Lectures on Symplectic Field Theory

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Preface

This book is an expanded version of the lecture notes I produced for a two-semester course taught at University College London in 2015–16, for Ph.D. students with a background in basic symplectic geometry and interest in symplectic topology and/or geometric analysis. For the most part, each chapter corresponds to a two-hour lecture in the original course, though the reader will quickly notice that in this "expanded" version, most individual chapters contain far more material than can reasonably fit into one lecture (or even two). In reality, much of that material was only sketched or mentioned in passing during lectures, and I ended up using the notes to discuss everything that I would like to have explained if I'd had unlimited time. This includes relatively detailed discussions of several important technical points (e.g. the definition of spectral flow, generic transversality in symplectizations, the punctured Riemann-Roch formula, finite energy and asymptotics with arbitrary stable Hamiltonian structures) which are either incompletely covered by the existing literature or, in my opinion, simply more difficult to learn from other sources than they should be. For topics that are, on the other hand, well covered elsewhere, I have usually not felt obliged to explain every detail, but have tried always to provide adequate references.

One of the interesting features of SFT is that its foundations are—at the time of this writing—not yet complete. When the original "propaganda paper" [**EGH00**] appeared in 2000, it was widely believed that the technical details would be filled in within a few years, and several papers introducing important applications of SFT to contact topology were written under this assumption. Since then, a certain realization has set in that the results in those papers cannot truly be regarded as "theorems" in the sense of mathematics, and it has become less socially acceptable to preface statements of results with caveats of the form, "this theorem is dependent on the foundations of SFT". At the same time, the need for a robust perturbation scheme to achieve transversality in SFT spawned the development of a whole new approach to infinite-dimensional differential geometry, the *polyfold* project [**Hof06**], which is intended for much more general applications but is not yet finished. Opinions vary among symplectic topologists as to how unsatisfied we should all be with this state of affairs, and what could be done about it—among other things, one could make an entire course out of the discussion of such issues, but I have not chosen to do that. My approach is instead to develop the *classical*[1] analysis of pseudoholomorphic curves in symplectizations and symplectic cobordisms, to explain how this would lead to a theory of algebraic contact invariants if transversality for multiple

[1]For the purposes of this discussion, the word "classical" may be defined as "not involving the words *polyfold*, *virtual* or *Kuranishi*".

covers were not an issue, and then to use the tools and insights gained from this discussion to prove *rigorous mathematical theorems* about contact manifolds. Typically, such theorems can be regarded informally as consequences of computations in a (not yet well-defined) theory called SFT, but in a rigorous sense, they are actually consequences of the methods used in those computations. Examples covered in these notes include distinguishing tight contact structures on the 3-torus that are homotopic but not isomorphic (Chapter 11), and the nonexistence of symplectic fillings or symplectic cobordisms between certain pairs of contact manifolds (Chapter 17). The choice of applications is of course biased somewhat toward my own research interests.

Prerequisites. The stated target audience for the original lecture course was "advanced Masters and Ph.D. students in differential geometry or related fields who are not afraid of analysis". More precisely, the notes assume some knowledge of the following topics:

- Differential geometry: manifolds and vector bundles, differential forms and Stokes' theorem, connections, basic familiarity with symplectic manifolds;
- Functional analysis: linear operators on Banach spaces, basics of Sobolev spaces, Fredholm operators;
- Differential topology: smooth mapping degree, intersection numbers, Sard's theorem;
- Algebraic topology: fundamental group, homology and cohomology of manifolds, Poincaré duality, first Chern class, homological intersection numbers.

The following topics are not considered formal prerequisites, but some knowledge of them is likely in any case to be helpful to the reader, who may want to have a good reference for them (as suggested below) within arm's reach:

- Contact manifolds (e.g. Geiges [**Gei08**]);
- Differential calculus on Banach spaces and Banach manifolds (e.g. these two books by Lang: [**Lan93**] and [**Lan99**]);
- Closed pseudoholomorphic curves (e.g. McDuff-Salamon [**MS12**] or my other book in preparation [**Wenb**]);
- Floer homology (e.g. Salamon [**Sal99**] or Audin-Damian [**AD14**]).

Acknowledgements. I would like to thank the students who have sat through various iterations of the course that gave rise to this book, notably Alexandru Cioba and Agustín Moreno for their assistance in editing the first several lectures, as well as Adrian Dawid, Milica Đukić, Shah Faisal, Solveig Hepp, Catalina Jurja, and Michael Rothgang for useful comments. My understanding of Taubes's approach to the Riemann-Roch formula (explained in Chapter 5) and its generalization to the punctured case emerged in part from discussions with Chris Gerig, and I am grateful also to Tim Perutz for helpful hints about Weitzenböck formulas, and Patrick Massot for patient discussions of singular integral operators and elliptic regularity. Thanks also to Michael Hutchings and Janko Latschev for helping me understand the combinatorial factors in Chapter 13, to Jo Nelson for helpful comments on coefficients and orbifold singularities, and to Sam Lisi and Barney Bramham for advice on the

Floer C_ϵ space. And also to Klaus Niederkrüger and Helmut Hofer for enlightening discussions on all manner of things.

About the current version

The version you see in front of you is being revised and updated regularly to accompany a Masters-level special topics course on symplectic field theory at the Humboldt-Universität zu Berlin in the 2026 summer semester.

I have tried to produce a manuscript that is relatively well polished, but I have not tried quite as diligently for that as I do with most of my research papers. As of the beginning of the 2026 summer semester, some of the later chapters that have been in planning for over a decade are not yet complete, and one or two additional chapters exist only as vague plans in my head, so if those chapters exist by the end of the semester, they are unlikely to be error-free. I apologize for any sloppiness that I may have failed so far to expunge. All comments and corrections are welcome,[2] and may be sent to `wendl@math.hu-berlin.de`. Updates on the publication of the book will be posted periodically on my website at

`https://www.mathematik.hu-berlin.de/~wendl/publications.html#notes`

Most recent update: **April 20, 2026**

[2]especially if those corrections are received before the book goes to press

CHAPTER 1

Introduction

Contents

Symplectic field theory is a general framework for defining invariants of contact manifolds and symplectic cobordisms between them via counts of "asymptotically cylindrical" pseudoholomorphic curves. In this first chapter, we'll summarize some of the historical background of the subject, and then sketch the basic algebraic formalism of SFT.

1.1. In the beginning, Gromov wrote a paper

Pseudoholomorphic curves first appeared in symplectic geometry in a 1985 paper of Gromov [Gro85]. The development was revolutionary for the field of symplectic topology, but it was not unprecedented: a few years before this, Donaldson had demonstrated the power of using elliptic PDEs in geometric contexts to define invariants of smooth 4-manifolds (see [DK90]). The PDE that Gromov used was a slight generalization of one that was already familiar from complex geometry.

Recall that if M is a smooth $2n$-dimensional manifold, an **almost complex structure** on M is a smooth linear bundle map $J : TM \to TM$ such that $J^2 = -\mathbb{1}$. This makes the tangent spaces of M into complex vector spaces and thus induces an orientation on M; the pair (M, J) is called an **almost complex manifold**. In this context, a **Riemann surface** is an almost complex manifold of real dimension 2 (hence complex dimension 1), and a **pseudoholomorphic curve** (also called J-**holomorphic**) is a smooth map

$$u : \Sigma \to M$$

satisfying the **nonlinear Cauchy-Riemann equation**

$$Tu \circ j = J \circ Tu, \tag{1.1}$$

where (Σ, j) is a Riemann surface and (M, J) is an almost complex manifold (of arbitrary dimension). The almost complex structure J is called **integrable** if M admits the structure of a complex manifold such that J is multiplication by i in holomorphic coordinate charts. By a basic theorem due to Gauss, every almost complex structure in real dimension two is integrable, hence one can always find local coordinates (s, t) on neighorhoods in Σ such that

$$j\partial_s = \partial_t, \qquad j\partial_t = -\partial_s.$$

In these coordinates, (1.1) takes the form

$$\partial_s u + J(u)\partial_t u = 0.$$

The fundamental insight of [Gro85] was that solutions to the equation (1.1) capture information about symplectic structures on M whenever they are related to J in the following way.

DEFINITION 1.1.1. Suppose (M, ω) is a symplectic manifold. An almost complex structure J on M is said to be **tamed** by ω if

$$\omega(X, JX) > 0 \quad \text{for all } X \in TM \text{ with } X \neq 0.$$

Additionally, J is **compatible** with ω if the pairing

$$g(X, Y) := \omega(X, JY)$$

defines a Riemannian metric on M.

EXERCISE 1.1.2. Show that an almost complex structure J is compatible with a symplectic form ω if and only if it is tame and ω is J-invariant.

We shall denote by $\mathcal{J}(M)$ the space of all smooth almost complex structures on M, with the C^∞_{loc}-topology, and if ω is a symplectic form on M, let

$$\mathcal{J}_\tau(M, \omega), \mathcal{J}(M, \omega) \subset \mathcal{J}(M)$$

denote the subsets consisting of almost complex structures that are tamed by or compatible with ω respectively. Notice that $\mathcal{J}_\tau(M, \omega)$ is an open subset of $\mathcal{J}(M)$, but $\mathcal{J}(M, \omega)$ is not. Proofs of the following may be found in [MS17, §2.5] or [Wenb, §2.2], among other places.

PROPOSITION 1.1.3. *On any symplectic manifold (M, ω), the spaces $\mathcal{J}_\tau(M, \omega)$ and $\mathcal{J}(M, \omega)$ are each nonempty and contractible.* □

Tameness implies that the **energy** of a J-holomorphic curve $u : \Sigma \to M$,

$$E(u) := \int_\Sigma u^*\omega,$$

is always nonnegative, and it is strictly positive unless u is constant. Notice moreover that if the domain Σ is closed, then $E(u)$ depends only on the cohomology class $[\omega] \in H^2_{\text{dR}}(M)$ and the homology class

$$[u] := u_*[\Sigma] \in H_2(M),$$

so in particular, any family of J-holomorphic curves in a fixed homology class satisfies a uniform energy bound. This basic observation is one of the key facts behind

Gromov's compactness theorem, which states that moduli spaces of closed curves in a fixed homology class are compact up to "nodal" degenerations.

The most famous application of pseudoholomorphic curves presented in [Gro85] is Gromov's *nonsqueezing theorem*, which was the first known example of an obstruction for embedding symplectic domains that is subtler than the obvious obstruction defined by volume. The technology introduced in [Gro85] also led directly to the development of the *Gromov-Witten invariants* (see [MS12, RT95, RT97]), which follow the same pattern as Donaldson's earlier smooth 4-manifold invariants: they use counts of J-holomorphic curves to define invariants of symplectic manifolds up to symplectic deformation equivalence.

Here is another sample application from [Gro85]. We denote by

$$A \cdot B \in \mathbb{Z}$$

the intersection number between two homology classes $A, B \in H_2(M)$ in a closed oriented 4-manifold M.

THEOREM 1.1.4. *Suppose (M, ω) is a closed and connected symplectic 4-manifold with the following properties:*

(i) (M, ω) does not contain any symplectic submanifold $S \subset M$ that is diffeomorphic to S^2 and satisfies $[S] \cdot [S] = -1$.

(ii) (M, ω) contains two symplectic submanifolds $S_1, S_2 \subset M$ which are both diffeomorphic to S^2, satisfy

$$[S_1] \cdot [S_1] = [S_2] \cdot [S_2] = 0,$$

and have exactly one intersection point with each other, which is transverse and positive.

Then (M, ω) is symplectomorphic to $(S^2 \times S^2, \sigma_1 \oplus \sigma_2)$, where for $i = 1, 2$, the σ_i are area forms on S^2 satisfying

$$\int_{S^2} \sigma_i = \langle [\omega], [S_i] \rangle.$$

SKETCH OF THE PROOF. Since S_1 and S_2 are both symplectic submanifolds, one can choose a compatible almost complex structure J on M for which both of them are the images of embedded J-holomorphic curves. One then considers the moduli spaces $\mathcal{M}_1(J)$ and $\mathcal{M}_2(J)$ of equivalence classes of J-holomorphic spheres homologous to S_1 and S_2 respectively, where any two such curves are considered equivalent if one is a reparametrization of the other (in the present setting this just means they have the same image). These spaces are both manifestly nonempty, and one can argue via Gromov's compactness theorem for J-holomorphic curves that both are compact. Moreover, an infinte-dimensional version of the implicit function theorem implies that both are smooth 2-dimensional manifolds, carrying canonical orientations, hence both are diffeomorphic to closed surfaces. Finally, one uses *positivity of intersections* to show that every curve in $\mathcal{M}_1(J)$ intersects every curve in $\mathcal{M}_2(J)$ exactly once, and this intersection is always transverse and positive; moreover, any two curves in the same space $\mathcal{M}_1(J)$ or $\mathcal{M}_2(J)$ are either identical or disjoint. It follows that both moduli spaces are diffeomorphic to S^2, and both consist of smooth families of J-holomorphic spheres that foliate M, hence defining a diffeomorphism

$$\mathcal{M}_1(J) \times \mathcal{M}_2(J) \to M$$

that sends (u_1, u_2) to the unique point in the intersection $\operatorname{im} u_1 \cap \operatorname{im} u_2$. This identifies M with $S^2 \times S^2$ such that each of the submanifolds $S^2 \times \{*\}$ and $\{*\} \times S^2$ are symplectic. The latter observation can be used to determine the symplectic form up to deformation, so that by the Moser stability theorem, ω is determined up to isotopy by its cohomology class $[\omega] \in H^2_{\mathrm{dR}}(S^2 \times S^2)$, which depends only on the evaluation of ω on $[S^2 \times \{*\}]$ and $[\{*\} \times S^2] \in H_2(S^2 \times S^2)$. □

For a detailed exposition of the above proof of Theorem 1.1.4, see [Wen18, Theorem E].

1.2. Hamiltonian Floer homology

Throughout the following, we write

$$S^1 := \mathbb{R}/\mathbb{Z},$$

so maps on S^1 are the same as 1-periodic maps on $\mathbb{R}$. One popular version of the *Arnol′d conjecture* on symplectic fixed points can be stated as follows. Suppose (M, ω) is a closed symplectic manifold and $H : S^1 \times M \to \mathbb{R}$ is a smooth function. Writing $H_t := H(t, \cdot) : M \to \mathbb{R}$, H determines a 1-periodic time-dependent Hamiltonian vector field X_t via the relation[1]

$$\omega(X_t, \cdot) = -dH_t. \tag{1.2}$$

CONJECTURE 1.2.1 (Arnol′d conjecture). *If all 1-periodic orbits of X_t are nondegenerate, then the number of these orbits is at least the sum of the Betti numbers of M.*

Here a 1-periodic orbit $\gamma : S^1 \to M$ of X_t is called **nondegenerate** if, denoting the flow of X_t by φ^t, the linearized time 1 flow

$$d\varphi^1(\gamma(0)) : T_{\gamma(0)}M \to T_{\gamma(0)}M$$

does not have 1 as an eigenvalue. This can be thought of as a Morse condition for an action functional on the loop space whose critical points are periodic orbits; like Morse critical points, nondegenerate periodic orbits occur in isolation. To simplify our lives, let's restrict attention to *contractible* orbits and also assume that (M, ω) is **symplectically aspherical**, which means

$$[\omega]|_{\pi_2(M)} = 0, \quad \text{i.e.} \quad \langle [\omega], [u] \rangle = 0 \text{ for all continuous maps } u : S^2 \to M.$$

Then if $C^\infty_{\mathrm{contr}}(S^1, M)$ denotes the space of all smoothly contractible smooth loops in M, the **symplectic action functional** can be defined by

$$\mathcal{A}_H : C^\infty_{\mathrm{contr}}(S^1, M) \to \mathbb{R} : \gamma \mapsto -\int_{\mathbb{D}} \bar{\gamma}^* \omega + \int_{S^1} H_t(\gamma(t))\, dt,$$

[1]Elsewhere in the literature, you will sometimes see (1.2) without the minus sign on the right hand side. If you want to know why I strongly believe that the minus sign belongs there, see [Wena], but to some extent this is just a personal opinion.

where $\bar{\gamma} : \mathbb{D} \to M$ is any smooth map on the closed unit disk $\mathbb{D} \subset \mathbb{C}$ satisfying

$$\bar{\gamma}(e^{2\pi it}) = \gamma(t),$$

and the symplectic asphericity condition guarantees that $\mathcal{A}_H(\gamma)$ does not depend on the choice of $\bar{\gamma}$.

EXERCISE 1.2.2. The **first variation** of a functional such as $\mathcal{A}_H : C^\infty_{\text{contr}}(S^1, M)$ at $\gamma \in C^\infty_{\text{contr}}(S^1, M)$ is by definition the unique linear map $d\mathcal{A}_H(\gamma) : \Gamma(\gamma^* TM) \to \mathbb{R}$ such that for any smooth 1-parameter family $\{\gamma_s \in C^\infty_{\text{contr}}(S^1, M)\}_{s \in (-\epsilon,\epsilon)}$ with $\gamma_0 = \gamma$ and $\partial_s \gamma_s|_{s=0} = \eta$, one has

$$\left.\frac{d}{ds}\mathcal{A}_H(\gamma_s)\right|_{s=0} = d\mathcal{A}_H(\gamma)\eta.$$

In other words, if we think of $C^\infty_{\text{contr}}(S^1, M)$ as an infinite-dimensional manifold[2] with tangent spaces $T_\gamma C^\infty_{\text{contr}}(S^1, M) = \Gamma(\gamma^* TM)$, then $d\mathcal{A}_H(\gamma)$ is simply the differential of $\mathcal{A}_H$ at γ. Prove the formula

$$d\mathcal{A}_H(\gamma)\eta = \int_{S^1} \left[\omega(\dot{\gamma}, \eta) + dH_t(\eta)\right] dt = \int_{S^1} \omega(\dot{\gamma} - X_t(\gamma), \eta)\, dt.$$

Using the nondegeneracy of ω, this shows that the critical points of $\mathcal{A}_H$ are precisely the contractible 1-periodic orbits of X_t.

A few years after Gromov's introduction of pseudoholomorphic curves, Floer proved the most important cases of the Arnol′d conjecture by developing a novel version of infinite-dimensional Morse theory for the functional $\mathcal{A}_H$. This approach mimicked the homological approach to Morse theory which has since been popularized in books such as [**AD14**, **Sch93**], but was apparently only known to experts at the time. In *Morse homology*, one considers a smooth Riemannian manifold (M, g) with a Morse function $f : M \to \mathbb{R}$, and defines a chain complex whose generators are the critical points of f, graded according to their Morse index. If we denote the generator corresponding to a given critical point $x \in \text{Crit}(f)$ by $\langle x \rangle$, the boundary map on this complex is defined by

$$\partial\langle x \rangle = \sum_{\text{Morse}(y) = \text{Morse}(x) - 1} \#\left(\mathcal{M}(x, y)/\mathbb{R}\right) \langle y \rangle,$$

where $\mathcal{M}(x, y)$ denotes the moduli space of negative gradient flow lines $u : \mathbb{R} \to M$, satisfying $\partial_s u = -\nabla f(u(s))$, $\lim_{s \to -\infty} u(s) = x$ and $\lim_{s \to +\infty} u(s) = y$. This space admits a natural $\mathbb{R}$-action by shifting the variable in the domain, and one can show that for generic choices of f and the metric g, $\mathcal{M}(x, y)/\mathbb{R}$ is a finite set whenever $\text{Morse}(x) - \text{Morse}(y) = 1$. The real magic, however, is contained in the following statement about the case $\text{Morse}(x) - \text{Morse}(y) = 2$:

[2] At this stage, it is best not to worry so much over exactly what kind of infinite-dimensional manifold $C^\infty_{\text{contr}}(S^1, M)$ is. It is locally modeled on open subsets of the space of smooth sections $\Gamma(\gamma^* TM)$, which is a Fréchet space, but the notion of a "Fréchet manifold" is not so straightforward to define precisely, and one can easily define the term "first variation" without worrying about it.

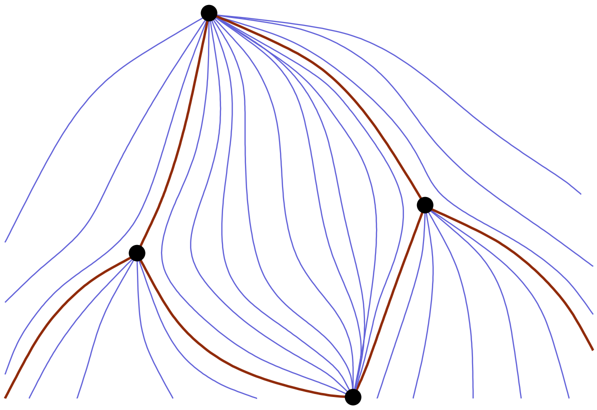

FIGURE 1.1. One-parameter families of gradient flow lines on a Riemannian manifold degenerate to broken flow lines.

PROPOSITION 1.2.3. *For generic choices of f and g and any two critical points $x, y \in \mathrm{Crit}(f)$ with* $\mathrm{Morse}(x) - \mathrm{Morse}(y) = 2$*, $\mathcal{M}(x,y)/\mathbb{R}$ is homeomorphic to a finite collection of circles and open intervals whose end points are canonically identified with the finite set*

$$\partial\overline{\mathcal{M}}(x,y) := \bigcup_{\mathrm{Morse}(z)=\mathrm{Morse}(x)-1} \mathcal{M}(x,z) \times \mathcal{M}(z,y).$$

We say that $\mathcal{M}(x,y)$ has a natural **compactification** $\overline{\mathcal{M}}(x,y)$, which has the topology of a compact 1-manifold with boundary, and its boundary is the set of all **broken flow lines** from x to y, cf. Figure 1.1. This set of broken flow lines is precisely what is counted if one computes the $\langle y \rangle$ coefficient of $\partial^2\langle x \rangle$, hence we deduce

$$\partial^2 = 0$$

as a consequence of the fact that compact 1-manifolds always have zero boundary points when counted with appropriate signs.[3] The homology of the resulting chain complex can be denoted by $HM_*(M\,;\,g,f)$ and is called the **Morse homology** of M. The well-known Morse inequalities can then be deduced from a fundamental theorem stating that $HM_*(M\,;\,g,f)$ is, for generic f and g, isomorphic to the singular homology of M.

With the above notion of Morse homology understood, Floer's approach to the Arnol′d conjecture can now be summarized as follows:

Step 1: Under suitable technical assumptions, construct a homology theory

$$HF_*(M,\omega\,;\,H,\{J_t\}),$$

depending *a priori* on the choices of a Hamiltonian $H : S^1 \times M \to \mathbb{R}$ with all 1-periodic orbits nondegenerate, and a generic S^1-parametrized family of ω-compatible almost complex structures $\{J_t\}_{t\in S^1}$. The generators of the

[3] Counting with signs presumes that we have chosen suitable orientations for the moduli spaces $\mathcal{M}(x,y)$, and this can always be done. Alternatively, one can avoid this issue by counting modulo 2, and thus define a homology theory with $\mathbb{Z}_2$ coefficients.

chain complex are the critical points of the symplectic action functional $\mathcal{A}_H$, i.e. 1-periodic orbits of the Hamiltonian flow, and the boundary map is defined by counting a suitable notion of gradient flow lines connecting pairs of orbits (more on this below).

Step 2: Prove that $HF_*(M, \omega) := HF_*(M, \omega\,;\, H, \{J_t\})$ is a *symplectic invariant*, i.e. it depends on ω, but not on the auxiliary choices H and $\{J_t\}$.

Step 3: Show that if H and $\{J_t\}$ are chosen to be time-independent and H is also C^2-small, then the chain complex for $HF_*(M, \omega\,;\, H, \{J_t\})$ is isomorphic (with a suitable grading shift) to the chain complex for Morse homology $HM_*(M\,;\, g, H)$ with $g := \omega(\cdot, J_t \cdot)$. The isomorphism between $HM_*(M\,;\, g, H)$ and singular homology thus implies that the Floer complex must have at least as many generators (i.e. periodic orbits) as there are generators of $H_*(M)$, proving the Arnol′d conjecture.

