, moreover, it satisfies $||f||_{L^\infty} \leq C||h||_{L^p}$.

Hint: The kernel of the Laplace operator $\Delta : W^{2,p}(X) \to L^p(X)$ consists of the constant functions and its range consists of the functions of mean value zero (to show this reflect on $\langle \Delta f, h \rangle = \langle df, dh \rangle$).

d. * Conclude the $\chi = 0$ case of the uniformization theorem.

The $\chi < 0$ case will follow after we show that equations (on f) of the form $\Delta f + 2e^f = h$ with $\int_\Sigma h > 0$ have a unique solution (note the use of Gauss-Bonnet needed to assume that the integral of h is positive). This will follow from the following theorem.

> Kazdan-Warner theorem. *Let $n \in \mathbb{Z}^{>0}$, $p \in \mathbb{Z}$ such that $2p > n$, and (X, g) a closed connected oriented Riemannian n-manifold. We say that a pair $\mu, h \in L^p(X)$ satisfies (P) if $\mu \geq 0$, $\int_X \mu_+ > 0$ and $\int_X h > 0$. Then, there exists a unique pair $\mu, h \in L^p(X)$ satisfying (P) and the equation*
>
> $$\Delta f + \mu e^f = h.$$

We sketch the proof of this theorem by fitting it into a "moduli space set-up". Define the parameter space $\mathcal{P} = \{(\mu, h) \in \big(L^p(X)\big)^2 : \text{they satisfy property}(P)\}$ and the moduli space of solutions

$$\mathcal{M} = \{(f, \mu, h) \in W^{2,p}(X) \times \mathcal{P} : \ \Delta f + \mu e^f = h\},$$

and the corresponding projection $\pi : \mathcal{M} \longrightarrow \mathcal{P}$.

e. * Show that $\mathcal{P}$ is a contractible Banach manifold.

f. Show that 0 is the unique solution $(\mu, h) = (1, 1)$, i.e. $\pi^{-1}(1, 1) = \{(0, 1, 1)\}$.

   Hint: At a maximum x of a function f we have that $\Delta f(x) \geq 0$ and similarly for a minimum reversing the inequality.

g. * The moduli space $\mathcal{M}$ is a Banach submanifold of $W^{2,p}(X) \times \big(L^2(X)\big)^2$ and, moreover, π is a submersion.

   Hint: Present the moduli space as a zero set of a smooth Banach map $s : W^{2,p} \to L^p$ for fixed (μ, h), compute its linearization $d_f s$ at a solution and show that $d_f s$ is an isomorphism (by showing that it is Fredholm, of index 0 and injective). For this part *c.* is useful.

From this, we conclude that for (μ, h) sufficiently close to $(1, 1)$ there is a unique solution f to the equation. This is the equivalent of our theorems for local structure of the moduli space. To be able to claim a unique solution exists for all $(\mu, h) \in \mathcal{P}$ we must establish some compactness properties.

h. * Assuming that π is proper, show that it is a diffeomorphism.

Let me *sketch* how properness works. We want to show that given a sequence $(f_n, \mu_n, h_n) \in \mathcal{M}$ such that the sequence of parameters $(\mu_n, h_n) \in \mathcal{P}$ converges, then we can find a convergent subsequence of (f_n, μ_n, h_n) in $\mathcal{M}$. Analogously to what we did in the lectures, since we have a second order elliptic problem on our hands, the above will be guaranteed as soon as we have uniform C^2-bounds. Using the elliptic estimate

$$||f_n||_{W^{2,p}} \leq C\big(||h_n||_{L^p} + ||f_n||_{L^p}\big)$$

we see that it would suffice if we had good enough *"a priori" bounds* on the L^∞-norm of solutions to the Kazdan-Warner equation (in terms of its parameters). This is something that we do not have in the holomorphic curve case, the gradient can blow up. In this case, the a priori estimates are a clever application of the maximum principle.

Remark. In this problem we see how the analysis and moduli space framework can be applied to other interesting problems, such as the uniformization theorem. The core idea, which is to solve a parametric equation for easy values of the parameters, show that the solutions found persist locally and then extend them globally via compactness, is a classical and known as the "continuation method" (which you may have seen in the context of PDE theory). What is truly remarkable about holomorphic curves is that compactness cannot be achieved naively but that a geometrically meaningful compactification exists *and* that it somehow holds a lot of *algebro-topological* information.[5] You can see this by how the boundary of the moduli space of Floer cylinders that come in 1 parameter families allows you prove $d^2 = 0$ (we will see something similar in SFT); or how the moduli spaces of closed holomorphic curves are expected to have a codimension 2 boundary and so intersection theory can be arranged for enumerative purposes; and many more subtler phenomena such as higher operations on Floer theory and SFT.

[5] There is no need to set-up subtle theories to extract difficult enumerative/algebraic invariants from moduli space to obtain interesting and worthy results. Indeed, the continuation method together with the nice compactness properties is one of the hands-on ways to work with holomorphic curves and it has yielded excellent results, such as McDuff's classification of rational/ruled 4-manifolds.