The implementation of Floer's idea required a different type of analysis than what is needed for Morse homology. The moduli space $\mathcal{M}(x, y)$ in Morse homology is simple to understand as the (generically transverse) intersection between the unstable manifold of x and the stable manifold of y with respect to the negative gradient flow. Conveniently, both of those are finite-dimensional manifolds, with their dimensions determined by the Morse indices of x and y. We will see in Chapter 3 that no such thing is true for the symplectic action functional: to the extent that $\mathcal{A}_H$ can be thought of as a Morse function on an infinite-dimensional manifold, its Morse index and its Morse "co-index" at every critical point are both infinite, hence the stable and unstable manifolds are not nearly as nice as finite-dimensional manifolds, providing no reason to expect that their intersection should be. There are additional problems since $C^\infty_{\text{contr}}(S^1, M)$ does not have a Banach space topology: in order to view the negative gradient flow of $\mathcal{A}_H$ as an ODE and make use of the usual local existence/uniqueness theorems (as in [**Lan99**, Chapter IV]), one would have to extend $\mathcal{A}_H$ to a smooth function on a suitable Hilbert manifold with a Riemannian metric. There is a very limited range of situations in which one can do this and obtain a reasonable formula for $\nabla\mathcal{A}_H$, e.g. [**HZ94**, §6.2] explains the case $M = \mathbb{T}^{2n}$, in which $\mathcal{A}_H$ can be defined on the Sobolev space $H^{1/2}(S^1, \mathbb{R}^{2n})$ and then studied using Fourier series. This approach is very dependent on the fact that the torus $\mathbb{T}^{2n}$ is a quotient of $\mathbb{R}^{2n}$. For general symplectic manifolds (M, ω), one cannot even define $H^{1/2}(S^1, M)$, since functions of class $H^{1/2}$ on S^1 need not be continuous ($H^{1/2}$ is a "Sobolev borderline case" in dimension one).

One of the novelties in Floer's approach was to refrain from viewing the gradient flow as an ODE in a Banach space setting, but instead to write down a formal version of the gradient flow equation and regard it as an elliptic PDE. To this end, let us regard $C^\infty_{\text{contr}}(S^1, M)$ formally as a manifold with tangent spaces

$$T_\gamma C^\infty_{\text{contr}}(S^1, M) := \Gamma(\gamma^* TM),$$

choose a formal Riemannian metric on this manifold (i.e. a smoothly varying family of L^2-inner products on the spaces $\Gamma(\gamma^* TM)$) and write down the resulting equation for the negative gradient flow. A suitable Riemannian metric can be defined by

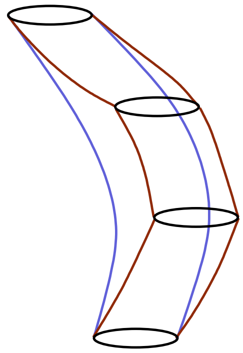

FIGURE 1.2. A family of smooth Floer trajectories can degenerate into a broken Floer trajectory.

choosing a smooth S^1-parametrized family of compatible almost complex structures

$$\{J_t \in \mathcal{J}(M,\omega)\}_{t \in S^1},$$

abbreviated in the following as $\{J_t\}$, and setting

$$\langle \xi, \eta \rangle_{L^2} := \int_{S^1} \omega(\xi(t), J_t \eta(t))\, dt$$

for $\xi, \eta \in \Gamma(\gamma^* TM)$. Exercise 1.2.2 then yields the formula

$$d\mathcal{A}_H(\gamma)\eta = \langle J_t(\dot{\gamma} - X_t(\gamma)), \eta \rangle_{L^2},$$

so that it seems reasonable to define the so-called *unregularized* gradient of $\mathcal{A}_H$ by

$$\nabla \mathcal{A}_H(\gamma) := J_t(\dot{\gamma} - X_t(\gamma)) \in \Gamma(\gamma^* TM). \tag{1.3}$$

Let us also think of a path $u : \mathbb{R} \to C^\infty_{\text{contr}}(S^1, M)$ as a map $u : \mathbb{R} \times S^1 \to M$, writing $u(s,t) := u(s)(t)$. The negative gradient flow equation $\partial_s u + \nabla \mathcal{A}_H(u(s)) = 0$ then becomes the elliptic PDE

$$\partial_s u + J_t(u)\left(\partial_t u - X_t(u)\right) = 0. \tag{1.4}$$

This is called the **Floer equation**, and its solutions are often called **Floer trajectories**. The relevance of Floer homology to our previous discussion of pseudoholomorphic curves should now be obvious. Indeed, the resemblance of the Floer equation to the nonlinear Cauchy-Riemann equation is not merely superficial—we will see in Chapter 6 that the former can always be viewed as a special case of the latter. In any case, one can use the same set of analytical techniques for both: elliptic regularity theory implies that Floer trajectories are always smooth, Fredholm theory and the implicit function theorem imply that (under appropriate assumptions) they form smooth finite-dimensional moduli spaces. Most importantly, the same "bubbling off" analysis that underlies Gromov's compactness theorem can be used to prove that spaces of Floer trajectories are compact up to "breaking", just as in Morse homology (see Figure 1.2)—this is the main reason for the relation $\partial^2 = 0$ in Floer homology.

We should mention one complication that does not arise either in the study of closed holomorphic curves or in finite-dimensional Morse theory. Since the gradient flow in Morse homology takes place on a closed manifold, it is obvious that every gradient flow line asymptotically approaches critical points at both $-\infty$ and $+\infty$. The following example shows that in the infinite-dimensional setting of Floer theory, this is no longer true.

EXAMPLE 1.2.4. Consider the Floer equation on $M := S^2 = \mathbb{C} \cup \{\infty\}$ with $H := 0$ and J_t defined as the standard complex structure i for every t. Then the orbits of X_t are all constant, and a map $u : \mathbb{R} \times S^1 \to S^2$ satisfies the Floer equation if and only if it is holomorphic. Identifying $\mathbb{R} \times S^1$ with $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$ via the biholomorphic map $(s,t) \mapsto e^{2\pi(s+it)}$, a solution u approaches periodic orbits as $s \to \pm\infty$ if and only if the corresponding holomorphic map $\mathbb{C}^* \to S^2$ extends continuously (and therefore holomorphically) over 0 and ∞. But this is not true for every holomorphic map $\mathbb{C}^* \to S^2$, e.g. take any entire function $\mathbb{C} \to \mathbb{C}$ that has an essential singularity at ∞.

EXERCISE 1.2.5. Show that in the above example with an essential singularity at ∞, the symplectic action $\mathcal{A}_H(u(s,\cdot))$ is unbounded as $s \to \infty$.

EXERCISE 1.2.6. Suppose $u : \mathbb{R} \times S^1 \to M$ is a solution to the Floer equation with $\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm$ uniformly for a pair of 1-periodic orbits $\gamma_\pm \in \mathrm{Crit}(\mathcal{A}_H)$. Show that

(1.5)
$$\mathcal{A}(\gamma_-) - \mathcal{A}(\gamma_+) = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, \partial_t u - X_t(u))\, ds\, dt = \int_{\mathbb{R}\times S^1} \omega(\partial_s u, J_t(u)\partial_s u)\, ds\, dt.$$

The right hand side of (1.5) is manifestly nonnegative since J_t is compatible with ω, and it is strictly positive unless $\gamma_- = \gamma_+$. It is therefore sensible to call this expression the **energy** $E(u)$ of a Floer trajectory. The following converse of Exercise 1.2.6 plays a crucial role in the compactness theory for Floer trajectories, as it guarantees that all the "levels" in a broken Floer trajectory are asymptotically well behaved. We will prove a variant of this result in the SFT context (see Prop. 1.3.12 below) in Chapter 7.

PROPOSITION 1.2.7. *If $u : \mathbb{R} \times S^1 \to M$ is a Floer trajectory with $E(u) < \infty$ and all 1-periodic orbits of X_t are nondegenerate, then there exist orbits $\gamma_-, \gamma_+ \in$ $\mathrm{Crit}(\mathcal{A}_H)$ such that $\lim_{s\to\pm\infty} u(s,\cdot) = \gamma_\pm$ uniformly.*

REMARK 1.2.8. It should be emphasized again that we have assumed $[\omega]|_{\pi_2(M)} =$ 0 throughout this discussion. Floer homology can also be defined under more general assumptions, but several details become more complicated.

For nice comprehensive treatments of Hamiltonian Floer homology—unfortunately not always with the same sign conventions as used here—see [**Sal99**, **AD14**]. Note that this is only one of a few "Floer homologies" that were introduced by Floer in the late 80's: the others include *Lagrangian intersection Floer homology* [**Flo88a**] (which has since evolved into the *Fukaya category*, see [**Sei08**, **FOOO09**]), and *instanton homology* [**Flo88c**], an extension of Donaldson's gauge-theoretic smooth 4-manifold invariants to dimension three. The development of new Floer-type theories has since become a major industry; see [**AS19**] for a survey.

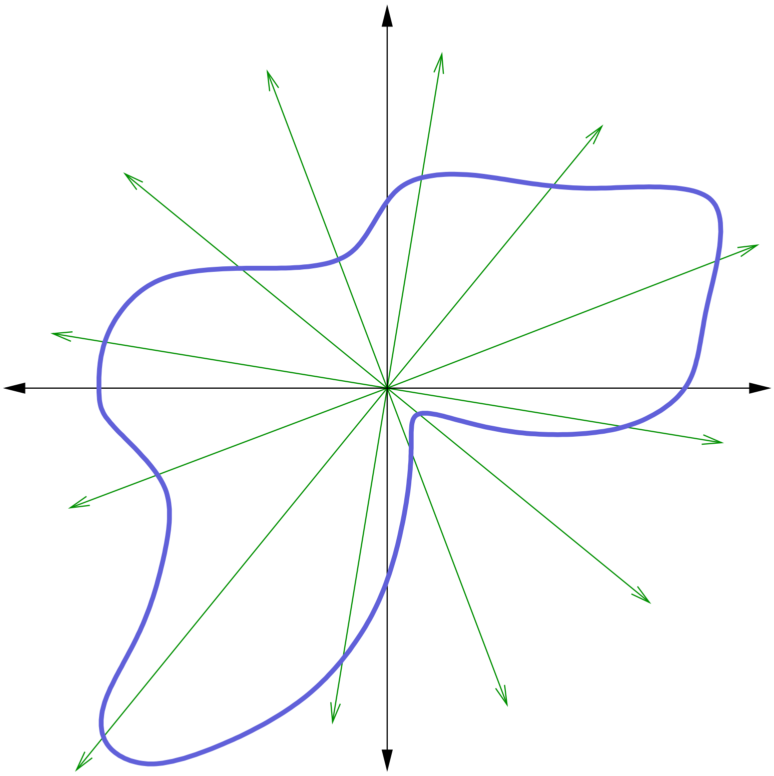

FIGURE 1.3. A star-shaped hypersurface in Euclidean space

1.3. Contact manifolds and the Weinstein conjecture

A Hamiltonian system on a symplectic manifold (W, ω) is called **autonomous** if the Hamiltonian $H : W \to \mathbb{R}$ does not depend on time. In this case, the Hamiltonian vector field X_H defined by

$$\omega(X_H, \cdot) = -dH$$

is time-independent and its orbits are confined to level sets of H. The images of these orbits on a given regular level set $H^{-1}(c)$ depend on the geometry of $H^{-1}(c)$, but not on H itself, as they are the integral curves (also known as **characteristics**) of the **characteristic line field** on $H^{-1}(c)$, defined as the unique direction spanned by a vector X such that $\omega(X, Y) = 0$ for all Y tangent to $H^{-1}(c)$. In 1978, Weinstein [**Wei78**] and Rabinowitz [**Rab78**] proved that certain kinds of regular level sets in symplectic manifolds are guaranteed to admit closed characteristics, hence implying the existence of periodic Hamiltonian orbits. In particular, this is true whenever $H^{-1}(c)$ is a *star-shaped* hypersurface in the standard symplectic $\mathbb{R}^{2n}$ (see Figure 1.3).

The following symplectic interpretation of the star-shaped condition provides both an intuitive reason to believe Rabinowitz's existence result and motivation for the more general conjecture of Weinstein. In any symplectic manifold (W, ω), a **Liouville vector field** is a smooth vector field V that satisfies

$$\mathcal{L}_V \omega = \omega.$$

By Cartan's formula for the Lie derivative, the 1-form λ defined by $\lambda := \omega(V, \cdot)$ satisfies $d\lambda = \omega$ if and only if V is a Liouville vector field; moreover, λ then also satisfies $\mathcal{L}_V \lambda = \lambda$, and it is referred to as a **Liouville form**. We sometimes say in this situation that the Liouville form λ and Liouville vector field V are **ω-dual** to each other. A hypersurface $M \subset (W, \omega)$ is said to be of **contact type** if it is transverse to a Liouville vector field defined on a neighborhood of M.

EXAMPLE 1.3.1. Using coordinates $(q_1, p_1, \dots, q_n, p_n)$ on $\mathbb{R}^{2n}$, the standard symplectic form is written as

$$\omega_{\text{std}} := \sum_{j=1}^{n} dp_j \wedge dq_j,$$

and the Liouville form $\lambda_{\text{std}} := \frac{1}{2}\sum_{j=1}^{n}(p_j\, dq_j - q_j\, dp_j)$ is dual to the radial Liouville vector field

$$V_{\text{std}} := \frac{1}{2}\sum_{j=1}^{n}\left(p_j \frac{\partial}{\partial p_j} + q_j \frac{\partial}{\partial q_j}\right).$$

Any star-shaped hypersurface is therefore of contact type.

EXERCISE 1.3.2. Suppose (W, ω) is a symplectic manifold of dimension $2n$, $M \subset W$ is a smoothly embedded and oriented hypersurface, V is a Liouville vector field defined near M and $\lambda := \omega(V, \cdot)$ is the dual Liouville form. Define a 1-form on M by $\alpha := \lambda|_{TM}$.

(a) Show that V is positively transverse to M if and only if α satisfies

$$\alpha \wedge (d\alpha)^{n-1} > 0. \tag{1.6}$$

(b) If V is positively transverse to M, choose $\epsilon > 0$ sufficiently small and consider the embedding

$$\Phi : (-\epsilon, \epsilon) \times M \hookrightarrow W : (r, x) \mapsto \varphi_V^r(x),$$

where φ_V^t denotes the time t flow of V. Show that

$$\Phi^*\lambda = e^r \alpha,$$

where we are abusing notation on the right hand side by identifying $\alpha \in \Omega^1(M)$ with its pullback under the projection $(-\epsilon, \epsilon) \times M \to M$. In particular, we obtain from this the formula $\Phi^*\omega = d(e^r \alpha)$.

The above exercise presents any contact-type hypersurface $M \subset (W, \omega)$ as one member of a smooth 1-parameter family of contact-type hypersurfaces $M_r := \varphi_V^r(M) \subset W$, each canonically identified with M such that $\omega|_{TM_r} = e^r\, d\alpha$. In particular, the characteristic line fields on M_r are the same for all r, thus the existence of a closed characteristic on any of these implies that there also exists one on M. This observation has sometimes been used to prove such existence theorems, e.g. it is used in [**HZ94**, Chapter 4] to reduce Rabinowitz's result to an "almost existence" theorem based on symplectic capacities. This discussion hopefully makes the following conjecture seem believable.

CONJECTURE 1.3.3 (Weinstein conjecture, symplectic version). *Any closed contact-type hypersurface in a symplectic manifold admits a closed characteristic.*

Weinstein's conjecture admits a natural rephrasing in the language of contact geometry. A 1-form α on an oriented $(2n-1)$-dimensional manifold M is called a (positive) **contact form** if it satisfies (1.6), and the resulting co-oriented hyperplane field

$$\xi := \ker \alpha \subset TM$$

is then called a (positive and co-oriented) **contact structure**.[4] We call the pair (M, ξ) a **contact manifold**, and refer to a diffeomorphism $\varphi : M \to M'$ as a **contactomorphism** from (M, ξ) to (M', ξ') if φ_* maps ξ to ξ' and also preserves the respective co-orientations. Equivalently, if ξ and ξ' are defined via contact forms α and α' respectively, this means

$$\varphi^*\alpha' = f\alpha \quad \text{for some } f \in C^\infty(M, (0, \infty)).$$

Contact topology studies the category of contact manifolds (M, ξ) up to contactomorphism. The following basic result provides one good reason to regard ξ rather than α as the geometrically meaningful data, as the result holds for contact *structures*, but not for contact *forms*.

THEOREM 1.3.4 (Gray's stability theorem). *If M is a closed $(2n-1)$-dimensional manifold and $\{\xi_t\}_{t \in [0,1]}$ is a smooth 1-parameter family of contact structures on M, then there exists a smooth 1-parameter family of diffeomorphisms $\{\varphi_t\}_{t \in [0,1]}$ such that $\varphi_0 = \mathrm{Id}$ and $(\varphi_t)_*\xi_0 = \xi_t$.*

PROOF. See [Gei08, §2.2] or [Wenb, Theorem 1.6.12]. □

A corollary is that while the contact form α induced on a contact-type hypersurface $M \subset (W, \omega)$ via Exercise 1.3.2 is not unique, its induced contact structure is unique up to isotopy. Indeed, the space of all Liouville vector fields transverse to M is very large (e.g. one can add to V any sufficiently small Hamiltonian vector field), but it is *convex*, hence any two choices of the induced contact form α on M are connected by a smooth 1-parameter family of contact forms, implying an isotopy of contact structures via Gray's theorem.

EXERCISE 1.3.5. If α is a nowhere zero 1-form on M and $\xi = \ker \alpha$, show that α is contact if and only if $d\alpha|_\xi$ defines a symplectic vector bundle structure on $\xi \to M$. Moreover, the orientation of ξ determined by this symplectic bundle structure is compatible with the co-orientation determined by α and the orientation of M for which $\alpha \wedge (d\alpha)^{n-1} > 0$.

The following definition is based on the fact that since $d\alpha|_\xi$ is nondegenerate when α is contact, $\ker d\alpha \subset TM$ is always 1-dimensional and transverse to ξ.

DEFINITION 1.3.6. Given a contact form α on M, the **Reeb vector field** is the unique vector field R_α that satisfies

$$d\alpha(R_\alpha, \cdot) \equiv 0, \quad \text{and} \quad \alpha(R_\alpha) \equiv 1.$$

[4]The adjective "positive" refers to the fact that the orientation of M agrees with the one determined by the volume form $\alpha \wedge (d\alpha)^{n-1}$; we call α a *negative* contact form if these two orientations disagree. It is also possible in general to define contact structures without co-orientations, but contact structures of this type will never appear in this book; for our purposes, the co-orientation is *always* considered to be part of the data of a contact structure.

EXERCISE 1.3.7. Show that the flow of any Reeb vector field R_α preserves both $\xi = \ker \alpha$ and the symplectic vector bundle structure $d\alpha|_\xi$.

CONJECTURE 1.3.8 (Weinstein conjecture, contact version). *On any closed contact manifold (M, ξ) with contact form α, the Reeb vector field R_α admits a periodic orbit.*

To see that this is equivalent to the symplectic version of the conjecture, observe that any contact manifold $(M, \xi = \ker \alpha)$ can be viewed as the contact-type hypersurface $\{0\} \times M$ in the open symplectic manifold

$$(\mathbb{R} \times M, d(e^r \alpha)),$$

called the **symplectization** of (M, ξ). Here, as in Exercise 1.3.2, we are abusing notation and identifying $\alpha \in \Omega^1(M)$ with its pullback to $\mathbb{R} \times M$ via the projection $\mathbb{R} \times M \to M$.

EXERCISE 1.3.9. Recall that for any smooth manifold M, the cotangent bundle T^*M carries a tautological 1-form $\lambda_{\text{std}} \in \Omega^1(T^*M)$ that locally takes the form $\lambda_{\text{std}} = \sum_{j=1}^n p_j \, dq_j$ in any choice of local coordinates $(q_1, \ldots, q_n)$ on a neighbood $\mathcal{U} \subset M$, with $(p_1, \ldots, p_n)$ denoting the induced coordinates on the cotangent fibers over $\mathcal{U}$. (We will discuss cotangent bundles in somewhat more detail in §3.8.) This defines a Liouville form, with $d\lambda_{\text{std}}$ defining the canonical symplectic structure of T^*M. Now if $\xi \subset TM$ is a co-oriented hyperplane field on M, consider the submanifold

$$S_\xi M := \left\{ p \in T^*M \;\middle|\; \ker p = \xi \text{ and } p(X) > 0 \;\forall X \in TM \text{ pos. transverse to } \xi \right\}.$$

Show that ξ is contact if and only if $S_\xi M$ is a symplectic submanifold of $(T^*M, d\lambda_{\text{std}})$, and the Liouville vector field on T^*M dual to λ_{std} is tangent to $S_\xi M$. Moreover, if ξ is contact, then any choice of contact form for ξ determines a diffeomorphism of $S_\xi M$ to $\mathbb{R} \times M$ identifying the Liouville form λ_{std} along $S_\xi M$ with $e^r \alpha$.

REMARK 1.3.10. Exercise 1.3.9 shows that up to symplectomorphism, our definition of the symplectization of (M, ξ) above actually depends only on ξ and not on α.

In 1993, Hofer [**Hof93**] introduced a new approach to the Weinstein conjecture that was based in part on ideas of Gromov and Floer. Fix a contact manifold (M, ξ) with contact form α, and let

$$\mathcal{J}(\alpha) \subset \mathcal{J}(\mathbb{R} \times M)$$

denote the nonempty and contractible space of all almost complex structures J on $\mathbb{R} \times M$ satisfying the following conditions:

(1) The natural translation action on $\mathbb{R} \times M$ preserves J;
(2) $J\partial_r = R_\alpha$ and $JR_\alpha = -\partial_r$, where r denotes the canonical coordinate on the $\mathbb{R}$-factor in $\mathbb{R} \times M$;
(3) $J\xi = \xi$ and $d\alpha(\cdot, J\cdot)|_\xi$ defines a bundle metric on ξ.

It is easy to check that any $J \in \mathcal{J}(\alpha)$ is compatible with the symplectic structure $d(e^r\alpha)$ on $\mathbb{R} \times M$. Moreover, if $\gamma : \mathbb{R} \to M$ is any periodic orbit of R_α with period

$T > 0$, then for any $J \in \mathcal{J}(\alpha)$, the so-called **trivial cylinder**

$$u : \mathbb{R} \times S^1 \to \mathbb{R} \times M : (s,t) \mapsto (Ts, \gamma(Tt))$$

is a J-holomorphic curve. Following Floer, one version of Hofer's idea would be to look for J-holomorphic cylinders that satisfy a finite energy condition as in Prop. 1.2.7, forcing them to approach trivial cylinders asymptotically—the existence of such a cylinder would then imply the existence of a closed Reeb orbit, and thus prove the Weinstein conjecture. The first hindrance is that the "obvious" definition of energy in this context,

$$\int_{\mathbb{R}\times S^1} u^* d(e^r \alpha),$$

is not very useful: this integral is infinite if u is a trivial cylinder. To circumvent this, notice that every $J \in \mathcal{J}(\alpha)$ is also compatible with any symplectic structure of the form

$$\omega_\varphi := d(e^{\varphi(r)}\alpha),$$

where φ is a function chosen freely from the set

$$\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; \varphi' > 0 \right\}. \tag{1.7}$$

Essentially, choosing ω_φ means identifying $\mathbb{R} \times M$ with a subset of the bounded region $(-1,1) \times M$, in which trivial cylinders have finite symplectic area. Since there is no preferred choice for the function φ, we define the **Hofer energy**[5] of a J-holomorphic curve $u : \Sigma \to \mathbb{R} \times M$ by

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_\Sigma u^* \omega_\varphi. \tag{1.8}$$

This has the desired property of being finite for trivial cylinders, and it is also nonnegative, with strict positivity whenever u is not constant.

Another useful observation from [**Hof93**] was that if the goal is to find periodic orbits, then we need not restrict our attention to J-holomorphic *cylinders* in particular. One can more generally consider curves defined on an arbitrary *punctured* Riemann surface

$$\dot{\Sigma} := \Sigma \backslash \Gamma,$$

where (Σ, j) is a closed connected Riemann surface and $\Gamma \subset \Sigma$ is a finite set of punctures. For any $\zeta \in \Gamma$, one can find coordinates identifying some punctured neighborhood of ζ biholomorphically with the closed punctured disk

$$\dot{\mathbb{D}} := \mathbb{D} \backslash \{0\} \subset \mathbb{C},$$

and then identify this with either the positive or negative half-cylinder

$$Z_+ := [0, \infty) \times S^1, \qquad Z_- := (-\infty, 0] \times S^1$$

via the biholomorphic maps

$$Z_+ \to \dot{\mathbb{D}} : (s,t) \mapsto e^{-2\pi(s+it)}, \qquad Z_- \to \dot{\mathbb{D}} : (s,t) \mapsto e^{2\pi(s+it)}.$$

[5] Strictly speaking, the energy defined in (1.8) is not identical to the notion introduced in [**Hof93**] and used in many of Hofer's papers, but it is equivalent to it in the sense that uniform bounds on either notion of energy imply uniform bounds on the other.

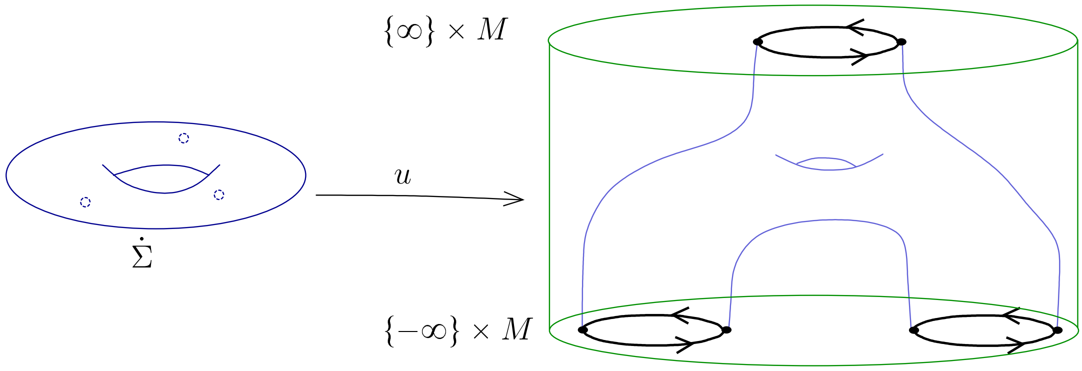

FIGURE 1.4. An asymptotically cylindrical holomorphic curve in a symplectization, with genus 1, one positive puncture and two negative punctures.

We will refer to such a choice as a (positive or negative) **holomorphic cylindrical coordinate** system near ζ, and in this way, we can present $(\dot{\Sigma}, j)$ as a *Riemann surface with cylindrical ends*, i.e. the union of some compact Riemann surface with boundary with a finite collection of half-cylinders $Z_\pm$ on which j takes the standard form $j\partial_s = \partial_t$. Note that the standard cylinder $\mathbb{R} \times S^1$ is a special case of this, as it can be identified biholomorphically with $S^2 \backslash \{0, \infty\}$. Another important special case is the plane, $\mathbb{C} = S^2 \backslash \{\infty\}$.

If $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve and $\zeta \in \Gamma$ is one of its punctures, we will say that u is **positively/negatively asymptotic** to a T-periodic Reeb orbit $\gamma : \mathbb{R} \to M$ at ζ if one can choose holomorphic cylindrical coordinates $(s, t) \in Z_\pm$ near ζ such that

$$u(s, t) = \exp_{(Ts, \gamma(Tt))} h(s, t) \quad \text{for } |s| \text{ sufficiently large,}$$

where $h(s, t)$ is a vector field along the trivial cylinder satisfying $h(s, \cdot) \to 0$ uniformly as $|s| \to \infty$, and the exponential map is defined with respect to any $\mathbb{R}$-invariant choice of Riemannian metric on $\mathbb{R} \times M$. We say that $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is **asymptotically cylindrical** if it is (positively or negatively) asymptotic to some closed Reeb orbit at each of its punctures. Note that this partitions the finite set of punctures $\Gamma \subset \Sigma$ into two subsets,

$$\Gamma = \Gamma^+ \cup \Gamma^-,$$

the *positive* and *negative* punctures respectively, see Figure 1.4.

EXERCISE 1.3.11. Suppose $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is an asymptotically cylindrical J-holomorphic curve, with the asymptotic orbit at each puncture $\zeta \in \Gamma^\pm$ denoted by γ_ζ, having period $T_\zeta > 0$. Show that

$$\sum_{\zeta \in \Gamma^+} T_\zeta - \sum_{\zeta \in \Gamma^-} T_\zeta = \int_{\dot{\Sigma}} u^* d\alpha \geqslant 0,$$

with equality if and only if the image of u is contained in that of a trivial cylinder. In particular, u must have at least one positive puncture unless it is constant. Show also that $E(u)$ is finite and satisfies an upper bound determined only by the periods of the positive asymptotic orbits.

The following analogue of Prop. 1.2.7 will be proved in Chapter 7. For simplicity, we shall state a weakened version of what Hofer proved in [**Hof93**], which did not require any nondegeneracy assumption. A T-periodic Reeb orbit $\gamma : \mathbb{R} \to M$ is called **nondegenerate** if the Reeb flow φ_α^t has the property that its linearization along the contact bundle (cf. Exercise 1.3.7),

$$d\varphi_\alpha^T(\gamma(0))|_{\xi_{\gamma(0)}} : \xi_{\gamma(0)} \to \xi_{\gamma(0)}$$

does not have 1 as an eigenvalue. Note that since R_α is not time-dependent, closed Reeb orbits are never completely isolated—they always exist in S^1-parametrized families—but these families are isolated in the nondegenerate case. A **nondegenerate contact form** is one for which every closed Reeb orbit is nondegenerate. One can show that this condition is generic, meaning for instance that on any closed manifold, the nondegenerate contact forms constitute a C^∞-dense subset of the space of all contact forms (see Remark 1.3.13 below). The following result is the contact analogue of Proposition 1.2.7.

PROPOSITION 1.3.12. *Suppose (M, ξ) is a closed contact manifold with a nondegenerate contact form α. If $u : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, J)$ is a J-holomorphic curve with $E(u) < \infty$ on a punctured Riemann surface such that none of the punctures are removable, then u is asymptotically cylindrical.*

The main results in [**Hof93**] state that under certain assumptions on a closed contact 3-manifold (M, ξ), namely if either ξ is *overtwisted* (as defined in [**Eli89**]) or $\pi_2(M) \neq 0$, one can find for any contact form α on (M, ξ) and any $J \in \mathcal{J}(\alpha)$ a finite-energy J-holomorphic plane. By Proposition 1.3.12, this implies the existence of a contractible periodic Reeb orbit and thus proves the Weinstein conjecture in these settings.

REMARK 1.3.13. The standard genericity result mentioned above for nondegenerate contact forms can be proved in various ways, e.g. it follows from a slightly more general result about generic regular level sets in Hamiltonian systems proved in [**Rob70**]. A more direct proof via the Sard-Smale theorem that is similar in spirit to the transversality arguments in Chapter 9 may be found in the appendix of [**ABW10**].

1.4. Symplectic cobordisms and their completions

After the developments described in the previous three sections, it seemed natural that one might define invariants of contact manifolds via a Floer-type theory generated by closed Reeb orbits and counting asymptotically cylindrical holomorphic curves in symplectizations. This theory is what is now called SFT, and its basic structure was outlined in a paper by Eliashberg, Givental and Hofer [**EGH00**] in 2000, though some of its analytical foundations remain unfinished as of 2026. The term "field theory" is an allusion to "topological quantum field theories," which associate vector spaces to certain geometric objects and morphisms to cobordisms between those objects. Thus in order to place SFT in its proper setting, we need to introduce symplectic cobordisms between contact manifolds.

Recall that if M_+ and M_- are smooth oriented closed manifolds of the same dimension, an oriented cobordism from M_- to M_+ is a compact smooth oriented manifold W with oriented boundary

$$\partial W \cong -M_- \coprod M_+,$$

where the symbol "$\cong$" in this setting means orientation-preserving diffeomorphism, and $-M_-$ denotes M_- with its orientation reversed. Given positive contact structures $\xi_\pm$ on $M_\pm$, we say that a symplectic manifold (W, ω) is a **symplectic cobordism from** (M_-, ξ_-) **to** (M_+, ξ_+) if W is an oriented cobordism[6] from M_- to M_+ such that both components of ∂W are contact-type hypersurfaces with induced contact structures isotopic to $\xi_\pm$. Note that our chosen orientation conventions imply that the Liouville vector field chosen near ∂W must point *outward* at M_+ and *inward* at M_-; we say in this case that M_+ is a symplectically **convex** boundary component, while M_- is symplectically **concave**. As important special cases, (W, ω) is a **symplectic filling** of (M_+, ξ_+) if $M_- = \emptyset$, and it is a **symplectic cap** of (M_-, ξ_-) if $M_+ = \emptyset$. In the literature, fillings and caps are sometimes also referred to as *convex fillings* or *concave fillings* respectively.

The contact-type condition implies the existence of a Liouville form λ near ∂W with $d\lambda = \omega$, such that by Exercise 1.3.2, neighborhoods of M_+ and M_- in W can be identified with the collars (see Figure 1.5)

$$(-\epsilon, 0] \times M_+ \quad \text{or} \quad [0, \epsilon) \times M_-$$

respectively for sufficiently small $\epsilon > 0$, with λ taking the form

$$\lambda = e^r \alpha_\pm,$$

where $\alpha_\pm := \lambda|_{TM_\pm}$ are contact forms for $\xi_\pm$, and r as usual denotes the canonical coordinate on the first factor in $\mathbb{R} \times M$. The **symplectic completion** of (W, ω) is the noncompact symplectic manifold $(\widehat{W}, \hat{\omega})$ defined by attaching cylindrical ends to these collar neighborhoods (Figure 1.6):

$$(\widehat{W}, \hat{\omega}) = ((-\infty, 0] \times M_-, d(e^r \alpha_-)) \cup_{M_-} (W, \omega) \cup_{M_+} ([0, \infty) \times M_+, d(e^r \alpha_+)) . \tag{1.9}$$

In this context, the symplectization $(\mathbb{R} \times M, d(e^r \alpha))$ is symplectomorphic to the completion of the **trivial symplectic cobordism** $([0,1] \times M, d(e^r \alpha))$ from $(M, \xi = \ker \alpha)$ to itself. More generally, the object in the following easy exercise can also sensibly be called a trivial symplectic cobordism:

EXERCISE 1.4.1. Suppose (M, ξ) is a closed contact manifold with contact form α, and $f_\pm : M \to \mathbb{R}$ is a pair of functions with $f_- < f_+$ everywhere. Show that the domain

$$\{(r, x) \in \mathbb{R} \times M \mid f_-(x) \leqslant r \leqslant f_+(x)\} \subset \mathbb{R} \times M$$

[6] We assume of course that W is assigned the orientation determined by its symplectic form.

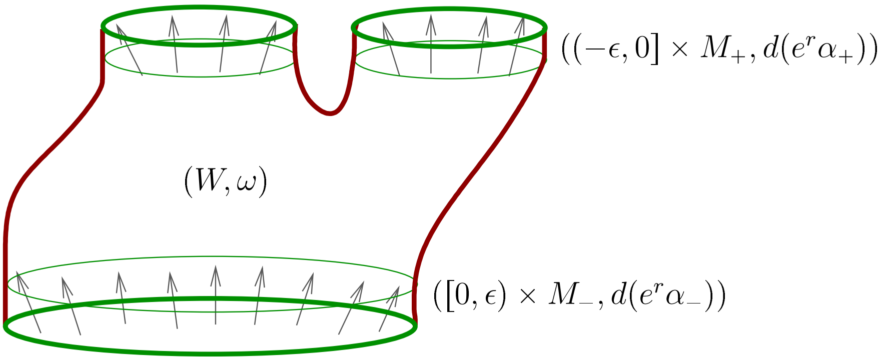

FIGURE 1.5. A symplectic cobordism with concave boundary (M_-, ξ_-) and convex boundary (M_+, ξ_+), with symplectic collar neighborhoods defined by flowing along Liouville vector fields near the boundary.

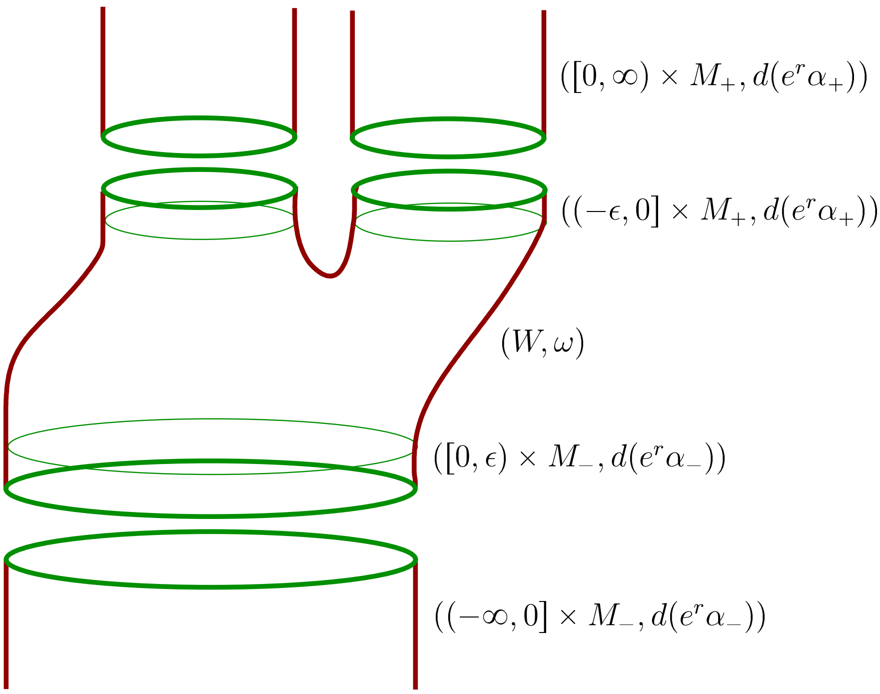

FIGURE 1.6. The completion of a symplectic cobordism

defines a symplectic cobordism from (M, ξ) to itself, with a global Liouville form $\lambda = e^r\alpha$ inducing contact forms $e^{f_-}\alpha$ and $e^{f_+}\alpha$ on its concave and convex boundaries respectively.

We say that (W, ω) is an **exact symplectic cobordism** or **Liouville cobordism** if the Liouville form λ can be extended from a neighborhood of ∂W to define a global primitive of ω on W. Equivalently, this means that ω admits a global Liouville vector field that points inward at M_- and outward at M_+. An **exact filling** of (M_+, ξ_+) is an exact cobordism whose concave boundary is empty. Observe that if (W, ω) is exact, then its completion $(\widehat{W}, \hat{\omega})$ also inherits a global Liouville form.

EXERCISE 1.4.2. Use Stokes' theorem to show that there is no such thing as an exact symplectic cap.

The above exercise hints at an important difference between cobordisms in the *symplectic* as opposed to the *oriented smooth* category: symplectic cobordisms are not generally reversible. If W is an oriented cobordism from M_- to M_+, then reversing the orientation of W produces an oriented cobordism from M_+ to M_-. But one cannot simply reverse orientations in the symplectic category, since the orientation is determined by the symplectic form. For example, many obstructions to the existence of symplectic fillings of given contact manifolds are known—some of them defined in terms of SFT—but there are no obstructions at all to symplectic caps, in fact it is known that all closed contact manifolds admit them (see [**EH02**, **CE20**, **Laz20**]).

The definitions for holomorphic curves in symplectizations in the previous section generalize to completions of symplectic cobordisms in a fairly straightforward way, since these completions look exactly like symplectizations outside of a compact subset. Define

$$\mathcal{J}(W, \omega, \alpha_+, \alpha_-) \subset \mathcal{J}(\widehat{W})$$

as the space of all almost complex structures J on $\widehat{W}$ such that

$$J|_W \in \mathcal{J}(W, \omega), \qquad J|_{[0,\infty)\times M_+} \in \mathcal{J}(\alpha_+) \quad \text{and} \quad J|_{(-\infty,0]\times M_-} \in \mathcal{J}(\alpha_-).$$

Occasionally it is useful to relax the compatibility condition on W to tameness,[7] i.e. $J|_W \in \mathcal{J}_\tau(W, \omega)$, producing a space that we shall denote by

$$\mathcal{J}_\tau(W, \omega, \alpha_+, \alpha_-) \subset \mathcal{J}(\widehat{W}).$$

As in Prop. 1.1.3, both of these spaces are nonempty and contractible. We can then consider asymptotically cylindrical J-holomorphic curves

$$u : (\dot{\Sigma} = \Sigma \backslash (\Gamma^+ \cup \Gamma^-), j) \to (\widehat{W}, J),$$

which are proper maps asymptotic to closed orbits of $R_{\alpha_\pm}$ in $M_\pm$ at punctures in $\Gamma^\pm$, see Figure 1.7.

One must again tinker with the symplectic form on $\widehat{W}$ in order to define a notion of energy that is finite when we need it to be. We generalize (1.7) as

$$\mathcal{T} := \left\{ \varphi \in C^\infty(\mathbb{R}, (-1,1)) \;\middle|\; \varphi' > 0 \text{ and } \varphi(r) = r \text{ near } r = 0 \right\},$$

[7]It seems natural to wonder whether one could not also relax the conditions on the cylindrical ends and require $J|_{\xi_\pm}$ to be tamed by $d\alpha_\pm|_{\xi_\pm}$ instead of compatible with it. I do not currently know whether this works, but in later chapters we will see some reasons to worry that it might not (see §6.7.2).

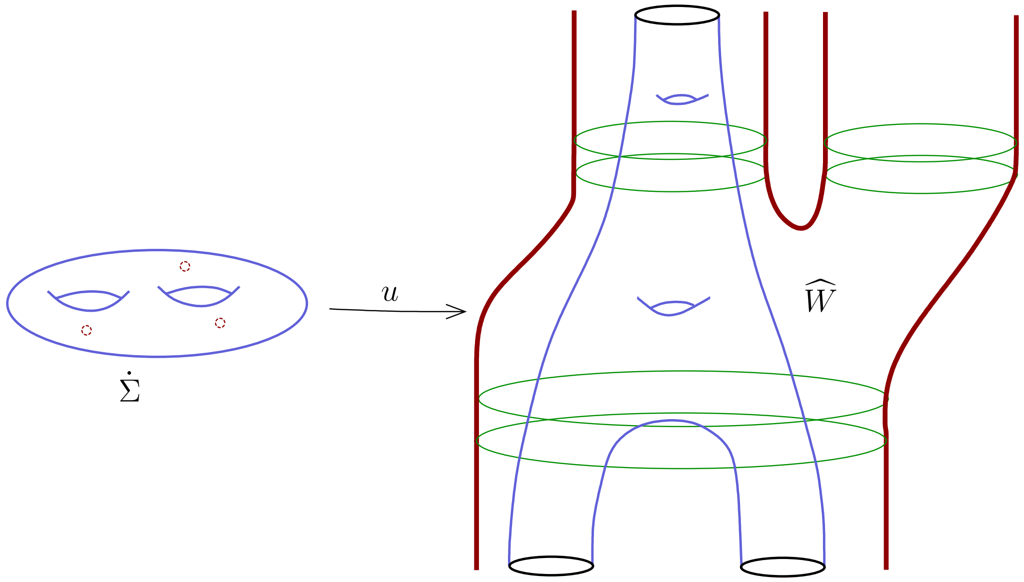

FIGURE 1.7. An asymptotically cylindrical holomorphic curve in a completed symplectic cobordism, with genus 2, one positive puncture and two negative punctures.

and associate to each $\varphi \in \mathcal{T}$ a symplectic form $\hat{\omega}_\varphi$ on $\widehat{W}$ defined by

$$\hat{\omega}_\varphi := \begin{cases} d(e^{\varphi(r)}\alpha_+) & \text{on } [0,\infty) \times M_+, \\ \omega & \text{on } W, \\ d(e^{\varphi(r)}\alpha_-) & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

One can again check that every $J \in \mathcal{J}(W, \omega, \alpha_+, \alpha_-)$ or $\mathcal{J}_\tau(W, \omega, \alpha_+, \alpha_-)$ is tamed by $\hat{\omega}_\varphi$ for every $\varphi \in \mathcal{T}$. Thus it makes sense to define the **energy** of $u : (\dot{\Sigma}, j) \to (\widehat{W}, J)$ by

$$E(u) := \sup_{\varphi \in \mathcal{T}} \int_{\dot{\Sigma}} u^* \hat{\omega}_\varphi.$$

It will be a straightforward matter to generalize Proposition 1.3.12 and show that finite energy implies asymptotically cylindrical behavior in completed cobordisms.

EXERCISE 1.4.3. Show that if (W, ω) is an exact cobordism, then every asymptotically cylindrical J-holomorphic curve in $\widehat{W}$ has at least one positive puncture.

1.5. Contact homology and SFT

We can now sketch the algebraic structure of SFT. We shall ignore or suppress several pesky details that are best dealt with later, some of them algebraic, others analytical. Due to analytical problems, some of the "theorems" that we shall (often imprecisely) state in this section are not yet provable at the current level of technology, though we expect that they will be in the foreseeable future. We shall use quotation marks to indicate this caveat wherever appropriate.

The standard versions of SFT all define homology theories with varying levels of algebraic structure which are meant to be invariants of a contact manifold (M, ξ). The chain complexes always depend on certain auxiliary choices, including a nondegenerate contact form α and a generic $J \in \mathcal{J}(\alpha)$. The generators consist of formal variables q_γ, one for each[8] closed Reeb orbit γ. In the most straightforward generalization of Hamiltonian Floer homology, the chain complex is simply a graded $\mathbb{Q}$-vector space generated by the variables q_γ, and the boundary map is defined by

$$\partial_{\mathrm{CCH}} q_\gamma = \sum_{\gamma'} \# \left(\mathcal{M}(\gamma, \gamma')/\mathbb{R} \right) q_{\gamma'},$$

where $\mathcal{M}(\gamma, \gamma')$ is the moduli space of J-holomorphic cylinders in $\mathbb{R} \times M$ with a positive puncture asymptotic to γ and a negative puncture asymptotic to γ', and the sum ranges over all orbits γ' for which this moduli space is 1-dimensional. The count $\# \left(\mathcal{M}(\gamma, \gamma')/\mathbb{R} \right)$ is rational, as it includes rational weighting factors that depend on combinatorial information and are best not discussed right now.[9]

"THEOREM" 1.5.1. *If α admits no contractible Reeb orbits, then $\partial_{\mathrm{CCH}}^2 = 0$, and the resulting homology is independent of the choices of α with this property and generic $J \in \mathcal{J}(\alpha)$.*

The invariant arising from this result is known as **cylindrical contact homology**, and it is sometimes quite easy to work with when it is well defined, though it has the disadvantage of not always being defined. Namely, the relation $\partial_{\mathrm{CCH}}^2 = 0$ can fail if α admits contractible Reeb orbits, because unlike in Floer homology, the compactification of the space of cylinders $\mathcal{M}(\gamma, \gamma')$ generally includes objects that are not broken cylinders. In fact, the objects arising in the "SFT compactification" of moduli spaces of finite-energy curves in completed cobordisms can be quite elaborate, see Figure 1.8. The combinatorics of the situation are not so bad however if the cobordism is exact, as is the case for a symplectization: Exercise 1.4.3 then prevents curves without positive ends from appearing. The only possible degenerations for cylinders then consist of broken configurations whose levels each have *exactly one positive puncture* and arbitrary negative punctures; moreover, all but one of the negative punctures must eventually be capped off by planes, which is why "Theorem" 1.5.1 holds in the absence of planes.

If planes do exist, then one can account for them by defining the chain complex as an *algebra* rather than a vector space, producing the theory known as **contact homology**. For this, the chain complex is taken to be a graded unital algebra over $\mathbb{Q}$, and we define

$$\partial_{\mathrm{CH}} q_\gamma = \sum_{(\gamma_1, \ldots, \gamma_m)} \# \left(\mathcal{M}(\gamma; \gamma_1, \ldots, \gamma_m)/\mathbb{R} \right) q_{\gamma_1} \ldots q_{\gamma_m},$$

with $\mathcal{M}(\gamma; \gamma_1, \ldots, \gamma_m)$ denoting the moduli space of punctured J-holomorphic spheres in $\mathbb{R} \times M$ with a positive puncture at γ and m negative punctures at the orbits

[8] Actually, I should be making a distinction here between "good" and "bad" Reeb orbits, but let's discuss that later; see Chapter 12.

[9] Similar combinatorial factors are hidden behind the symbol "#" in our definitions of ∂_{CH} and $\mathbf{H}$, and will be discussed in earnest in Chapter 13.

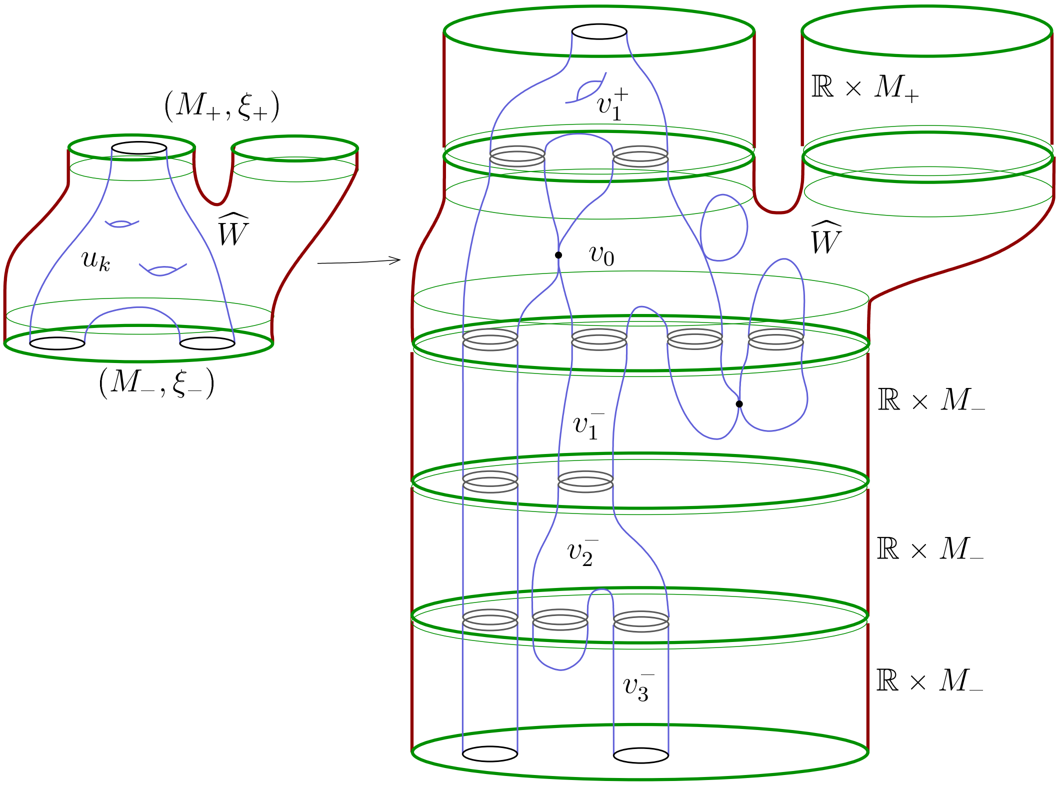

FIGURE 1.8. Degeneration of a sequence u_k of finite energy punctured holomorphic curves with genus 2, one positive puncture and two negative punctures in a symplectic cobordism. The limiting holomorphic building $(v_1^+, v_0, v_1^-, v_2^-, v_3^-)$ in this example has one upper level living in the symplectization $\mathbb{R} \times M_+$, a main level living in $\widehat{W}$, and three lower levels, each of which is a (possibly disconnected) finite-energy punctured nodal holomorphic curve in $\mathbb{R} \times M_-$. The building has arithmetic genus 2 and the same numbers of positive and negative punctures as u_k.

$\gamma_1, \ldots, \gamma_m$, and the sum ranges over all integers $m \geqslant 0$ and all m-tuples of orbits for which the moduli space is 1-dimensional. The action of ∂_{CH} is then extended to the whole algebra via a graded Leibniz rule

$$\partial_{\mathrm{CH}}(q_\gamma q_{\gamma'}) := (\partial_{\mathrm{CH}} q_\gamma)\, q_{\gamma'} + (-1)^{|\gamma|} q_\gamma\, (\partial_{\mathrm{CH}} q_{\gamma'})\,.$$

The general compactness and gluing theory for genus zero curves with one positive puncture now implies:

"THEOREM" 1.5.2. *$\partial_{\mathrm{CH}}^2 = 0$, and the resulting homology is (as a graded unital $\mathbb{Q}$-algebra) independent of the choices α and J.*

Maybe you've noticed the pattern: in order to accommodate more general classes of holomorphic curves, we need to add more algebraic structure. The **full SFT** algebra counts all rigid holomorphic curves in $\mathbb{R} \times M$, including all combinations of positive and negative punctures and all genera. Here is a brief picture of what it looks like. Counting all the 1-dimensional moduli spaces of J-holomorphic curves modulo $\mathbb{R}$-translation in $\mathbb{R} \times M$ produces a formal power series

$$\mathbf{H} := \sum \# \left(\mathcal{M}_g(\gamma_1^+, \ldots, \gamma_{m_+}^+ ; \gamma_1^-, \ldots, \gamma_{m_-}^-) \big/ \mathbb{R} \right) q_{\gamma_1^-} \ldots q_{\gamma_{m_-}^-} p_{\gamma_1^+} \ldots p_{\gamma_{m_+}^+} \hbar^{g-1},$$

where the sum ranges over all integers $g, m_+, m_- \geqslant 0$ and tuples of orbits, $\hbar$ and p_γ (one for each orbit γ) are additional formal variables, and

$$\mathcal{M}_g(\gamma_1^+, \ldots, \gamma_{m_+}^+ ; \gamma_1^-, \ldots, \gamma_{m_-}^-)$$

denotes the moduli space of J-holomorphic curves in $\mathbb{R} \times M$ with genus g, m_+ positive punctures at the orbits $\gamma_1^+, \ldots, \gamma_{m_+}^+$, and m_- negative punctures at the orbits $\gamma_1^-, \ldots, \gamma_{m_-}^+$. We can regard $\mathbf{H}$ as an operator on a graded algebra $\mathfrak{W}$ of formal power series in the variables $\{p_\gamma\}$, $\{q_\gamma\}$ and $\hbar$, equipped with a graded bracket operation that satisfies the quantum mechanical commutation relation

$$[p_\gamma, q_\gamma] = \kappa_\gamma \hbar,$$

where κ_γ is a combinatorial factor that is best ignored for now. Note that due to the signs that accompany the grading, odd elements $\mathbf{F} \in \mathfrak{W}$ need not satisfy $[\mathbf{F}, \mathbf{F}] = 0$, and $\mathbf{H}$ itself is an odd element, thus the following statement is nontrivial; in fact, it is the algebraic manifestation of the general compactness and gluing theory for punctured holomorphic curves in symplectizations.

"THEOREM" 1.5.3. $[\mathbf{H}, \mathbf{H}] = 0$, *hence by the graded Jacobi identity,* $\mathbf{H}$ *determines an operator*

$$D_{\mathrm{SFT}} : \mathfrak{W} \to \mathfrak{W} : \mathbf{F} \mapsto [\mathbf{H}, \mathbf{F}]$$

satisfying $D_{\mathrm{SFT}}^2 = 0$. *The resulting homology depends on* (M, ξ) *but not on the auxiliary choices* α *and* J.

It takes some time to understand how pictures such as Figure 1.8 translate into algebraic relations like $[\mathbf{H}, \mathbf{H}] = 0$, but this is a subject we'll come back to. There is also an intermediate theory between contact homology and full SFT, called **rational SFT**, which counts only genus zero curves with arbitrary positive and negative punctures. Algebraically, it is obtained from the full SFT algebra as a "semiclassical approximation" by discarding higher-order factors of $\hbar$, so that the commutation bracket in $\mathfrak{W}$ becomes a graded Poisson bracket. We will discuss all of this in Chapter 13.

1.6. Two applications

We briefly mention two applications that we will be able to establish rigorously using the methods developed in this book. Since SFT itself is not yet well defined in full generality, this sometimes means using SFT for *inspiration*, while proving corollaries via more direct methods.

1.6.1. Tight contact structures on $\mathbb{T}^3$. The 3-torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ with coordinates (t, θ, ϕ) admits a sequence of contact structures

$$\xi_k := \ker\left(\cos(2\pi kt)\, d\theta + \sin(2\pi kt)\, d\phi\right),$$

one for each $k \in \mathbb{N}$. These cannot be distinguished from each other by any classical invariants, e.g. they all have the same Euler class, in fact they are all homotopic as co-oriented 2-plane fields. Nonetheless:

THEOREM 1.6.1. *For $k \neq \ell$, $(\mathbb{T}^3, \xi_k)$ and $(\mathbb{T}^3, \xi_\ell)$ are not contactomorphic.*

We will be able to prove this in Chapter 11 by rigorously defining and computing cylindrical contact homology for a suitable choice of contact forms on $(\mathbb{T}^3, \xi_k)$.

1.6.2. Filling and cobordism obstructions. Consider a closed connected and oriented surface Σ presented as $\Sigma_+ \cup_\Gamma \Sigma_-$, where $\Sigma_\pm \subset \Sigma$ are each (not necessarily connected) compact surfaces with a common boundary Γ. By an old result of Lutz [Lut77], the 3-manifold $S^1 \times \Sigma$ admits a unique isotopy class of S^1-invariant contact structures ξ_Γ such that the loops $S^1 \times \{z\}$ are positively/negatively transverse to ξ_Γ for $z \in \mathring{\Sigma}_\pm$ and tangent to ξ_Γ for $z \in \Gamma$. Now for each $k \in \mathbb{N}$, define

$$(V_k, \xi_k) := (S^1 \times \Sigma, \xi_\Gamma)$$

where $\Sigma = \Sigma_+ \cup_\Gamma \Sigma_-$ is chosen such that Γ has k connected components, Σ_- is connected with genus zero, and Σ_+ is connected with positive genus (see Figure 1.9).

THEOREM 1.6.2. *The contact manifolds (V_k, ξ_k) do not admit any symplectic fillings. Moreover, if $k > \ell$, then there exists no exact symplectic cobordism from (V_k, ξ_k) to (V_ℓ, ξ_ℓ).*

For these examples, one can use explicit constructions from [Wen13, Avd21] to show that non-exact cobordisms from (V_k, ξ_k) to (V_ℓ, ξ_ℓ) do exist, and so do exact cobordisms from (V_ℓ, ξ_ℓ) to (V_k, ξ_k), thus both the directionality of the cobordism relation and the distinction between exact and non-exact are crucial. The proof of the theorem, due to the author with Latschev and Hutchings [LW11], uses a numerical contact invariant based on the full SFT algebra—in particular, the curves that cause this phenomenon have multiple positive ends and are thus not seen by contact homology. We will introduce the relevant numerical invariant in Chapter 14 and compute it for these examples in Chapter 17.

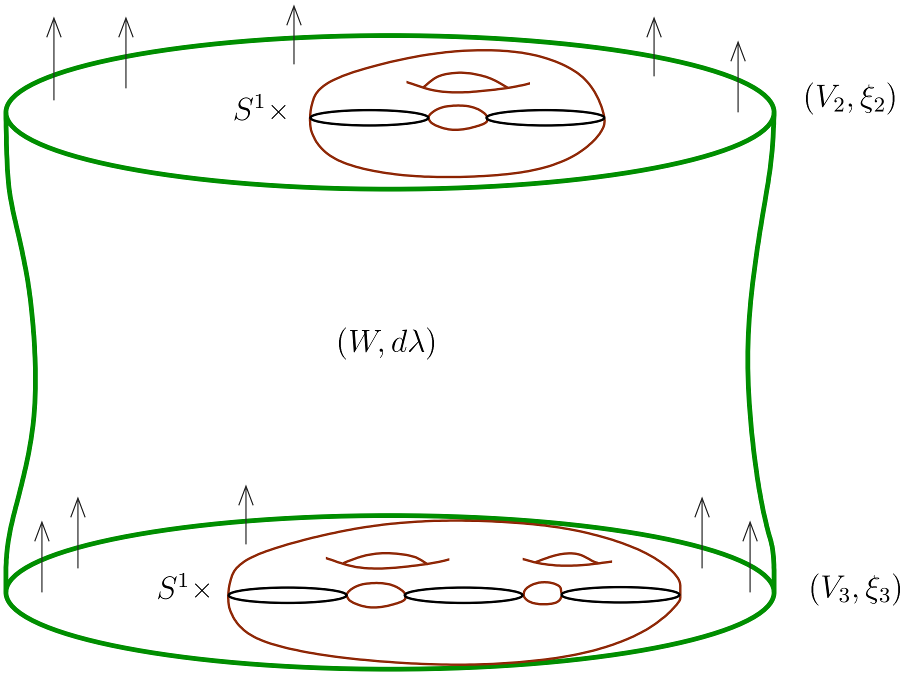

Figure 1.9. This exact symplectic cobordism does not exist.

CHAPTER 2

Basics on holomorphic curves

Contents

In this chapter we begin studying the analysis of J-holomorphic curves. The coverage will necessarily be a bit sparse in some places, but more detailed proofs of most things in this chapter can be found in [**Wenb**].

2.1. Linearized Cauchy-Riemann operators

In order to motivate the study of linear Cauchy-Riemann type operators, we begin with a formal discussion of the nonlinear Cauchy-Riemann equation and its linearization.

Fix a Riemann surface (Σ, j) and almost complex manifold (W, J). The nonlinear Cauchy-Riemann equation for maps $u : \Sigma \to W$ then takes the form

$$Tu \circ j = J \circ Tu,$$

which in any choice of local holomorphic coordinates (s, t) on suitably small neighborhoods in Σ is equivalent to

$$\partial_s u + J(u)\partial_t u = 0,$$

where we've explicitly written the dependence of $J : T_{u(z)}W \to T_{u(z)}W$ on $u(z)$ at each point $z \in \Sigma$ in order to emphasize the nonlinearity of the equation. The linearized equation at a given solution $u : \Sigma \to W$ is obtained by considering a smooth 1-parameter family of solutions $u_\rho : \Sigma \to W$ for $\rho \in (-\epsilon, \epsilon)$, with $u_0 = u$. Writing $\partial_\rho u_\rho|_{\rho=0} = \eta \in \Gamma(u^*TW)$, choosing a connection ∇ on W and taking the covariant derivative of the nonlinear equation with respect to the parameter gives

$$0 = \nabla_\rho \left[\partial_s u_\rho + J(u_\rho)\partial_t u_\rho\right]\big|_{\rho=0} = \nabla_\rho \partial_s u_\rho\big|_{\rho=0} + J(u)\nabla_\rho \partial_t u_\rho\big|_{\rho=0} + (\nabla_\eta J)\partial_t u.$$

Note that since $\partial_s u + J(u)\partial_t u = 0$, the expression on the right does not depend on the choice of connection. In particular, if we choose ∇ to be symmetric, then we can replace $\nabla_\rho \partial_s$ and $\nabla_\rho \partial_t$ with $\nabla_s \partial_\rho$ and $\nabla_t \partial_\rho$ respectively, so that the linearized equation takes the more appealing form

$$\nabla_s \eta + J(u)\nabla_t \eta + (\nabla_\eta J)\partial_t u = 0,$$

or in coordinate-free terms,

$$\nabla \eta + J(u)\nabla \eta \circ j + (\nabla_\eta J) \circ Tu \circ j = 0.$$

This is a globally well-defined linear first-order PDE for sections $\eta \in \Gamma(u^*TW)$. We will often abbreviate it in the form $\mathbf{D}_u \eta = 0$, defining the so-called **linearized Cauchy-Riemann operator at** u by

$$\begin{aligned} \mathbf{D}_u : \Gamma(u^*TW) &\to \Omega^{0,1}(\Sigma, u^*TW) \\ \eta &\mapsto \nabla \eta + J(u)\nabla \eta \circ j + (\nabla_\eta J) \circ Tu \circ j. \end{aligned} \tag{2.1}$$

Here we have used a bit of standard notation from complex geometry: $\Omega^{0,1}(\Sigma, u^*TW)$ denotes the space of u^*TW-valued $(0,1)$-forms on Σ, where "$(0,1)$" means 1-forms that are complex-*antilinear*.[1] Equivalently, elements of $\Omega^{0,1}(\Sigma, u^*TW)$ are smooth sections of $\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW) = T^{0,1}\Sigma \otimes_{\mathbb{C}} u^*TW$, where $T^{0,1}\Sigma$ denotes the $(0,1)$-part of the complexified cotangent bundle.[2]

The linearized Cauchy-Riemann operator arises in the following application. Suppose we wish to understand the structure of some space of the form

$$\{u : \Sigma \to W \mid Tu \circ j = J \circ Tu \text{ plus further conditions}\}, \tag{2.2}$$

where the "further conditions" (which we will for now leave unspecified) may impose constraints on e.g. the regularity of u, as well as its boundary and/or asymptotic behavior. The standard approach in global analysis can be summarized as follows:

Step 1: Construct a smooth Banach manifold $\mathcal{B}$ of maps $u : \Sigma \to W$ such that all the solutions we're interested in will be elements of $\mathcal{B}$. The tangent spaces $T_u\mathcal{B}$ are then Banach spaces of sections of u^*TW.

Step 2: Construct a smooth Banach space bundle $\mathcal{E} \to \mathcal{B}$ such that for each $u \in \mathcal{B}$, the fiber $\mathcal{E}_u$ is a Banach space of sections of the vector bundle

$$\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW) \to \Sigma$$

of complex-antilinear bundle maps $(T\Sigma, j) \to (u^*TW, J)$. Since our purpose is to study a first-order PDE, we need the sections in $\mathcal{E}_u$ to be "one step less regular" than the maps in $\mathcal{B}$, e.g. if $\mathcal{B}$ consists of maps of Sobolev class $W^{k,p}$, then the sections in $\mathcal{E}_u$ should be of class $W^{k-1,p}$.

[1] Complex-linear 1-forms are similarly called $(1,0)$-forms.

[2] In more straightforward terms, $T^{0,1}\Sigma \to \Sigma$ is a complex line bundle whose fiber at any given point $z \in \Sigma$ is the space of complex-antilinear maps $T_z\Sigma \to \mathbb{C}$. Similarly, fibers of $T^{1,0}\Sigma \to \Sigma$ are spaces of complex-linear maps $T_z\Sigma \to \mathbb{C}$. The direct sum of these two bundles is the complexification of $T^*\Sigma$, whose fiber at $z \in \Sigma$ consists of all *real*-linear maps $T_z\Sigma \to \mathbb{C}$.

Step 3: Show that

$$\bar{\partial}_J : \mathcal{B} \to \mathcal{E} : u \mapsto du + J(u) \circ du \circ j$$

defines a smooth section of $\mathcal{E} \to \mathcal{B}$, whose zero set is precisely the space of solutions (2.2).

Step 4: Show that under suitable assumptions (e.g. on regularity and asymptotic behavior), one can arrange such that for every $u \in \bar{\partial}_J^{-1}(0)$, the **linearization** of $\bar{\partial}_J$,[3]

$$D\bar{\partial}_J(u) : T_u\mathcal{B} \to \mathcal{E}_u$$

is a Fredholm operator and is generically surjective. (In geometric terms, this would mean that $\bar{\partial}_J$ is *transverse to the zero section.*)

Step 5: Using the implicit function theorem in Banach spaces (see [**Lan93**]), the surjectivity and Fredholm property of $D\bar{\partial}_J(u)$ imply that $\bar{\partial}_J^{-1}(0)$ is a smooth finite-dimensional manifold, with its tangent space at each $u \in \bar{\partial}_J^{-1}(0)$ canonically identified with $\ker D\bar{\partial}_J(u)$, hence the dimension of $\bar{\partial}_J^{-1}(0)$ near u equals the Fredholm index of $D\bar{\partial}_J(u)$.

In this context, the linearization of the section $\bar{\partial}_J$ at a point $u \in \bar{\partial}_J^{-1}(0)$ will be given by the natural extension of $\mathbf{D}_u : \Gamma(u^*TW) \to \Omega^{0,1}(\Sigma, u^*TW)$ to a suitable Banach space setting, e.g. if $\mathcal{B}$ consists of maps $\Sigma \to W$ of Sobolev class $W^{k,p}$, then $\mathbf{D}_u$ will be extended to a map from the $W^{k,p}$-sections of u^*TW to the $W^{k-1,p}$-sections of $\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW)$.

DEFINITION 2.1.1. Fix a complex vector bundle E over a Riemann surface (Σ, j). A (real) linear **Cauchy-Riemann type operator** on E is a real-linear first-order differential operator

$$\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$$

such that for every $f \in C^\infty(\Sigma, \mathbb{R})$ and $\eta \in \Gamma(E)$,

$$\mathbf{D}(f\eta) = (\bar{\partial} f)\eta + f\mathbf{D}\eta, \tag{2.3}$$

where $\bar{\partial} f$ denotes the complex-valued $(0,1)$-form $df + i\, df \circ j$.

Observe that $\mathbf{D}$ is complex linear if and only if the Leibniz rule (2.3) also holds for all smooth complex-valued functions f, not just real-valued. It is a standard result in complex geometry that choosing a complex-linear Cauchy-Riemann type operator $\mathbf{D}$ on E is equivalent to endowing it with the structure of a *holomorphic* vector bundle, where local sections η are defined to be holomorphic if and only if $\mathbf{D}\eta = 0$. Indeed, every holomorphic bundle comes with a canonical Cauchy-Riemann operator that is expressed as $\bar{\partial}$ in holomorphic trivializations, and in the

[3]The **linearization** of a section $s : B \to E$ of a smooth vector bundle $E \to B$ at a point $x \in s^{-1}(0) \subset B$ is a linear map $Ds(x) : T_xB \to E_x$ that can be computed by choosing any connection ∇ on E and setting $Ds(x)v := \nabla_v s$. The result is independent of the choice of connection since $s(x) = 0$. Equivalently, one could choose a local chart and trivialization near x, compute the differential of the section at x in coordinates, and argue in the same way that the resulting map $T_xB \to E_x$ is independent of choices.

other direction, the equivalence follows from a local existence result for solutions to the equation $\mathbf{D}\eta = 0$, proved in §2.5 below.[4]

EXERCISE 2.1.2. If $\mathbf{D}$ is a linear Cauchy-Riemann type operator on E, show that for every smooth linear bundle map $A : E \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$, or equivalently every $A \in \Omega^{0,1}(\Sigma, \operatorname{End}_{\mathbb{R}}(E))$, $\mathbf{D} + A$ also defines a linear Cauchy-Riemann type operator on E, and every linear Cauchy-Riemann type operator on E is of this form. Using this, show that in suitable local trivializations over a subset $\mathcal{U} \subset \Sigma$ identified biholomorphically with an open set in $\mathbb{C}$, every Cauchy-Riemann type operator $\mathbf{D}$ takes the form

$$\mathbf{D} = \bar{\partial} + A : C^\infty(\mathcal{U}, \mathbb{C}^m) \to C^\infty(\mathcal{U}, \mathbb{C}^m),$$

where $\bar{\partial} = \partial_s + i\partial_t$ in complex coordinates $z = s + it$ and $A \in C^\infty(\mathcal{U}, \operatorname{End}_{\mathbb{R}}(\mathbb{C}^m))$.

EXERCISE 2.1.3. Show that for any (not necessarily complex) connection ∇ on a complex vector bundle E over a Riemann surface Σ, $\mathbf{D}\eta := \nabla\eta + i\,\nabla\eta \circ j$ defines a linear Cauchy-Riemann type operator on E. Deduce from this that the operator $\mathbf{D}_u$ of (2.1) is a real-linear Cauchy-Riemann type operator on u^*TW.

EXERCISE 2.1.4. Suppose $\mathbf{D}$ is a linear Cauchy-Riemann type operator on a bundle E over a Riemann surface (Σ, j), and (Σ', j') is another Riemann surface with a holomorphic map $\varphi : (\Sigma', j') \to (\Sigma, j)$. Show that the pullback bundle φ^*E over (Σ', j') admits a unique linear Cauchy-Riemann type operator $\varphi^*\mathbf{D} : \Gamma(\varphi^*E) \to \Omega^{0,1}(\Sigma', \varphi^*E)$ satisfying the condition

$$(\varphi^*\mathbf{D})(\eta \circ \varphi) = \varphi^*(\mathbf{D}\eta) \qquad \text{for all } \eta \in \Gamma(E).$$

Prove moreover that if $\mathbf{D}$ is given by $\mathbf{D}\eta = \nabla\eta + i\,\nabla\eta \circ j + A\eta$ for some connection ∇ on E and $A \in \Omega^{0,1}(\Sigma, \operatorname{End}_{\mathbb{R}}(E))$, then $\varphi^*\mathbf{D}$ is given by

$$(\varphi^*\mathbf{D})\xi = \nabla\xi + i\,\nabla\xi \circ j + (\varphi^*A)\xi,$$

where ∇ in this case denotes the pullback connection on $\varphi^*E \to \Sigma'$ determined by our chosen connection on E. *Hint: Locally, you can write any section of φ^*E as a linear combination of sections of the form $\eta \circ \varphi$ for sections η of E.*

EXERCISE 2.1.5. Suppose E, F are two complex vector bundles over (Σ, j) and $\mathbf{D} : \Gamma(E \oplus F) \to \Omega^{0,1}(\Sigma, E \oplus F)$ is a linear Cauchy-Riemann type operator on $E \oplus F$, which can therefore be written in block form as

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}^E & \mathbf{D}^{EF} \\ \mathbf{D}^{FE} & \mathbf{D}^F \end{pmatrix} : \Gamma(E) \oplus \Gamma(F) \to \Omega^{0,1}(\Sigma, E) \oplus \Omega^{0,1}(\Sigma, F).$$

Show that $\mathbf{D}^E$ and $\mathbf{D}^F$ are then linear Cauchy-Riemann type operators on E and F respectively, while the off-diagonal operators $\mathbf{D}^{EF}$ and $\mathbf{D}^{FE}$ are tensorial, i.e. they can be expressed as smooth real-linear bundle maps $F \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$ and $E \to \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, F)$ respectively.

[4]This statement about the existence of holomorphic vector bundle structures is true when the base is a Riemann surface, but not if it is a higher-dimensional complex manifold. In higher dimensions there are obstructions, see e.g. [Kob87].

EXERCISE 2.1.6. Suppose E is a complex vector bundle over (Σ, j), $F \subset E$ is a complex subbundle, and $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ is a linear Cauchy-Riemann type operator on E such that $\mathbf{D}(\Gamma(F)) \subset \Omega^{0,1}(\Sigma, F)$, so $\mathbf{D}$ restricts to a linear Cauchy-Riemann type operator on F as well. Show that the induced map

$$\Gamma(E)/\Gamma(F) \xrightarrow{\mathbf{D}} \Omega^{0,1}(\Sigma, E)/\Omega^{0,1}(\Sigma, F)$$

can then be interpreted as defining a linear Cauchy-Riemann type operator on the quotient bundle E/F, using the natural identifications

$$\Gamma(E/F) = \Gamma(E)/\Gamma(F) \qquad \text{and} \qquad \Omega^{0,1}(\Sigma, E/F) = \Omega^{0,1}(\Sigma, E)/\Omega^{0,1}(\Sigma, F).$$

2.2. Some useful Sobolev inequalities

In this section, we review a few general properties of Sobolev spaces that are essential for applications in nonlinear analysis. The results stated here are explained in more detail in Appendix A.

We consider functions with values in $\mathbb{C}$ unless otherwise specified, and defined on an open domain $\mathcal{U}$ in either $\mathbb{R}^n$ or a quotient of $\mathbb{R}^n$ on which the Lebesgue measure is well defined. Certain regularity assumptions must generally be placed on the boundary of $\overline{\mathcal{U}}$ in order for all the results stated below to hold; we will ignore this detail except to mention that the necessary assumptions are satisfied for the two classes of domains that we are most interested in, which are

$$\mathcal{U} = \mathring{\mathbb{D}} \subset \mathbb{C},$$
$$\mathcal{U} = (0, L) \times S^1 \subset \mathbb{C}/i\mathbb{Z}, \qquad 0 < L \leqslant \infty.$$

Here $\mathbb{D}$ denotes the closed unit disk, $\mathring{\mathbb{D}}$ is its interior, and the identification of $(0, L) \times S^1 = (0, L) \times (\mathbb{R}/\mathbb{Z})$ with a subset of $\mathbb{C}/i\mathbb{Z}$ arises from the obvious identification of $\mathbb{R}^2$ with $\mathbb{C}$. Certain results will be specified to hold only for *bounded* domains, which means in practice that they hold on $\mathring{\mathbb{D}}$ and $(0, L) \times S^1$ for any $L > 0$, but not on $(0, \infty) \times S^1$.

Recall that for $p \in [1, \infty)$ we define the L^p norm of a measurable function $f : \mathcal{U} \to \mathbb{R}^m$ to be

$$\|f\|_{L^p} = \left(\int_{\mathcal{U}} |f|^p \right)^{1/p}.$$

For the space L^∞ we define the norm to be the essential supremum of f over $\mathcal{U}$. Denote by

$$C_0^\infty(\mathcal{U}) \subset C^\infty(\mathcal{U})$$

the space of smooth functions with compact support in $\mathcal{U}$. We say a function f has a **weak j-th partial derivative** g if the *integration by parts* formula holds for all so-called **test functions** $\varphi \in C_0^\infty(\mathcal{U})$:

$$\int_{\mathcal{U}} g\varphi = -\int_{\mathcal{U}} f\, \partial_j \varphi.$$

Equivalently, this means that g is a partial derivative of f **in the sense of distributions** (see e.g. [LL01]). Higher order weak partial derivatives are defined similarly:

recall that for a multiindex $\alpha = (i_1, ...i_n)$ we denote

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}},$$

where $|\alpha| := \sum_j i_j$. We then write $\partial^\alpha f = g$ if for all $\varphi \in C_0^\infty(\mathcal{U})$,

$$\int_{\mathcal{U}} g\varphi = (-1)^{|\alpha|} \int_{\mathcal{U}} f\, \partial^\alpha \varphi.$$

Now we may define $W^{k,p}(\mathcal{U})$ to be the set of functions on $\mathcal{U}$ with weak partial derivatives up to order k lying in L^p, and define the norm of such a function by

$$\|f\|_{W^{k,p}} = \sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^p}.$$

This definition gives $W^{0,p}(\mathcal{U}) = L^p(\mathcal{U})$, and in general $W^{k,p}(\mathcal{U})$ can be identified with a closed subset of a product of finitely many copies of $L^p(\mathcal{U})$, one for each multiindex of order at most k. This identification shows that is a Banach space; moreover, it can be shown to be reflexive and separable for $1 < p < \infty$.

While the Sobolev spaces $W^{k,p}(\mathcal{U})$ are generally defined on *open* domains, we often consider the closure $\overline{\mathcal{U}}$ as the domain for spaces of differentiable functions $C^k(\overline{\mathcal{U}})$ and $C^\infty(\overline{\mathcal{U}})$. For instance, $C^k(\overline{\mathcal{U}})$ is the Banach space of k-times differentiable functions on $\mathcal{U}$ whose derivatives up to order k are bounded and uniformly continuous on $\mathcal{U}$; note that uniform continuity implies the existence of continuous extensions to the closure $\overline{\mathcal{U}}$. Given suitable regularity assumptions for the boundary of $\overline{\mathcal{U}}$, one can show (with some effort—cf. [**AF03**, proof of Theorem 5.19]) that $C^k(\overline{\mathcal{U}})$ is precisely the set of functions which admit k-times differentiable extensions to some open set containing $\overline{\mathcal{U}}$.

EXERCISE 2.2.1. Show that if f is a continuous function on the closed disk $\mathbb{D} \subset \mathbb{C}$ that is continuously differentiable on $\dot{\mathbb{D}} = \mathbb{D}\backslash\{0\}$ and its first derivative is Lebesgue integrable on $\dot{\mathbb{D}}$, then f also has a weak first derivative on $\mathbb{D}$, which is equal to its classical derivative almost everywhere.

The following result is an amalgamation of frequently used special cases of the Sobolev embedding theorem and the Rellich-Kondrachov compactness theorem. See Theorems A.1.6 and A.1.10 in Appendix A for the more general versions, proofs of which may be found e.g. in [**AF03**].

PROPOSITION 2.2.2 (embedding/compactness). *Assume $1 \leqslant p < \infty$ and $k \in \mathbb{N}$.*

(1) If $kp > n$, then for every integer $d \geqslant 0$, there exists a continuous inclusion

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^d(\overline{\mathcal{U}}),$$

which is compact if $\mathcal{U}$ is bounded.

(2) If $1 \leqslant q < \infty$ and $m \geqslant 0$ is another integer such that $k \geqslant m$, $p \leqslant q$ and $k - \frac{n}{p} \geqslant m - \frac{n}{q}$, then there exists a continuous inclusion

$$W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U}),$$

which is compact if $\mathcal{U}$ is bounded and the inequality $k - \frac{n}{p} \geqslant m - \frac{n}{q}$ is strict.

□

The most important case of the second inclusion is $W^{k+1,p}(\mathcal{U}) \hookrightarrow W^{k,p}(\mathcal{U})$, whose continuity is obvious, and the compactness in the case of bounded $\mathcal{U}$ can be regarded as a natural analogue of the fact (arising from the Arzelà-Ascoli theorem) that the inclusions $C^{k+1}(\overline{\mathcal{U}}) \hookrightarrow C^k(\overline{\mathcal{U}})$ are compact when $\overline{\mathcal{U}}$ is compact. A useful way to remember the hypotheses in Proposition 2.2.2 is by thinking of $W^{k,p}(\mathcal{U})$ as a space of functions that have "$k - \frac{n}{p}$ continuous derivatives".

Exercise 2.2.3. Show that the compactness of the inclusions in Proposition 2.2.2 fails in general for unbounded domains, e.g. for $\mathbb{R}$.

The next three results for the case $kp > n$ are proved in §A.2 as corollaries of the Sobolev embedding theorem. The first is a Sobolev analogue of the fact that the product of a C^m-function with a C^k-function for $k \geqslant m$ is also of class C^m.

Proposition 2.2.4 (Banach algebra property). *Suppose $1 \leqslant p, q < \infty$, $kp > n$, $k \geqslant m$ and $k - \frac{n}{p} \geqslant m - \frac{n}{q}$. Then the product pairing $(f, g) \mapsto fg$ defines a continuous bilinear map*

$$W^{k,p}(\mathcal{U}) \times W^{m,q}(\mathcal{U}) \to W^{m,q}(\mathcal{U}).$$

In particular this applies when $m = k$ and $q = p$, hence $W^{k,p}(\mathcal{U})$ is a Banach algebra. □

The continuity statements above translate into inequalities between the norms in the respective spaces. For example, continuous inclusions $W^{k+d,p} \hookrightarrow C^d$ or $W^{k,p} \hookrightarrow W^{m,q}$ respectively imply that

$$\|f\|_{C^d} \leqslant c\|f\|_{W^{k+d,p}} \qquad \text{or} \qquad \|f\|_{W^{m,q}} \leqslant c\|f\|_{W^{k,p}}$$

for some constants $c > 0$ which may depend on d, k, p, m, q or $\mathcal{U}$, but not f. Similarly, the Banach algebra property means there is an inequality

$$\|fg\|_{W^{k,p}} \leqslant c\|f\|_{W^{k,p}}\|g\|_{W^{k,p}}$$

whenever $kp > n$, where again the constant c is independent of g and f.

We state the next result only for the case of bounded domains; it does have an extension to unbounded domains, but the statement becomes more complicated (cf. Theorem A.2.6). Given an open set $\Omega \subset \mathbb{R}^n$, we denote

$$W^{k,p}(\mathcal{U}, \Omega) := \left\{ u \in W^{k,p}(\mathcal{U}, \mathbb{R}^n) \;\middle|\; \overline{u(\mathcal{U})} \subset \Omega \right\}.$$

Note that this is an open subset if $kp > n$, due to the Sobolev embedding theorem.

Proposition 2.2.5 (C^k-continuity property). *Assume $1 \leqslant p < \infty$, $kp > n$, $\mathcal{U}$ is bounded and $\Omega \subset \mathbb{R}^n$ is an open set. Then there is a well-defined and continuous map*

$$W^{k,p}(\mathcal{U}, \Omega) \xrightarrow{T} \mathscr{L}\left(C^k(\Omega, \mathbb{R}^N), W^{k,p}(\mathcal{U}, \mathbb{R}^N)\right)$$
$$T(u)f := f \circ u,$$

where the space $\mathscr{L}\big(C^k(\Omega,\mathbb{R}^N), W^{k,p}(\mathcal{U},\mathbb{R}^N)\big)$ *of bounded linear maps from* $C^k(\Omega,\mathbb{R}^N)$ *to* $W^{k,p}(\mathcal{U},\mathbb{R}^N)$ *is equipped with the operator norm. It follows in particular that the map*

$$C^k(\Omega,\mathbb{R}^N) \times W^{k,p}(\mathcal{U},\Omega) \to W^{k,p}(\mathcal{U},\mathbb{R}^N) : (f,u) \mapsto f \circ u$$

is well defined and continuous. □

REMARK 2.2.6. In the settings of Propositions 2.2.4 and 2.2.5, it is also often important to know that the classical formulas for computing derivatives of fg or $f \circ u$ via the product or chain rules remain valid for computing *weak* derivatives of functions that are not necessarily classically differentiable. This is not true in general, but does hold in these specific settings due to the fact that Sobolev spaces contain dense subspaces of smooth functions. For details, see Proposition A.2.4 and Theorem A.2.6 in Appendix A.

REMARK 2.2.7. In later chapters, it will become important to be aware that Propositions 2.2.2, 2.2.4 and 2.2.5 are the essential conditions needed for defining smooth Banach manifold structures on spaces of $W^{k,p}$-smooth maps from one manifold to another, cf. Proposition 2.7.4 and [**Elĭ67**, **Pal68**]. This only works under the condition $kp > n$, as the smooth category is not well equipped to deal with discontinuous maps!

The following rescaling result will be needed for nonlinear regularity arguments; see Theorem A.2.9 in Appendix A for a proof.

PROPOSITION 2.2.8. *Assume* $p \in [1,\infty)$ *and* $k \in \mathbb{N}$ *satisfy* $kp > n$*, let* $\mathring{\mathbb{D}}^n$ *denote the open unit ball in* $\mathbb{R}^n$*,* $x_0 \in \mathring{\mathbb{D}}^n$ *a fixed point, and for each* $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ *and* $\epsilon > 0$ *sufficiently small define* $f_\epsilon \in W^{k,p}(\mathring{\mathbb{D}}^n)$ *by*

$$f_\epsilon(x) := f(x_0 + \epsilon x).$$

Then for any $\alpha \in (0,1)$ *satisfying* $\alpha \leqslant k - n/p$*, there exists a constant* $C > 0$ *such that for every* $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ *and all* $\epsilon > 0$ *smaller than the distance from* x_0 *to* $\partial\mathbb{D}^n$*,*

$$\|f_\epsilon - f_\epsilon(0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \leqslant C\epsilon^\alpha \|f - f(x_0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)}.$$

□

EXERCISE 2.2.9. Working on a 2-dimensional domain with $kp > 2$, prove directly that for any multiindex α of positive order k,

$$\|\partial^\alpha f_\epsilon\|_{L^p(\mathring{\mathbb{D}})} \leqslant \epsilon^{k-2/p} \|\partial^\alpha f\|_{L^p(\mathring{\mathbb{D}})}$$

for $f \in W^{k,p}(\mathring{\mathbb{D}})$. Find examples (e.g. in $W^{1,2}(\mathring{\mathbb{D}})$) to show that no estimate of the form

$$\|\partial^\alpha f_\epsilon\|_{L^p(\mathring{\mathbb{D}})} \leqslant C_\epsilon \|f - f(x_0)\|_{W^{k,p}(\mathring{\mathbb{D}})}$$

with $\lim_{\epsilon\to 0^+} C_\epsilon = 0$ is possible when $kp \leqslant 2$.

2.3. The fundamental elliptic estimate

We will make considerable use of the fact that the linear first-order differential operator

$$\bar{\partial} := \partial_s + i\partial_t : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$$

is *elliptic*. We will briefly touch upon the general notion of ellipticity in a bit, but in practice, the main consequence we need to be aware of is the following pair of analytical results.

THEOREM 2.3.1. *If* $1 < p < \infty$*, then* $\bar{\partial} : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ *admits a bounded right inverse* $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$*.*

THEOREM 2.3.2. *If* $1 < p < \infty$ *and* $k \in \mathbb{N}$*, then there exists a constant* $c > 0$ *such that for all* $f \in W_0^{k,p}(\mathring{\mathbb{D}})$*,*

$$\|f\|_{W^{k,p}} \leqslant c\|\bar{\partial} f\|_{W^{k-1,p}}.$$

Here $W_0^{k,p}(\mathring{\mathbb{D}})$ denotes the $W^{k,p}$-closure of the space $C_0^\infty(\mathring{\mathbb{D}})$ of smooth functions with compact support in $\mathring{\mathbb{D}}$.

The complete proofs of the two theorems above are rather lengthy, and we shall refer to [**Wenb**, §2.6 and §2.A] for the details, but we can at least explain why they hold in the case $p = 2$. First, it is straightforward to show that the function $K \in L^1_{\text{loc}}(\mathbb{C})$ defined by

$$K(z) = \frac{1}{2\pi z}$$

is a **fundamental solution** for the equation $\bar{\partial} u = f$, meaning it satisfies

$$\bar{\partial} K = \delta$$

in the sense of distributions, where δ denotes the Dirac δ-function. Hence for any $f \in C_0^\infty(\mathbb{C})$, one finds a smooth solution $u : \mathbb{C} \to \mathbb{C}$ to the equation $\bar{\partial} u = f$ as the convolution

$$u(z) = (K * f)(z) := \int_{\mathbb{C}} K(z - \zeta) f(\zeta)\, d\mu(\zeta),$$

where $d\mu(\zeta)$ denotes the Lebesgue measure with respect to the variable $\zeta \in \mathbb{C}$. It will be useful to observe that whenever f has compact support on $\mathbb{C}$, $K * f$ also has decaying behavior at infinity:

LEMMA 2.3.3. *For any* $f \in C_0^\infty(\mathbb{C})$*,* $K * f$ *satisfies* $|(K * f)(z)| \leqslant \frac{C}{|z|}$ *for some constant* $C > 0$*.*

PROOF. Choose $R > 0$ large enough so that f is supported in the disk of radius R, and suppose $|z| \geqslant 2R$. Then for all $\zeta \in \mathbb{C}$ such that $f(\zeta) \neq 0$, we have $|z - \zeta| \geqslant |z| - R \geqslant \frac{|z|}{2}$, thus

$$\begin{aligned} |(K * f)(z)| &= \frac{1}{2\pi} \left| \int_{\mathbb{C}} \frac{f(\zeta)}{z - \zeta}\, d\mu(\zeta) \right| \leqslant \frac{1}{2\pi} \int_{\mathbb{C}} \frac{|f(\zeta)|}{|z - \zeta|}\, d\mu(\zeta) \\ &\leqslant \frac{1}{\pi |z|} \int_{\mathbb{C}} |f(\zeta)|\, d\mu(\zeta) = \frac{\|f\|_{L^1}}{\pi |z|}. \end{aligned}$$

□

If $u \in C_0^\infty(\mathbb{C})$ and $\bar\partial u = f$, it follows from Lemma 2.3.3 that $u - K * f$ is a holomorphic function on $\mathbb{C}$ that decays at infinity, hence $u \equiv K * f$. Since $C_0^\infty(\mathring{\mathbb{D}})$ is dense in $L^p(\mathring{\mathbb{D}})$ for all $p < \infty$, Theorem 2.3.1 now follows from the claim that for all $f \in C_0^\infty(\mathring{\mathbb{D}})$, there exist estimates of the form

$$\|K * f\|_{L^p(\mathring{\mathbb{D}})} \leqslant c\|f\|_{L^p(\mathring{\mathbb{D}})}, \qquad \|\partial_j(K * f)\|_{L^p(\mathring{\mathbb{D}})} \leqslant c\|f\|_{L^p(\mathring{\mathbb{D}})}, \tag{2.4}$$

with $\partial_j = \partial_s$ or ∂_t for $j = 1, 2$ respectively, and the constant $c > 0$ independent of f.

EXERCISE 2.3.4. Use Theorem 2.3.1 and the remarks above to prove Theorem 2.3.2 for the case $k = 1$ with $f \in C_0^\infty(\mathring{\mathbb{D}})$, then extend it to $f \in W_0^{1,p}(\mathring{\mathbb{D}})$ by a density argument. Then extend it to the general case by differentiating both f and $\bar\partial f$.

The first estimate in (2.4) is not too hard if you remember your introductory measure theory class: it follows from a general "potential inequality" for convolution operators (see [**Wenb**, Lemma 2.6.10]), similar to Young's inequality, the key points being that K is locally of class L^1 and $\mathring{\mathbb{D}}$ has finite measure. For the second inequality, observe that $\bar\partial(K * f) = f$, and the rest of the first derivative of $K * f$ is determined by $\partial(K * f)$, where

$$\partial := \partial_s - i\partial_t.$$

Differentiating K in the sense of distributions provides a formula for $\partial(K * f)$ as a principal value integral, namely

$$\partial(K * f)(z) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|\zeta - z| \geqslant \epsilon} \frac{f(\zeta)}{(z - \zeta)^2} \, d\mu(\zeta).$$

This is a so-called **singular integral operator**: it is similar to our previous convolution operator, but more difficult to handle because the kernel $\frac{1}{z^2}$ is not of class L^1_{loc} on $\mathbb{C}$. The proof of the estimate

$$\|\partial(K * f)\|_{L^p} \leqslant c\|f\|_{L^p} \quad \text{for all } f \in C_0^\infty(\mathring{\mathbb{D}}) \tag{2.5}$$

follows from a rather difficult general estimate on singular integral operators, known as the *Calderón-Zygmund inequality*, cf. [**Wenb**, §2.A] and the references therein. The good news however is that the first step in that proof is not hard: that is the case $p = 2$.

As is the case for all elliptic operators with constant coefficients, the L^2-estimate on the fundamental solution of $\bar\partial$ admits an easy proof using Fourier transforms. In general, a sufficiently nice function $u : \mathbb{R}^n \to \mathbb{C}$ is related to its Fourier transform $\mathscr{F}u = \hat{u} : \mathbb{R}^n \to \mathbb{C}$ by

$$u(x) = \int_{\mathbb{R}^n} \hat{u}(p) e^{2\pi i (x \cdot p)} \, d\mu(p),$$

where $x \cdot p$ denotes the standard Euclidean inner product on $\mathbb{R}^n$. It thus satisfies the identity

$$\mathscr{F}(\partial_j u)(p) = 2\pi i p_j \hat{u}(p). \tag{2.6}$$

It follows more generally that for any differential operator D of order $k \in \mathbb{N}$ with constant coefficients acting on complex-valued functions on $\mathbb{R}^n$, there is a unique polynomial $P^D : \mathbb{R}^n \to \mathbb{C}$ of degree k such that

$$\mathscr{F}(Du)(p) = P^D(p)\hat{u}(p)$$

for reasonable functions $u : \mathbb{R}^n \to \mathbb{C}$. We call D an **elliptic** operator if $P^D(p) = P^D_k(p) + O(|p|^{k-1})$ and the homogeneous kth-order part P^D_k satisfies[5]

$$P^D_k(p) \neq 0 \quad \text{for all} \quad p \neq 0.$$

Since P^D_k is homogeneous with degree k, this condition implies that P^D satisfies an estimate of the form

$$|P^D(p)| \geqslant c|p|^k \quad \text{for all } p \in \mathbb{R}^n \text{ outside of some compact subset.}$$

Now if α is any multiindex of order $|\alpha| \leqslant k$, (2.6) implies $\mathscr{F}(\partial^\alpha u)(p) = (2\pi i p)^\alpha \hat{u}(p)$ with $|(2\pi i p)^\alpha| \leqslant c|p|^{|\alpha|} \leqslant c'|P^D(p)|$ for all $|p| \gg 0$ and some constant $c' > 0$. Since $(2\pi i p)^\alpha / P^D(p)$ is now a bounded function outside of some compact subset $K \subset \mathbb{R}^n$, one therefore obtains via Plancherel's theorem a bound of the form

$$\begin{aligned}\|\partial^\alpha u\|_{L^2(\mathbb{R}^n)} &= \|\mathscr{F}(\partial^\alpha u)\|_{L^2(\mathbb{R}^n)} = \|(2\pi i p)^\alpha \hat{u}\|_{L^2(\mathbb{R}^n)} \\ &= \|(2\pi i p)^\alpha \hat{u}\|_{L^2(K)} + \|(2\pi i p)^\alpha \hat{u}\|_{L^2(\mathbb{R}^n \setminus K)} \\ &\leqslant c\|\hat{u}\|_{L^2(K)} + c\|P^D(p)\hat{u}\|_{L^2(\mathbb{R}^n\setminus K)} \leqslant c\|u\|_{L^2(\mathbb{R}^n)} + c\|Du\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

In the case of $D := \bar{\partial}$ and $\partial^\alpha := \partial$ on $\mathbb{R}^2 = \mathbb{C}$, this story becomes especially simple since

$$\mathscr{F}(\bar{\partial} u)(\zeta) = 2\pi i \zeta \hat{u}(\zeta), \qquad \mathscr{F}(\partial u)(\zeta) = 2\pi i \bar{\zeta} \hat{u}(\zeta), \tag{2.7}$$

i.e. both $\bar{\partial}$ and ∂ are first-order elliptic operators.

PROPOSITION 2.3.5. *For all $f \in C^\infty_0(\mathbb{C})$, we have $\|\partial(K * f)\|_{L^2} = \|f\|_{L^2}$.*

PROOF. We write $u = K * f$, so $\bar{\partial} u = f$, and combining (2.7) with Plancherel's theorem gives

$$\begin{aligned}\|\partial(K * f)\|_{L^2} &= \|\partial u\|_{L^2} = \|\mathscr{F}(\partial u)\|_{L^2} = \|2\pi i \bar{\zeta} \hat{u}\|_{L^2} \\ &= \left\| \frac{\bar{\zeta}}{\zeta} 2\pi i \zeta \hat{u} \right\|_{L^2} = \|2\pi i \zeta \hat{u}\|_{L^2} = \|\hat{f}\|_{L^2} = \|f\|_{L^2}.\end{aligned}$$

□

COROLLARY 2.3.6. *The estimate (2.5) holds in the case $p = 2$.* □

[5] In the more general setting of a differential operator sending sections of one vector bundle to sections of another, the polynomial P^D in this discussion would take values in a space of linear maps from one finite-dimensional vector space to another. One then calls D elliptic if and only if the linear transformation $P^D(p)$ is invertible for all $p \neq 0$. More details on the general notion of ellipticity can be found e.g. in [Wenb, §2.B].

2.4. Regularity

We will now use the estimate $\|u\|_{W^{k,p}} \leqslant c\|\bar{\partial}u\|_{W^{k-1,p}}$ from the previous section to prove three types of results about solutions to Cauchy-Riemann type equations:

(1) All solutions of reasonable Sobolev-type regularity are smooth.
(2) Every sequence of solutions satisfying uniform bounds in certain Sobolev norms has a C^∞_{loc}-convergent subsequence.
(3) All reasonable Sobolev-type topologies on spaces of solutions are (locally) equivalent to the C^∞-topology.

In the following,

$$\mathbb{D}_r \subset \mathbb{C}$$

denotes the closed disk of radius $r > 0$, and $\mathring{\mathbb{D}}_r$ denotes its interior. Note that functions of class $C^\infty(\mathbb{D}_r)$ are assumed to be smooth up to the boundary (or equivalently, on some open neighborhood of $\mathbb{D}_r$ in $\mathbb{C}$), not just on $\mathring{\mathbb{D}}_r$.

2.4.1. The linear case. Recall from Exercise 2.1.2 that every linear Cauchy-Riemann type operator on a vector bundle of complex rank n locally takes the form $\bar{\partial} + A$, where $\bar{\partial} = \partial_s + i\partial_t$, and A is a smooth function with values in $\mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Using the Sobolev embedding theorem, the following result implies by induction that weak solutions of class L^p_{loc} for $1 < p < \infty$ to linear Cauchy-Riemann type equations are always smooth. The associated local estimate will also play a major role in our proof of the Fredholm property in Chapter 4.

THEOREM 2.4.1 (Linear regularity). *Assume $1 < p < \infty$, m and k are integers with $m \geqslant k \geqslant 0$, $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is a C^m-smooth function, $f \in W^{m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ and $u \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ is a weak solution to the equation*

$$(\bar{\partial} + A)u = f.$$

Then:

(1) u is of class $W^{m+1,p}$ on every compact subset of $\mathring{\mathbb{D}}$.
(2) For every $r \in (0, 1)$, there exists a constant $c > 0$ dependent on the Sobolev parameters k, m, p, the radius r and the zeroth-order term A, but not on u or f, such that

$$\|u\|_{W^{m+1}(\mathring{\mathbb{D}}_r)} \leqslant c\|u\|_{W^{k,p}(\mathring{\mathbb{D}})} + c\|f\|_{W^{m,p}(\mathring{\mathbb{D}})}.$$

REMARK 2.4.2. A portion of the results in this section can be generalized to allow weaker regularity hypotheses on the zeroth-order term $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. In the proof of Theorem 2.4.1 below, for instance, the importance of the assumption $m \geqslant k$ lies in the fact that there is a continuous product pairing $C^m \times W^{k,p} \to W^{k,p}$, which in the case $k \geqslant 1$ remains true (by Proposition 2.2.4) if C^m is replaced by $W^{m,q}$ for any $q \in [p, \infty)$ satisfying $mq > 2$, and for $k = 0$ it is also true if C^m is replaced by L^∞. The theorem therefore also holds for zeroth-order terms A of suitable Sobolev-type regularity, including some cases where A is not even continuous. This level of generality is occasionally useful for technical reasons, e.g. it plays a role in our proof of the similarity principle (Theorem 2.5.3) in the next section.

PROOF OF THEOREM 2.4.1 (EXCLUDING (1) FOR $k = 0$). We first prove statement (2), assuming that statement (1) is already known. It will suffice to prove the estimate for the case $m = k$, because if $m > k$, one can then repeat the same argument $m - k + 1$ times, shrinking to a slightly smaller compact subset of $\mathring{\mathbb{D}}$ each time. With this understood, let us fix an integer $k \geqslant 0$ and consider a weak solution $u \in W^{k,p}(\mathring{\mathbb{D}})$ to the equation $(\bar{\partial} + A)u = f$ with $f \in W^{k,p}(\mathring{\mathbb{D}})$ and $A \in C^k(\mathbb{D}, \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n))$. For any $r \in (0,1)$, statement (1) in the theorem implies $u \in W^{k+1,p}(\mathbb{D}_r)$, and our objective is to bound $\|u\|_{W^{k+1,p}(\mathring{\mathbb{D}}_R)}$ in terms of $\|u\|_{W^{k,p}(\mathring{\mathbb{D}})}$ and $\|f\|_{W^{k,p}(\mathring{\mathbb{D}})}$.

In order to apply the fundamental elliptic estimate, we need to work with functions with compact support in the interior of $\mathbb{D}$, thus we choose a smooth bump function

$$\beta \in C_0^\infty(\mathring{\mathbb{D}}, [0,1])$$

that satisfies $\beta|_{\mathbb{D}_r} \equiv 1$. Using this choice, we now give two slightly different proofs of the required estimate. The first is based on the observation that since u is locally of class $W^{k+1,p}$ on $\mathring{\mathbb{D}}$, $\beta u \in W_0^{k+1,p}(\mathring{\mathbb{D}})$, so Theorem 2.3.2 can be applied to βu, giving

$$\begin{aligned}\|u\|_{W^{k+1,p}(\mathring{\mathbb{D}}_r)} &\leqslant \|\beta u\|_{W^{k+1,p}(\mathring{\mathbb{D}})} \leqslant c\|\bar{\partial}(\beta u)\|_{W^{k,p}} \leqslant c\|(\bar{\partial}\beta)u\|_{W^{k,p}} + c\|\beta(f - Au)\|_{W^{k,p}} \\ &\leqslant c'\|u\|_{W^{k,p}} + c'\|f\|_{W^{k,p}},\end{aligned}$$

where the use of the Leibniz rule to compute $\bar{\partial}(\beta u)$ is unproblematic since β is smooth, and we have absorbed the C^k-norms of β, $\bar{\partial}\beta$ and A into the constant $c' > 0$. Note that the latter makes use of the continuous product pairing $C^k \times W^{k,p} \to W^{k,p}$ (cf. Remark 2.4.2).

The following alternative proof of this estimate is valid only if $k \geqslant 1$ and is slightly less direct, but contains useful ideas that we will need to recycle in the proof of statement 1. By assumption, we already have a bound on $\|u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$, so the required $W^{k+1,p}$-bound will follow if we can also find $W^{k,p}$-bounds over $\mathring{\mathbb{D}}_r$ for the weak partial derivatives $\partial_j u$, $j = 1, 2$. These functions are (according to statement 1) of class $W^{k,p}_{\text{loc}}$, and since $k \geqslant 1$ and $\beta\, \partial_j u \in W_0^{k,p}(\mathring{\mathbb{D}})$, we can now apply Theorem 2.3.2 to $\beta\, \partial_j u$, giving

$$\begin{aligned}\|\partial_j u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} &\leqslant \|\beta\, \partial_j u\|_{W^{k,p}(\mathring{\mathbb{D}})} \leqslant c\left\|\bar{\partial}\,(\beta\, \partial_j u)\right\|_{W^{k-1,p}(\mathring{\mathbb{D}})} \\ &\leqslant c\|(\bar{\partial}\beta)(\partial_j u)\|_{W^{k-1,p}} + c\|\beta\, \bar{\partial}(\partial_j u)\|_{W^{k-1,p}}.\end{aligned} \tag{2.8}$$

The first term on the right hand side is bounded by $c'\|u\|_{W^{k,p}}$ for some constant $c' > 0$ that depends on the C^{k-1}-norm of $\bar{\partial}\beta$. To control the second term, we differentiate the equation $\bar{\partial}u = -Au + f$, giving

$$\bar{\partial}(\partial_j u) = -(\partial_j A)u - A\, \partial_j u + \partial_j f,$$

where the Leibniz rule has been used to compute $\partial_j(Au)$ in light of Remark A.2.5 and the continuous product pairing $C^k \times W^{k,p} \to W^{k,p}$. The $W^{k-1,p}$-norm of $\beta\, \bar{\partial}(\partial_j u)$ is now bounded by a constant times $\|u\|_{W^{k-1,p}} + \|\partial_j u\|_{W^{k-1,p}} + \|\partial_j f\|_{W^{k-1,p}} \leqslant 2\|u\|_{W^{k,p}} + \|f\|_{W^{k,p}}$, where the constant in question depends only on $\|\beta\|_{C^{k-1}}$ and $\|A\|_{C^k}$.

We now prove statement (1) in the case $k \geqslant 1$; the case $k = 0$ requires a different argument and will be dealt with as an addendum at the end of this subsection. For

$k \geqslant 1$, we can use an adaptation of the second proof of statement 2 above, where instead of proving bounds on partial derivatives $\partial_j u$, we consider the corresponding **difference quotients**

$$D_j^h u(z) := \frac{u(z + he_j) - u(z)}{h}, \qquad j = 1, 2.$$

Here $e_1 := \partial_s$, $e_2 := \partial_t$, and the domain of $D_j^h u$ can be taken to be $\mathbb{D}_r$ for any $r \in (0,1)$ if $h \in \mathbb{R}\backslash\{0\}$ is sufficiently close to 0. It suffices again to consider only the case $m = k$, so let us suppose $u, f \in W^{k,p}(\mathring{\mathbb{D}})$ and $A \in C^k(\mathbb{D})$. The difference quotients $D_j^h u$ are then also of class $W^{k,p}_{\text{loc}}$ on their domains, so for the smooth cutoff function $\beta \in C_0^\infty(\mathring{\mathbb{D}})$ with $\beta|_{\mathbb{D}_r} \equiv 1$, we can assume for all $|h| > 0$ sufficiently small that $\beta D_j^h u$ is in $W_0^{k,p}(\mathring{\mathbb{D}})$. The analogue of (2.8) in this context is then

$$\begin{aligned} \|D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} &\leqslant \|\beta D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}})} \leqslant c \left\| \bar{\partial}\left(\beta\, D_j^h u\right)\right\|_{W^{k-1,p}(\mathring{\mathbb{D}})} \\ &\leqslant c\|(\bar{\partial}\beta)(D_j^h u)\|_{W^{k-1,p}} + c\|\beta\, \bar{\partial}(D_j^h u)\|_{W^{k-1,p}}. \end{aligned}$$

The first term is bounded independently of h since $\partial_j u \in W^{k-1,p}(\mathring{\mathbb{D}})$, implying a uniform $W^{k-1,p}$-bound on $D_j^h u$ as $h \to 0$; cf. Appendix A.3. To control the second term, we can apply the operator D_j^h to the equation $\bar{\partial} u = -Au + f$, giving

$$\bar{\partial}(D_j^h u) = D_j^h(\bar{\partial} u) = -(D_j^h A)u - A\, D_j^h u + D_j^h f.$$

Since $A \in C^k(\mathbb{D})$, $D_j^h A$ is uniformly C^{k-1}-bounded as $h \to 0$, and $\partial_j u, \partial_j f \in W^{k-1,p}(\mathring{\mathbb{D}})$ similarly implies uniform $W^{k-1,p}$-bounds on $D_j^h u$ and $D_j^h f$, thus the whole expression is uniformly $W^{k-1,p}$-bounded on some open disk containing the support of β, implying

$$\|D_j^h u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \leqslant c$$

for some constant $c > 0$ that does not change as $h \to 0$. This implies $u \in W^{k+1,p}(\mathring{\mathbb{D}}_r)$ via a standard application of the Banach-Alaoglu theorem. Indeed, the latter implies that if there is a uniform bound on $\|D_j^h u\|_{L^p}$ as $h \to 0$, then any decaying sequence $h_\nu \to 0$ has a subsequence for which $D_j^{h_\nu} u$ is weakly L^p-convergent. The limit of this subsequence belongs to $L^p(\mathring{\mathbb{D}}_r)$, and it is straightforward to show using the definition of weak derivatives that this limit is $\partial_j u$. One finds a similar result in the presence of uniform $W^{k,p}$-bounds for any $k \in \mathbb{N}$ by applying this argument to higher-order derivatives of $\partial_j u$; for details, see Theorem A.3.1 in Appendix A.3. ☐

EXERCISE 2.4.3. Deduce from Theorem 2.4.1 the following corollaries for a sequence of weak solutions $u_\nu \in W^{k,p}(\mathring{\mathbb{D}})$ to $(\bar{\partial} + A_\nu)u_\nu = f_\nu$, assuming $f_\nu \in W^{m,p}(\mathring{\mathbb{D}})$ and $A_\nu \in C^m(\mathring{\mathbb{D}})$ for all $\nu \in \mathbb{N}$, with $m \geqslant k \geqslant 0$ and $1 < p < \infty$.

(a) If $\|u_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})}$, $\|f_\nu\|_{W^{m,p}(\mathring{\mathbb{D}})}$ and $\|A_\nu\|_{C^m(\mathbb{D})}$ are uniformly bounded, then u_ν is also uniformly $W^{k+1,p}$-bounded on compact subsets of $\mathring{\mathbb{D}}$.

(b) If u_ν is $W^{k,p}$-convergent, f_ν is $W^{m,p}$-convergent and A_ν is C^m-convergent on $\mathbb{D}$, then u_ν is also $W^{m+1,p}_{\text{loc}}$-convergent on $\mathring{\mathbb{D}}$.

Remark 2.4.4. Combining the Sobolev embedding theorem with the Arzelà-Ascoli theorem, the result of Exercise 2.4.3(a) proves that if the f_ν and A_ν are C^∞-bounded on $\mathbb{D}$, then a $W^{k,p}$-bounded sequence of solutions u_ν has a C^∞_{loc}-convergent subsequence. Part (b) implies moreover that for every $k \geqslant 0$ and $p \in (1,\infty)$, the $W^{k,p}$-topology on spaces of solutions to linear Cauchy-Riemann type equations is locally equivalent to the C^∞-topology.

Exercise 2.4.5. Use Theorem 2.4.1 to generalize Theorem 2.3.1 to the existence of a bounded right inverse for

$$\bar\partial : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k-1,p}(\mathring{\mathbb{D}}).$$

for every $k \in \mathbb{N}$ and $1 < p < \infty$. *Hint: For any $R > 1$, there exists a bounded linear extension operator $E : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k,p}(\mathring{\mathbb{D}}_R)$ with the property $(Ef)|_{\mathring{\mathbb{D}}} = f$ for all $f \in W^{k,p}(\mathring{\mathbb{D}})$; see Theorem A.1.4 and Corollary A.1.5.*

It remains to prove the case $k = 0$ of Theorem 2.4.1(1). As preparation for this, we start with a classical result about "weakly holomorphic" functions:

Lemma 2.4.6. *If $u \in L^1(\mathring{\mathbb{D}})$ satisfies $\bar\partial u = 0$ in the sense of distributions, then u is smooth and holomorphic on the open disk $\mathring{\mathbb{D}}$.*

Proof. Taking real and imaginary parts, it suffices to prove that the same statement holds for the Laplace equation. By mollification, any weakly harmonic function can be approximated in L^1 with smooth harmonic functions. The latter satisfy the mean value property, which behaves well under L^1-convergence, so the result follows from the mean value characterization of harmonic functions; see [**Wenb**, Lemma 2.6.26] for more details. □

Lemma 2.4.7. *Suppose $1 < p < \infty$ and $u \in L^1(\mathring{\mathbb{D}})$ is a weak solution to $\bar\partial u = f$ for some $f \in L^p(\mathring{\mathbb{D}})$. Then u is of class $W^{1,p}$ on every compact subset of $\mathring{\mathbb{D}}$.*

Proof. Let $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$ denote the bounded right inverse of $\bar\partial : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ provided by Theorem 2.3.1. Then $u - Tf \in L^1(\mathring{\mathbb{D}})$ is a weak solution to $\bar\partial(u - Tf) = 0$ and is thus smooth by Lemma 2.4.6. In particular, $u - Tf$ restricts to $\mathring{\mathbb{D}}_r$ for every $r < 1$ as a function of class $W^{1,p}$, implying that u also has a restriction in $W^{1,p}(\mathring{\mathbb{D}}_r)$. □

Proof of Theorem 2.4.1(1) for $k = 0$. Suppose $(\bar\partial + A)u = f$, where A is continuous on $\mathbb{D}$ and $u, f \in L^p(\mathring{\mathbb{D}})$. Then $\bar\partial u = -Au + f \in L^p(\mathring{\mathbb{D}})$, so Lemma 2.4.7 implies $u \in W^{1,p}_{\text{loc}}(\mathring{\mathbb{D}})$. If $m \geqslant 1$, one can now shrink the disk slightly and plug in the case $k = 1$ of the theorem to conclude $u \in W^{m+1,p}_{\text{loc}}(\mathring{\mathbb{D}})$. □

Corollary 2.4.8. *If $A : \mathbb{D} \to \text{End}_{\mathbb{R}}(\mathbb{C}^n)$ is of class C^m for $0 \leqslant m \leqslant \infty$, then every weak solution $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ to $(\bar\partial + A)u = 0$ of class L^p_{loc} for a given $p \in (1,\infty)$ is also in $W^{k,q}_{\text{loc}}(\mathring{\mathbb{D}})$ for every $k \leqslant m + 1$ and $q \in (1,\infty)$. In particular, u is of class C^m.*

Proof. Assume for simplicity $m < \infty$, as the case $m = \infty$ will then immediately follow. Theorem 2.4.1(1) implies $u \in W^{m+1,p}(\mathring{\mathbb{D}}_r)$ for any $r < 1$. If $p > 2$, this implies via the Sobolev embedding theorem that $u \in C^m(\mathbb{D}_r)$. In particular, u is then continuous and bounded on the closed disk $\mathbb{D}_r$, so it is in $L^q(\mathring{\mathbb{D}}_r)$ for every $q \in (1,\infty)$, and feeding it into Theorem 2.4.1(1) again gives the desired result on $\mathbb{D}_r$. Since $r < 1$ was arbitrary, the result is therefore true on any compact subset of $\mathring{\mathbb{D}}$.

To finish, it will now suffice to show that if $u \in L^p(\mathring{\mathbb{D}})$ for some $p \leqslant 2$, then u is also in $L^q_{\text{loc}}(\mathring{\mathbb{D}})$ for some $q > 2$. Here Theorem 2.4.1(1) again implies $u \in W^{1,p}(\mathring{\mathbb{D}}_r)$ for any $r < 1$, and according to the Sobolev embedding theorem, there is a continuous inclusion $W^{1,p} \hookrightarrow L^q$ whenever $p \leqslant q < p^*$, where $p^* \in (p,\infty]$ is determined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$; see Theorem A.1.6. Since $p > 1$, this implies $\frac{1}{p^*} < \frac{1}{2}$ and thus $p^* > 2$, so we can choose any $q \in (2,p^*)$ and conclude $u \in L^q(\mathring{\mathbb{D}}_r)$. □

2.4.2. The nonlinear case: bootstrapping. The regularity argument in the previous subsection was inductive in nature: if the data A and f in the equation $(\bar{\partial} + A)u = f$ are smooth, then assuming $u \in W^{k,p}$ leads via elliptic estimates to the conclusion that u is actually of class $W^{k+1,p}$, so by induction, u is smooth. This technique is known in the PDE literature as *elliptic bootstrapping*. We will now prove a similar bootstrapping result for the nonlinear Cauchy-Riemann equation.

Locally, every J-holomorphic curve can be regarded as a map $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ satisfying $\partial_s u(z) + J(u(z))\partial_t u(z) = 0$ in coordinates $z = s + it \in \mathbb{D} \subset \mathbb{C}$, where J is an almost complex structure on $\mathbb{C}^n$, or equivalently, a function[6]

$$J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n) := \left\{ K \in \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n) \;\middle|\; K^2 = -\mathbb{1} \right\}.$$

A nonlinear analogue of the equation considered in Theorem 2.4.1 is then the *inhomogeneous* nonlinear Cauchy-Riemann equation

$$\partial_s u + J(u)\partial_t u = f \tag{2.9}$$

for functions $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$, where $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ and $f : \mathring{\mathbb{D}} \to \mathbb{C}^n$ are given.

REMARK 2.4.9. It is worth mentioning that while other nonlinear analogues of the equation $(\bar{\partial} + A)u = 0$ are possible, the equations of interest can all be reduced to (2.9) in practice. For example, in Floer homology and Gromov-Witten theory, one often considers equations that locally take the form

$$\partial_s u(z) + J(z, u(z))\partial_t u(z) = f(z, u(z)),$$

where $J : \mathring{\mathbb{D}} \times \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ is now allowed to depend explicitly on points in the domain of $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$, and $f : \mathring{\mathbb{D}} \times \mathbb{C}^n \to \mathbb{C}^n$ is likewise a function of both z and the value $u(z)$. As was observed by Gromov already in [Gro85, 1.4.C], the solutions $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ to this equation are equivalent to honest $\bar{J}$-holomorphic curves in $\mathring{\mathbb{D}} \times \mathbb{C}^n$

[6]Here the reader should beware of a minor ambiguity in notation: while we used $\mathcal{J}(M)$ in Chapter 1 to mean the space of smooth almost complex structures on a manifold M, one can just as sensibly define $\mathcal{J}(V)$ for each real $2n$–dimensional vector space V to be the space of *linear* complex structures on V, topologized as a subset of the finite-dimensional vector space $\operatorname{End}_{\mathbb{R}}(V) \cong \mathbb{R}^{2n \times 2n}$. It is not hard to show that $\mathcal{J}(V)$ is then a smooth submanifold of $\operatorname{End}_{\mathbb{R}}(V)$; in fact, the ability to choose J-complex bases for each $J \in \mathcal{J}(V)$ gives $\mathcal{J}(V)$ a natural identification with the homogeneous space $\operatorname{Aut}_{\mathbb{R}}(V)/\operatorname{Aut}_{\mathbb{C}}(V,J) \cong \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$. In the present discussion, the notation $\mathcal{J}(\mathbb{C}^n)$ views $\mathbb{C}^n$ as a real $2n$-dimensional vector space rather than as a manifold.

of the form $\bar{u}(z) := (z, u(z))$ if one defines an almost complex structure $\bar{J}$ in block form on $\mathring{\mathbb{D}} \times \mathbb{C}^n$ by

$$\bar{J}(z,x) := \begin{pmatrix} i & 0 \\ f(z,x)i & J(z,x) \end{pmatrix}.$$

For this reason, all theorems about regularity of "honest" J-holomorphic curves imply similar results about the nonlinear inhomogeneous equations in Floer homology and Gromov-Witten theory.

The following is *not* the most general theorem provable about regularity for J-holomorphic curves, though it is the one that is most closely analogous to the linear result in the previous subsection, and is also the one that we will need most often. A partial improvement with weaker hypotheses will be discussed in §2.4.3 below.

THEOREM 2.4.10 (Nonlinear regularity, $kp > 2$ version). *Assume $1 < p < \infty$, and m and k are integers with $m \geqslant k$ and $kp > 2$.*

(1) If $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ is of class C^m and $u \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ is a weak solution to the equation

$$\partial_s u + J(u)\partial_t u = f$$

for some $f \in W^{m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$, then u is of class $W^{m+1,p}$ on every compact subset of $\mathring{\mathbb{D}}$.

(2) Consider a C^m_{loc}-convergent sequence $J_\nu \to J$ of C^m-smooth almost complex structures $\mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$, together with sequences $f_\nu \in W^{m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ and $u_\nu \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ such that for each $\nu \in \mathbb{N}$, u_ν is a weak solution to the equation

$$\partial_s u_\nu + J_\nu(u_\nu)\partial_t u_\nu = f_\nu.$$

(a) If the norms $\|f_\nu\|_{W^{m,p}}$ and $\|u_\nu\|_{W^{k,p}}$ on $\mathring{\mathbb{D}}$ are uniformly bounded as $\nu \to \infty$, then u_ν is also uniformly $W^{m+1,p}$-bounded on every compact subset of $\mathring{\mathbb{D}}$.

(b) If f_ν is $W^{m,p}$-convergent and u_ν is $W^{k,p}$-convergent on $\mathring{\mathbb{D}}$, then u_ν is also $W^{m+1,p}$-convergent on every compact subset of $\mathring{\mathbb{D}}$.

Combining this result with the Sobolev embedding theorem and the Arzelà-Ascoli theorem yields:

COROLLARY 2.4.11. *If J is a smooth almost complex structure on $\mathbb{C}^n$, then every J-holomorphic map $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ that is of class $W^{k,p}$ for some $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$ is smooth. Moreover, if $J_\nu \to J$ is a C^∞_{loc}-convergent sequence of almost complex structures on $\mathbb{C}^n$ and $u_\nu : \mathring{\mathbb{D}} \to \mathbb{C}^n$ is a sequence of J_ν-holomorphic maps, then for any $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$, uniform $W^{k,p}$-bounds for u_ν imply C^∞_{loc}-convergence of a subsequence of u_ν, and similarly, $W^{k,p}$-convergence of u_ν implies C^∞_{loc}-convergence.* □

REMARK 2.4.12. We will take pains to avoid dealing with non-smooth almost complex structures in this book, but in some applications they are unavoidable for technical reasons. In such cases, one gets the most mileage out of Theorem 2.4.10 by choosing $p > 2$, as the Sobolev embedding theorem then implies that J-holomorphic curves of class $W^{1,p}$ have at least as many continuous derivatives as J does. If one instead starts with a curve u of class $W^{k,p}_{\mathrm{loc}}$ for some $p \leqslant 2$ but $kp > 2$, then since $k \geqslant 2$, one can use the Sobolev embedding theorem to argue (cf. Corollary 2.4.8) that u is therefore also of class $W^{1,q}_{\mathrm{loc}}$ for some $q > 2$, which leads to the same result. To summarize: if J is of class C^m, then any J-holomorphic curve of class $W^{k,p}_{\mathrm{loc}}$ for some k, p with $kp > 2$ is also of class $W^{m+1,q}_{\mathrm{loc}}$ for every $q \in (1, \infty)$, and in particular it is of class C^m.

Our proof of Theorem 2.4.10 will follow a similar outline to the proof of Theorem 2.4.1, which can be interpreted as the special case where $J_\nu \equiv i$ for all ν. The reason it works more generally is that if we zoom in on a sufficiently small neighborhood of one point in $\mathbb{C}^n$, then J can be viewed as a C^m-small perturbation of i. To make this precise, we shall use the following rescaling trick.

Associate to any C^m-smooth map $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ the function

$$Q := i - J \in C^m(\mathbb{C}^n, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)).$$

In terms of Q, the equation $\partial_s u + J(u)\partial_t u = f$ then becomes

$$\bar{\partial} u - Q(u)\partial_t u = f. \tag{2.10}$$

For any given point $z_0 \in \mathring{\mathbb{D}}$, we can assume without loss of generality after an affine change of coordinates on $\mathbb{C}^n$ that $u(z_0) = 0$ and $J(0) = i$, so in particular $Q(0) = 0$. For any $\epsilon \in (0, \mathrm{dist}(z_0, \partial\mathbb{D}))$ and a fixed constant $\alpha \in (0, 1)$ to be specified further below, we now associate to J, u and f the functions

$$\begin{aligned}
\widehat{J} : \mathbb{C}^n &\to \mathcal{J}(\mathbb{C}^n), & \widehat{J}(x) &:= J(\epsilon^\alpha x), \\
\widehat{Q} : \mathbb{C}^n &\to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n), & \widehat{Q}(x) &:= Q(\epsilon^\alpha x) = i - \widehat{J}(x), \\
\hat{u} : \mathring{\mathbb{D}} &\to \mathbb{C}^n, & \hat{u}(z) &:= \frac{1}{\epsilon^\alpha} u(z_0 + \epsilon z), \\
\hat{f} : \mathring{\mathbb{D}} &\to \mathbb{C}^n, & \hat{f}(z) &:= \epsilon^{1-\alpha} f(z_0 + \epsilon z).
\end{aligned} \tag{2.11}$$

Then u satisfies (2.10) if and only if $\hat{u}$ satisfies

$$\bar{\partial}\hat{u} - \widehat{Q}(\hat{u})\partial_t \hat{u} = \hat{f}. \tag{2.12}$$

The rescaled almost complex structure has the convenient feature that since $J(0)$ is the standard complex structure i, choosing $\epsilon > 0$ small makes $\widehat{J}$ arbitrarily C^m-close to i on the compact set[7] $\mathbb{D}^{2n} \subset \mathbb{C}^n$, which means $\|\widehat{Q}\|_{C^m(\mathbb{D}^{2n})}$ can be made arbitrarily small. By Proposition 2.2.8, $\|\hat{u}\|_{W^{k,p}(\mathring{\mathbb{D}})}$ will likewise stay under control for $\epsilon \to 0$ if we choose $\alpha \in (0, 1)$ such that $\alpha \leqslant k - 2/p$, and in fact, choosing α to be slightly smaller then ensures that we can make $\|\hat{u}\|_{W^{k,p}}$ an arbitrarily small multiple of $\|u\|_{W^{k,p}}$ by choosing $\epsilon > 0$ small. Since $kp > 2$, this will also make $\|\hat{u}\|_{C^0}$ arbitrarily small, and we can therefore assume that $\hat{u}$ has image in $\mathbb{D}^{2n}$. By the assumption $m \geqslant k$ and the continuity of the map $C^k \times W^{k,p} \to W^{k,p}$ in Proposition 2.2.5, the function

[7] Here $\mathbb{D}^{2n}$ denotes the closed unit ball in $\mathbb{C}^n = \mathbb{R}^{2n}$.

$\mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n) : z \mapsto \widehat{Q}(\hat{u}(z))$ can then likewise be assumed to be arbitrarily $W^{k,p}$-small by choosing $\epsilon > 0$ small. The effect is that (2.12) can now be viewed as a $W^{k,p}$-close approximation of the linear equation $\bar{\partial}\hat{u} = \hat{f}$.

The price we pay for this rescaling is that if we are able to prove e.g. a uniform bound on the norms $\|\hat{u}_\nu\|_{W^{k+1,p}(\mathring{\mathbb{D}}_r)}$ for some sequence $u_\nu \in W^{k,p}(\mathring{\mathbb{D}})$ and $r \in (0,1)$, then the resulting $W^{k+1,p}$-bound for u_ν will be valid only on the ϵ-disk around the point z_0. But this point was chosen arbitrarily in $\mathring{\mathbb{D}}$, so the result is a uniform bound over some neighborhood of *any* interior point of $\mathbb{D}$, and since a compact subset of $\mathring{\mathbb{D}}$ can be covered by finitely many such neighborhoods, that is enough to achieve uniform bounds over compact subsets.

REMARK 2.4.13. The rescaling trick described above is one of several reasons why the condition $kp > 2$ will be needed in the proof of Theorem 2.4.10, while it was irrelevant in the linear case. Another reason is of course the Sobolev embedding theorem, which guarantees that the solutions we consider are always continuous. We will see when we study compactness in Chapter 7 that the statement in Theorem 2.4.10 about uniform bounds is generally false when $kp \leqslant 2$, and even when we extend the statement about smoothness to allow $kp \leqslant 2$ in §2.4.3, continuity will have to be imposed as an explicit extra hypothesis.

PROOF OF THEOREM 2.4.10. We will prove statement (2a) assuming that statement (1) is already known, and leave the rest as exercises.

Since $m \geqslant k$, it suffices to prove the statement for the case $k = m$, as otherwise the argument can always be repeated on slightly smaller disks at each step to increase k until it reaches m. We therefore assume that a C^k_{loc}-convergent sequence $J_\nu \to J$ of functions $\mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ and sequences $u_\nu, f_\nu \in W^{k,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ satisfying uniform bounds

$$\|u_\nu\|_{W^{k,p}} \leqslant M, \qquad \|f_\nu\|_{W^{k,p}} \leqslant M$$

are given such that $\partial_s u_\nu + J_\nu(u_\nu)\partial_t u_\nu = f_\nu$, and we need to establish that u_ν is also uniformly $W^{k+1,p}$-bounded over compact subsets. (Note that we can assume due to statement 1 in the theorem that each u_ν is of class $W^{k+1,p}_{\mathrm{loc}}$.) It suffices in fact to prove that every *subsequence* of u_ν has a further subsequence for which such uniform bounds hold; indeed, if the bound for the whole sequence did not exist, then we would be able to find a subsequence with norms blowing up to infinity over some compact subset, and no further subsequence of this subsequence could satisfy a uniform bound. With this understood, we can appeal to the compactness of the inclusion $W^{k,p}(\mathring{\mathbb{D}}) \hookrightarrow C^0(\mathbb{D})$ for $kp > 2$ (see Proposition 2.2.2), and replace u_ν with a subsequence (still denoted by u_ν) that is C^0-convergent on $\mathbb{D}$ to some continuous map $u : \mathbb{D} \to \mathbb{C}^n$.

For any given point $z_0 \in \mathring{\mathbb{D}}$, we can now apply a converging sequence of affine transformations to $\mathbb{C}^n$ in order to assume without loss of generality

$$u_\nu(z_0) = 0 \text{ for all } \nu, \qquad \text{and} \qquad J(0) = i.$$

We then choose

$$\alpha \in (0,1) \quad \text{with} \quad \alpha < k - \frac{2}{p}, \tag{2.13}$$

and apply the rescaling trick outlined above to replace u_ν, f_ν and J_ν with the corresponding rescalings $\hat{u}_\nu$, $\hat{f}_\nu$ and $\hat{J}_\nu$ as defined in (2.11), defining also the related functions $\widehat{Q}_\nu = i - \hat{J}_\nu$. We then have the equation $\bar{\partial}\hat{u}_\nu - \widehat{Q}_\nu(\hat{u}_\nu)\partial_t\hat{u}_\nu = \hat{f}_\nu$, with C^k-convergence $\widehat{Q}_\nu \to \widehat{Q}$ over $\mathbb{D}^{2n}$, where $\widehat{Q}$ may be assumed arbitrarily C^k-small on this set by choosing $\epsilon > 0$ small. Since $\hat{u}_\nu(0) = u_\nu(z_0) = 0$ for all ν, we can choose some $\beta > \alpha$ that also satisfies the conditions in (2.13) and then apply Propostion 2.2.8 to obtain a bound

$$\|\hat{u}_\nu\|_{W^{k,p}} \leqslant C\epsilon^{\beta-\alpha}\|u_\nu\|_{W^{k,p}} \leqslant C\epsilon^{\beta-\alpha}M \tag{2.14}$$

for some constant $C > 0$ that is independent of ν and ϵ. We can therefore impose an arbitrarily small uniform $W^{k,p}$-bound (and therefore a similarly small C^0-bound) on $\hat{u}_\nu$ by choosing $\epsilon > 0$ small enough. For f_ν, it will suffice to know that the uniform bound $\|f_\nu\|_{W^{k,p}} \leqslant M$ implies a similar uniform bound

$$\|\hat{f}_\nu\|_{W^{k,p}} \leqslant M_\epsilon$$

for some constant $M_\epsilon > 0$ which may depend on ϵ, but not on ν. Our goal is now to prove that for some fixed choice of the rescaling parameter $\epsilon > 0$, $\|\partial_j\hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)}$ is uniformly bounded for $j = 1, 2$ and some $r \in (0, 1)$.

The argument begins exactly the same as in the linear case: choose a smooth bump function

$$\beta \in C_0^\infty(\mathring{\mathbb{D}}, [0, 1])$$

that satisfies $\beta|_{\mathbb{D}_r} \equiv 1$. We then have $\beta\,\partial_j\hat{u}_\nu \in W_0^{k,p}(\mathring{\mathbb{D}})$, so by Theorem 2.3.2,

$$\|\partial_j\hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \leqslant \|\beta\,\partial_j\hat{u}_\nu\|_{W^{k,p}(\mathring{\mathbb{D}})} \leqslant c\left\|\bar{\partial}\left(\beta\,\partial_j\hat{u}_\nu\right)\right\|_{W^{k-1,p}(\mathring{\mathbb{D}})}. \tag{2.15}$$

If this were still the proof of Theorem 2.4.1, we would now apply the Leibniz rule to write $\bar{\partial}(\beta\,\partial_j\hat{u}_\nu)$ as a sum of two terms, but the nonlinear case requires something slightly cleverer at this step. Let us instead derive a PDE satisfied by $\beta\,\partial_j\hat{u}_\nu$. Differentiating the equation $\bar{\partial}\hat{u}_\nu = \widehat{Q}_\nu(\hat{u}_\nu)\partial_t\hat{u}_\nu + \hat{f}_\nu$ gives

$$\bar{\partial}(\partial_j\hat{u}_\nu) = \partial_j(\bar{\partial}\hat{u}_\nu) = \widehat{Q}_\nu(\hat{u}_\nu)\partial_t\partial_j\hat{u}_\nu + D\widehat{Q}_\nu(\hat{u}_\nu)\left(\partial_j\hat{u}_\nu, \partial_t\hat{u}_\nu\right) + \partial_j\hat{f}_\nu,$$

where $D\widehat{Q}_\nu$ denotes the differential of $\widehat{Q}_\nu$. In this calculation we have assumed that the product and chain rules are universally valid, but this requires some care since we are dealing with weak rather than classical derivatives: in fact, the chain rule can be used for differentiating $\widehat{Q}_\nu(\hat{u}_\nu)$ according to Theorem A.2.6 since $\hat{u}_\nu$ is of class $W^{k,p}$ with $kp > 2$ and $\widehat{Q}_\nu$ is of class C^k, and Proposition A.2.4 then justifies the product rule for $\widehat{Q}_\nu(\hat{u}_\nu)\partial_t\hat{u}_\nu$ since $\widehat{Q}_\nu(\hat{u}_\nu) \in W^{k,p}$, $\partial_t\hat{u}_\nu \in W^{k-1,p}$, and the product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ is continuous. Returning to the formula itself, we now have

$$\begin{aligned}
\bar{\partial}(\beta\,\partial_j\hat{u}_\nu) &= \beta\widehat{Q}_\nu(\hat{u}_\nu)\partial_t\partial_j\hat{u}_\nu + \beta D\widehat{Q}_\nu(\hat{u}_\nu)\left(\partial_j\hat{u}_\nu, \partial_t\hat{u}_\nu\right) + \beta\,\partial_j\hat{f}_\nu + (\bar{\partial}\beta)\partial_j\hat{u}_\nu \\
&= \widehat{Q}_\nu(\hat{u}_\nu)\partial_t(\beta\,\partial_j\hat{u}_\nu) + D\widehat{Q}_\nu(\hat{u}_\nu)\left(\beta\,\partial_j\hat{u}_\nu, \partial_t\hat{u}_\nu\right) \\
&\qquad + \beta\,\partial_j\hat{f}_\nu + (\bar{\partial}\beta)\partial_j\hat{u}_\nu - \widehat{Q}_\nu(\hat{u}_\nu)(\partial_t\beta)\partial_j\hat{u}_\nu,
\end{aligned}$$

so that $\beta\,\partial_j \hat{u}_\nu$ satisfies

$$\bar{\partial}(\beta\,\partial_j \hat{u}_\nu) - \widehat{Q}_\nu(\hat{u}_\nu)\partial_t(\beta\,\partial_j \hat{u}_\nu) = D\widehat{Q}_\nu(\hat{u}_\nu)(\beta\,\partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu) + \left(\bar{\partial}\beta - \widehat{Q}_\nu(\hat{u}_\nu)\partial_t\beta\right)\partial_j \hat{u}_\nu + \beta\,\partial_j \hat{f}_\nu.$$

Combining this with (2.15) gives

$$\|\beta\,\partial_j \hat{u}_\nu\|_{W^{k,p}} \leqslant c\left\|\widehat{Q}_\nu(\hat{u}_\nu)\partial_t(\beta\,\partial_j \hat{u}_\nu)\right\|_{W^{k-1,p}} + c\left\|D\widehat{Q}_\nu(\hat{u}_\nu)(\beta\,\partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu)\right\|_{W^{k-1,p}} + c\left\|\left(\bar{\partial}\beta - \widehat{Q}_\nu(\hat{u}_\nu)\partial_t\beta\right)\partial_j \hat{u}_\nu + \beta\,\partial_j \hat{f}_\nu\right\|_{W^{k-1,p}}. \tag{2.16}$$

It is important to note that the constant $c > 0$ in this expression comes from the elliptic estimate $\|g\|_{W^{k,p}} \leqslant c\|\bar{\partial}g\|_{W^{k-1,p}}$, so it is the same constant regardless of our choice of the scaling parameter ϵ. Let's look at each of the three terms on the right hand side separately.

Step 1: The third term.
We claim that the term on the second line of (2.16) satisfies a uniform bound. For the terms in this expression that only involve products of $\partial_j \hat{u}_\nu$ or $\partial_j \hat{f}_\nu$ with smooth functions, this follows immediately from the uniform $W^{k,p}$-bounds on $\hat{u}_\nu$ and $\hat{f}_\nu$. For the term involving $\widehat{Q}_\nu(\hat{u}_\nu)$ we observe that since $\widehat{Q}_\nu \to \widehat{Q}$ in C^k on $\mathbb{D}^{2n}$ and $\hat{u}_\nu$ can be assumed to lie in a $W^{k,p}$-small neighborhood of 0 for every ν, Proposition 2.2.5 places $\widehat{Q}_\nu(\hat{u}_\nu)$ into a $W^{k,p}$-small neighborhood of 0 for ν sufficiently large, meaning this term is uniformly $W^{k,p}$-bounded. Its product with $\partial_j \hat{u}_\nu$ is then uniformly $W^{k-1,p}$-bounded due to the continuous product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ from Prop. 2.2.4.

Step 2: The first term.
The tricky aspect of the first term in (2.16) is that it involves kth derivatives of $\beta\,\partial_j \hat{u}_\nu$, which are actually what we were trying to bound in the first place. What saves the situation is the *smallness* of $\widehat{Q}_\nu(\hat{u}_\nu)$: indeed, we have seen above that this term can be assumed arbitrarily $W^{k,p}$-small as $\nu \to \infty$ if $\epsilon > 0$ is chosen sufficiently small. The continuous product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ gives a bound

$$\begin{aligned} c\left\|\widehat{Q}_\nu(\hat{u}_\nu)\partial_t(\beta\,\partial_j \hat{u}_\nu)\right\|_{W^{k-1,p}} &\leqslant c'\left\|\widehat{Q}_\nu(\hat{u}_\nu)\right\|_{W^{k,p}} \cdot \|\partial_t(\beta\,\partial_j \hat{u}_\nu)\|_{W^{k-1,p}} \\ &\leqslant c'\left\|\widehat{Q}_\nu(\hat{u}_\nu)\right\|_{W^{k,p}} \cdot \|\beta\,\partial_j \hat{u}_\nu\|_{W^{k,p}}, \end{aligned}$$

where $c' > 0$ is yet another constant that does not depend on ϵ. With this in mind, let us now choose $\epsilon > 0$ small enough to ensure

$$\left\|\widehat{Q}_\nu(\hat{u}_\nu)\right\|_{W^{k,p}} < \frac{1}{3c'}.$$

Step 3: The second term.
We observe first that $D\widehat{Q}_\nu \to D\widehat{Q}$ in C^{k-1}, and depending on whether $p > 2$ or $p \leqslant 2$, slightly different tricks can now be used to bound $D\widehat{Q}_\nu(\hat{u}_\nu)$. If $p > 2$, then $W^{k,p}$ has a continuous inclusion into C^{k-1} and we can therefore assume the $\hat{u}_\nu$ all lie in a fixed C^{k-1}-small neighborhood of 0, implying that $D\widehat{Q}_\nu(\hat{u}_\nu)$ is uniformly C^{k-1}-bounded. If on the other hand $p \leqslant 2$, then the condition $kp > 2$ requires $k \geqslant 2$,

and we can instead make use of a Sobolev embedding of the form $W^{k,p} \hookrightarrow W^{k-1,q}$. Indeed, choose any $q \in [p, \infty)$ such that the condition

$$0 < k - 1 - \frac{2}{q} \leqslant k - \frac{2}{p}$$

is satisfied; this is clearly possible since $k-1-\frac{2}{p} < k-\frac{2}{p}$ and $k-1-\frac{2}{\infty} = k-1 \geqslant k-\frac{2}{p}$ and $p \leqslant 2$. Proposition 2.2.2 now provides a continuous inclusion $W^{k,p} \hookrightarrow W^{k-1,q}$, and since $(k-1)q > 2$, there is also a continuous pairing $C^{k-1} \times W^{k-1,q} \to W^{k-1,q}$ from Proposition 2.2.5, implying that $D\widehat{Q}_\nu(\hat{u}_\nu)$ is uniformly $W^{k-1,q}$-bounded. In either case, the bounds can be assumed independent of the scaling parameter ϵ, and since both C^{k-1} and $W^{k-1,q}$ admit continuous product pairings with $W^{k-1,p}$, combining this with the product pairing $W^{k,p} \times W^{k-1,p} \to W^{k-1,p}$ then leads to a bound of the form

$$c\left\|D\widehat{Q}_\nu(\hat{u}_\nu)(\beta\, \partial_j \hat{u}_\nu, \partial_t \hat{u}_\nu)\right\|_{W^{k-1,p}} \leqslant c' \|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \cdot \|\partial_t \hat{u}_\nu\|_{W^{k-1,p}}$$

for a constant $c' > 0$ that is independent of ν and ϵ. By (2.14), we can now choose $\epsilon > 0$ small enough so that

$$\|\partial_t \hat{u}_\nu\|_{W^{k-1,p}} \leqslant \|\hat{u}_\nu\|_{W^{k,p}} < \frac{1}{3c'}$$

for all ν.

Conclusion.

Combining the three estimates above for the terms on the right hand side of (2.16) now gives an inequality of the form

$$\|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \leqslant c'' + \frac{2}{3}\|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}},$$

where $c'' > 0$ is the bound obtained in step 1. We conclude $\|\beta\, \partial_j \hat{u}_\nu\|_{W^{k,p}} \leqslant 3c''$, and have thus found a uniform bound for $\|\hat{u}_\nu\|_{W^{k+1,p}(\mathring{\mathbb{D}}_r)}$. ☐

EXERCISE 2.4.14. Use an analogous argument via difference quotients to prove statement (1) in Theorem 2.4.10. *Hint: If you're anything like me, you might get stuck trying to estimate the difference quotient analogues of the terms in* (2.16) *that involve derivatives of* $\widehat{Q}_\nu$. *The difficulty is that this expression was derived using the chain rule for derivatives, and there is no similarly simple chain rule for difference quotients. The trick is to remember that difference quotients only differ from the corresponding derivatives by a remainder term. The remainder will produce extra terms in the difference quotient version of* (2.16)*, but the extra terms can be bounded.*

2.4.3. The nonlinear case: from $W^{1,p} \cap C^0$ to $W^{1,q}$. The proof of Gromov's removable singularity theorem in Chapter 7 will require a stronger variant of Theorem 2.4.10(1) for honest J-holomorphic curves (with no inhomogeneous term), in which the hypothesis $kp > 2$ is relaxed. In the absence of this condition, it is no longer automatic from the Sobolev embedding theorem that our weak solution $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ is continuous, but continuity will be needed in the proof, so we impose it as an explicit hypothesis. For this statement we can also get away with allowing J to be continuous but not differentiable, though the conclusion in that case is correspondingly modest.

THEOREM 2.4.15 (Nonlinear regularity, $kp \leqslant 2$ version). *Assume $1 < p < \infty$, $m \geqslant 0$ is an integer, and $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ is of class C^m. Then every weak solution $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ to the nonlinear Cauchy-Riemann equation $\partial_s u + J(u)\partial_t u = 0$ that is continuous and of class $W^{1,p}$ on $\mathring{\mathbb{D}}$ is also of class $W^{m+1,q}_{\mathrm{loc}}$ on $\mathring{\mathbb{D}}$ for every $q \in (1,\infty)$. In particular, u is of class C^m.*

In light of the bootstrapping result in the previous subsection, Theorem 2.4.15 will follow immediately if we can prove it in the case $m = 0$, where the statement is really that a solution of class $W^{1,p} \cap C^0$ is also of class $W^{1,q}_{\mathrm{loc}}$ for any $q > p$, in particular for some $q > 2$. The following lemma to that effect is adapted from an argument due to Sikorav, cf. [Sik94, Prop. 2.3.6(i)].

LEMMA 2.4.16. *Assume $1 < p, q < \infty$ and J is a continuous almost complex structure on $\mathbb{C}^n$. If $u \in C^0(\mathbb{D}) \cap W^{1,p}(\mathring{\mathbb{D}})$ is a weak solution to the equation $\partial_s u + J(u)\partial_t u = 0$, then u is also of class $W^{1,q}$ on all compact subsets of $\mathring{\mathbb{D}}$.*

PROOF. There is nothing to prove if $q \leqslant p$, so we assume throughout that $q > p$, and that $u : \mathbb{D} \to \mathbb{C}^n$ is in $W^{1,p} \cap C^0$ and is J-holomorphic. Given $z_0 \in \mathring{\mathbb{D}}$, we can assume after changing coodinates on $\mathbb{C}^n$ that $u(z_0) = 0$ and $J(0) = i$. As in the proof of Theorem 2.4.10, we then write $Q := i - J : \mathbb{C}^n \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ and consider rescaled functions of the form

$$\begin{aligned} \widehat{J} : \mathbb{C}^n &\to \mathcal{J}(\mathbb{C}^n), & \widehat{J}(x) &:= J(x/R), \\ \widehat{Q} : \mathbb{C}^n &\to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n), & \widehat{Q}(x) &:= Q(x/R) = i - \widehat{J}(x), \\ \hat{u} : \mathbb{D} &\to \mathbb{C}^n, & \hat{u}(z) &:= Ru(z_0 + \epsilon z), \end{aligned} \tag{2.17}$$

where $\epsilon \in (0,1]$ and $R \geqslant 1$ are constants, so that u is J-holomorphic if and only if $\hat{u}$ satisfies

$$\bar{\partial}\hat{u} - \widehat{Q}(\hat{u})\partial_t \hat{u} = 0. \tag{2.18}$$

Choosing $R \geqslant 1$ sufficiently large makes $\widehat{Q}$ arbitrarily C^0-small on the unit disk $\mathbb{D}^{2n} \subset \mathbb{C}^n$, and after fixing R in this way, we can (since u is continuous) choose $\epsilon \in (0,1]$ sufficiently small to ensure $\hat{u}(\mathbb{D}) \subset \mathbb{D}^{2n}$. In this way, we are allowed to assume

$$\|\widehat{Q}(\hat{u})\|_{C^0(\mathbb{D})} < \delta$$

for some small constant $\delta > 0$, which can always be made smaller if necessary by adjusting R and ϵ. Consider the bounded linear operator

$$D_Q := \bar{\partial} - \widehat{Q}(\hat{u})\partial_t : W^{1,p}(\mathring{\mathbb{D}}, \mathbb{C}^n) \to L^p(\mathring{\mathbb{D}}, \mathbb{C}^n),$$

which has $\hat{u} \in \ker D_Q$ by (2.18), and observe that D_Q is close to $\bar{\partial} : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ in the operator norm if δ is sufficiently small. Fix $r \in (0,1)$ and a smooth compactly supported function $\beta \in C^\infty_0(\mathring{\mathbb{D}})$ with $\beta|_{\mathbb{D}_r} \equiv 1$. The Leibniz rule gives

$$D_Q(\beta\hat{u}) = \left(\bar{\partial}\beta - \widehat{Q}(\hat{u})\partial_t\beta\right)\hat{u} \in C^0(\mathbb{D}),$$

hence $D_Q(\beta\hat{u}) \in L^q(\mathring{\mathbb{D}})$. The rough outline of our argument will now be as follows. Recall from §2.3 that $\bar\partial : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ has a bounded right inverse $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$, given by the convolution $Tf := K * f$ with a fundamental solution $K \in L^1_{\text{loc}}(\mathbb{C})$ to the $\bar\partial$-equation. We saw also via Lemma 2.3.3 that whenever f is smooth with compact support on $\mathbb{C}$, one has $f = T(\bar\partial f)$, so by density, the same is true for every $f \in W^{1,p}_0(\mathring{\mathbb{D}})$. Since $L^q(\mathring{\mathbb{D}}) \subset L^p(\mathring{\mathbb{D}})$ for $q > p$, the same convolution operator restricts to $L^q(\mathring{\mathbb{D}})$ as a bounded right inverse of $\bar\partial : W^{1,q}(\mathring{\mathbb{D}}) \to L^q(\mathring{\mathbb{D}})$, and also satisfies $T\bar\partial(\beta\hat{u}) = \beta\hat{u}$ since $\beta\hat{u} \in W^{1,p}_0(\mathring{\mathbb{D}})$. The fact that $D_Q : W^{1,p} \to L^p$ is close to $\bar\partial$ implies that it also has a bounded right inverse

$$T_Q : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}}),$$

which we expect should similarly restrict to L^q as a right inverse of $D_Q : W^{1,q} \to L^q$ and satisfy $\beta\hat{u} = T_Q D_Q(\beta\hat{u})$. If we can justify that expectation, then it implies $\beta\hat{u} \in W^{1,q}(\mathring{\mathbb{D}})$ and thus $\hat{u} \in W^{1,q}(\mathring{\mathbb{D}}_r)$, as we've already seen that $D_Q(\beta\hat{u})$ is in $L^q(\mathring{\mathbb{D}})$. The consequence for the original map $u \in W^{1,p}(\mathring{\mathbb{D}})$ will be that its restriction to a sufficiently small disk around the arbitrarily chosen point $z_0 \in \mathring{\mathbb{D}}$ is of class $W^{1,q}$.

To put this discussion on solid ground, let us write down $T_Q : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$ more explicitly. The relation $\bar\partial \circ T = \mathbb{1}$ gives

$$D_Q \circ T = \mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T : L^p(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}}),$$

and this operator is clearly invertible if δ is sufficiently small; note that the necessary threshold for δ depends only on the norm of $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$, and not in any way on u, ϵ or R. In fact, we can also assume (possibly after shrinking δ further) that $\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T$ is an invertible operator on $L^q(\mathring{\mathbb{D}})$. A natural definition for T_Q is then

$$T_Q := T\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)^{-1} : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}}),$$

which has the desired property of restricting to $L^q(\mathring{\mathbb{D}})$ as a bounded right inverse of $D_Q : W^{1,q}(\mathring{\mathbb{D}}) \to L^q(\mathring{\mathbb{D}})$. Now using the relations $T\bar\partial(\beta\hat{u}) = \beta\hat{u}$ and $\bar\partial T = \mathbb{1}$, we compute,

$$\begin{aligned}
T_Q D_Q(\beta\hat{u}) &= T\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)^{-1}(\bar\partial - \widehat{Q}(\hat{u})\partial_t)(\beta\hat{u}) \\
&= T\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)^{-1}(\bar\partial - \widehat{Q}(\hat{u})\partial_t)T\bar\partial(\beta\hat{u}) \\
&= T\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)^{-1}\left(\bar\partial(\beta\hat{u}) - \widehat{Q}(\hat{u})\partial_t T\bar\partial(\beta\hat{u})\right) \\
&= T\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)^{-1}\left(\mathbb{1} - \widehat{Q}(\hat{u})\partial_t \circ T\right)\bar\partial(\beta\hat{u}) \\
&= T\bar\partial(\beta\hat{u}) = \beta\hat{u}.
\end{aligned}$$

This validates the argument outlined above: since $D_Q(\beta\hat{u})$ is in both L^p and L^q, $\beta\hat{u} = T_Q D_Q(\beta\hat{u})$ is in both $W^{1,p}$ and $W^{1,q}$, proving the first statement in the lemma. □

EXERCISE 2.4.17. Adapt the argument in the proof above to establish the following variant of Theorem 2.4.10(2b): for a C^m_{loc}-convergent sequence of almost complex structures $J_\nu \to J$ with $m \geqslant 0$, any C^0_{loc}-convergent sequence u_ν of J_ν-holomorphic curves that are also in $W^{1,p}_{\mathrm{loc}}$ for some $p > 1$ actually converges in $W^{m+1,q}_{\mathrm{loc}}$ for every $q \in (1, \infty)$. In particular, C^0_{loc}-convergence of u_ν implies C^m_{loc}-convergence.

REMARK 2.4.18. Why is there no variant of Theorem 2.4.10(2a) for $kp \leqslant 2$? Well, the first step in the proof of Theorem 2.4.10(2a) was to use the compactness of the Sobolev embedding $W^{k,p} \hookrightarrow C^0$ to replace u_ν with a C^0-convergent subsequence, without which the local rescaling trick in that proof would not have worked. If every $W^{1,p}$-bounded sequence similarly had a C^0-convergent subsequence when $p \leqslant 2$, then we could plug it into Exercise 2.4.17 and conclude that there are uniform $W^{1,q}$-bounds for some $q > 2$, so that Theorem 2.4.10(2a) would then apply. But indeed, the phenomenon of "bubbling" will demonstrate clearly in Chapter 7 that uniform $W^{1,2}$-bounds do not guarantee a C^0-convergent subsequence.

2.5. Linear local existence and the similarity principle

The following lemma can be applied in the case $A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$ to prove the aforementioned standard fact that complex-linear Cauchy-Riemann type operators induce holomorphic structures on vector bundles. The version with weakened regularity will be applied below to prove a useful "unique continuation" result about solutions to $(\bar\partial + A)f = 0$ in the real-linear case.

LEMMA 2.5.1. *Assume $2 < p < \infty$ and $A \in L^p(\mathring{\mathbb{D}}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$. Then for sufficiently small $\epsilon > 0$, the problem*

$$\bar\partial u + Au = 0$$
$$u(0) = u_0$$

has a solution $u \in W^{1,p}(\mathring{\mathbb{D}}_\epsilon, \mathbb{C}^n)$.

REMARK 2.5.2. Note that $u : \mathring{\mathbb{D}}_\epsilon \to \mathbb{C}^n$ in the above statement is only a *weak* solution to $\bar\partial u + Au = 0$, as it is not necessarily differentiable, but by the Sobolev embedding theorem, it is at least continuous.

PROOF OF LEMMA 2.5.1. The main idea is that if we take $\epsilon > 0$ sufficiently small, then the restriction of $\bar\partial + A$ to $\mathring{\mathbb{D}}_\epsilon$ can be regarded as a small perturbation of $\bar\partial$ in the space of bounded linear operators $W^{1,p} \to L^p$. Since the latter has a bounded right inverse by Theorem 2.3.1, the same will be true for the perturbation.

Since $p > 2$, the Sobolev embedding theorem implies that functions $u \in W^{1,p}$ are also continuous and bounded by $\|u\|_{W^{1,p}}$, thus we can define a bounded linear operator

$$\Phi : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n : u \mapsto (\bar\partial u, u(0)).$$

Theorem 2.3.1 implies that this operator is also surjective and has a bounded right inverse, namely

$$L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n \to W^{1,p}(\mathring{\mathbb{D}}) : (f, u_0) \mapsto Tf - Tf(0) + u_0,$$

where $T : L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$ is a right inverse of $\bar\partial$. Thus any operator sufficiently close to Φ in the norm topology also has a right inverse. Now define $\chi_\epsilon : \mathbb{D} \to \mathbb{R}$ to be the function that equals 1 on $\mathbb{D}_\epsilon$ and 0 outside of it, and let

$$\Phi_\epsilon : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}}) \times \mathbb{C}^n : u \mapsto ((\bar{\partial} + \chi_\epsilon A)u, u(0)).$$

To see that this is a bounded operator, it suffices to check that $W^{1,p} \to L^p : u \mapsto Au$ is bounded if $A \in L^p$; indeed,

$$\|Au\|_{L^p} \leqslant \|A\|_{L^p}\|u\|_{C^0} \leqslant c\|A\|_{L^p}\|u\|_{W^{1,p}},$$

again using the Sobolev embedding theorem. Now by this same trick, we find

$$\|\Phi_\epsilon u - \Phi u\| = \|\chi_\epsilon A u\|_{L^p(\mathring{\mathbb{D}})} \leqslant c\|A\|_{L^p(\mathring{\mathbb{D}}_\epsilon)}\|u\|_{W^{1,p}(\mathring{\mathbb{D}})},$$

thus $\|\Phi_\epsilon - \Phi\|$ is small if ϵ is small, and it follows that in this case Φ_ϵ is surjective. Our desired solution is therefore the restriction of any $u \in \Phi_\epsilon^{-1}(0, u_0)$ to $\mathring{\mathbb{D}}_\epsilon$. □

Here is a corollary, which says that every solution to a real-linear Cauchy-Riemann type equation looks locally like a holomorphic function in some *continuous* local trivialization.

THEOREM 2.5.3 (Similarity principle). *Suppose $A : \mathbb{D} \to \mathrm{End}_\mathbb{R}(\mathbb{C}^n)$ is of class L^p for some $p > 2$ and $u \in W^{1,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ is a weak solution to the equation $\bar{\partial}u + Au = 0$ with $u(0) = 0$. Then for sufficiently small $\epsilon > 0$, there exist maps $\Phi \in C^0(\mathbb{D}_\epsilon, \mathrm{End}_\mathbb{C}(\mathbb{C}^n))$ and $f \in C^\infty(\mathring{\mathbb{D}}_\epsilon, \mathbb{C}^n)$ such that*

$$u(z) = \Phi(z)f(z), \qquad \bar{\partial}f = 0, \qquad \text{and} \qquad \Phi(0) = \mathbb{1}.$$

PROOF. The solution $u : \mathring{\mathbb{D}} \to \mathbb{C}^n$ is necessarily continuous and bounded, by the Sobolev embedding theorem. Choose a function $C : \mathbb{D} \to \mathrm{End}_\mathbb{C}(\mathbb{C}^n)$ satisfying $C(z)u(z) = A(z)u(z)$ and $|C(z)| \leqslant |A(z)|$ for almost every $z \in \mathbb{D}$. Then $C \in L^p(\mathring{\mathbb{D}}, \mathrm{End}_\mathbb{C}(\mathbb{C}^n))$ and u is a weak solution to $(\bar{\partial} + C)u = 0$. Note that even if A were assumed to be smooth, we do not yet know anything about the zero set of u and thus could not assume C is continuous, though we have no trouble achieving $C \in L^p(\mathring{\mathbb{D}})$ for some $p > 2$.

Since $\bar{\partial} + C$ is now complex linear, we can use Lemma 2.5.1 to find n weak solutions of class $W^{1,p}$ to $(\bar{\partial}+C)v = 0$ on $\mathring{\mathbb{D}}_\epsilon$ that define the standard complex basis of $\mathbb{C}^n$ at 0, and these solutions are continuous by the Sobolev embedding theorem. This gives rise to a map $\Phi \in W^{1,p}(\mathring{\mathbb{D}}_\epsilon, \mathrm{End}_\mathbb{C}(\mathbb{C}^n))$ that satisfies $(\bar{\partial} + C)\Phi = 0$ in the sense of distributions and $\Phi(0) = \mathbb{1}$. Since Φ is continuous, we can assume without loss of generality that $\Phi(z)$ is invertible everywhere on $\mathring{\mathbb{D}}_\epsilon$. Setting $f := \Phi^{-1}u : \mathring{\mathbb{D}}_\epsilon \to \mathbb{C}^n$, the Leibniz rule then implies

$$0 = (\bar{\partial} + C)u = (\bar{\partial} + C)(\Phi f) = \left[(\bar{\partial} + C)\Phi\right] f + \Phi(\bar{\partial}f) = \Phi(\bar{\partial}f).$$

Note that the use of the Leibniz rule in this situation is justified by Proposition A.2.4 in light of the continuous product pairing $W^{1,p} \times W^{1,p} \to W^{1,p}$. It follows that $\bar{\partial}f = 0$, and f is smooth by Lemma 2.4.6. □

COROLLARY 2.5.4 (Unique continuation). *Suppose $\mathbf{D}$ is a linear Cauchy-Riemann type operator on a vector bundle E over a connected Riemann surface, and $\eta \in \Gamma(E)$ satisfies $\mathbf{D}\eta = 0$. Then either η is identically zero or its zeroes are isolated.* □

The similarity principle also has many nice applications for the nonlinear Cauchy-Riemann equation. Here is another "unique continuation" type result for the non-linear case.

PROPOSITION 2.5.5. *Suppose J is a smooth almost complex structure on $\mathbb{C}^n$ and $u, v : \mathring{\mathbb{D}} \to \mathbb{C}^n$ are smooth J-holomorphic curves such that $u(0) = v(0) = 0$ and u and v have matching partial derivatives of all orders at 0. Then $u \equiv v$ on a neighborhood of 0.*

PROOF. Let $h = v - u : \mathring{\mathbb{D}} \to \mathbb{C}^n$. We have

$$\partial_s u + J(u(z))\partial_t u = 0 \tag{2.19}$$

and

$$\begin{aligned} \partial_s v + J(u(z))\partial_t v &= \partial_s v + J(v(z))\partial_t v + [J(u(z)) - J(v(z))]\,\partial_t v \\ &= -\left[J(u(z) + h(z)) - J(u(z))\right]\partial_t v \\ &= -\left(\int_0^1 \frac{d}{d\tau} J(u(z) + \tau h(z))\,d\tau\right)\partial_t v \\ &= -\left(\int_0^1 dJ(u(z) + \tau h(z)) \cdot h(z)\,d\tau\right)\partial_t v =: -A(z)h(z), \end{aligned} \tag{2.20}$$

where the last step defines a smooth family of linear maps $A(z) \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. Subtracting (2.19) from (2.20) gives the linear equation

$$\partial_s h(z) + \bar{J}(z)\partial_t h(z) + A(z)h(z) = 0,$$

where $\bar{J}(z) := J(u(z))$. This is a linear Cauchy-Riemann type equation on a trivial complex vector bundle over $\mathring{\mathbb{D}}$ with complex structure $\bar{J}(z)$ on the fiber at z. The similarity principle thus implies $h(z) = \Phi(z)f(z)$ near 0 for some holomorphic function $f(z) \in \mathbb{C}^n$ and some continuous map $\Phi(z) \in \mathrm{GL}(2n, \mathbb{R})$ representing a change of trivialization. Now if h has vanishing derivatives of all orders at 0, Taylor's formula implies

$$\lim_{z \to 0} \frac{|\Phi(z)f(z)|}{|z|^k} = 0$$

for all $k \in \mathbb{N}$, so f must also have a zero of infinite order and thus $f \equiv 0$. □

REMARK 2.5.6. For most applications of the similarity principle, the zeroth-order term $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ can be assumed smooth, but it is occasionally useful to know that weaker regularity hypotheses are also sufficient. One situation that arises very naturally in SFT, for instance, is when the equation $(\bar{\partial} + A)u = 0$ on $(\mathbb{D}, i)$ is derived from a similar equation on the half-cylinder $([0,\infty) \times S^1, i)$ via the biholomorphic transformation $[0,\infty) \times S^1 \to \mathbb{D}\backslash\{0\} : (s,t) \mapsto e^{-2\pi(s+it)}$, in which case the zeroth-order term is defined almost everywhere on $\mathbb{D}$ but may be unbounded near 0. In this context, the condition $A \in L^p$ with $p > 2$ in Theorem 2.5.3 becomes crucial, and the statement turns out to be false without it; see Exercise 4.8.1 for a hint on how to derive explicit counterexamples.

2.6. Simple curves and multiple covers

We now prove a global result about the structure of closed J-holomorphic curves. In Chapter 6 we will be able to generalize it in a straightforward way for punctured holomorphic curves with asymptotically cylindrical behavior.

THEOREM 2.6.1. *Assume (Σ, j) is a closed connected Riemann surface, (W, J) is a smooth almost complex manifold and $u : (\Sigma, j) \to (W, J)$ is a nonconstant pseudoholomorphic curve. Then there exists a factorization $u = v \circ \varphi$, where*

- *$\varphi : (\Sigma, j) \to (\Sigma', j')$ is a holomorphic map of positive degree to another closed and connected Riemann surface (Σ', j');*
- *$v : (\Sigma', j') \to (W, J)$ is a pseudoholomorphic curve which is embedded except at a finite set of self-intersections and non-immersed points.*[8]

Note that holomorphic maps $(\Sigma, j) \to (\Sigma', j')$ of degree 1 are always diffeomorphisms, so the factorization $u = v \circ \varphi$ in this case is just a reparametrization, and u is then called a **simple** curve. In all other cases, $k := \deg(\varphi) \geqslant 2$ and φ is in general a branched cover; we then call u a **k-fold branched cover** of the simple curve v.

The main idea in the proof is to construct Σ' (minus some punctures) explicitly as the image of u after removing finitely many singular points, so that we can take v to be the inclusion $\Sigma' \hookrightarrow W$. The map $\varphi : \Sigma \to \Sigma'$ is then uniquely determined. In order to carry out this program, we need some information on what the image of u can look like near each of its singularities. These come in two types, each type corresponding to one of the lemmas below, both of which should seem immediately plausible if your intuition comes from complex analysis.

LEMMA 2.6.2 (Intersections). *Suppose $u : (\Sigma, j) \to (W, J)$ and $v : (\Sigma', j') \to (W, J)$ are two nonconstant pseudoholomorphic curves with an intersection $u(z) = v(z')$. Then there exist neighborhoods $z \in \mathcal{U} \subset \Sigma$ and $z' \in \mathcal{U}' \subset \Sigma'$ such that*

$$\text{either} \quad u(\mathcal{U}) = v(\mathcal{U}') \qquad \text{or} \qquad u(\mathcal{U}\setminus\{z\}) \cap v(\mathcal{U}') = u(\mathcal{U}) \cap v(\mathcal{U}'\setminus\{z'\}) = \emptyset.$$

□

PROOF IN THE SPECIAL CASE $du(z) \neq 0$. While the proof of this lemma in full generality is somewhat involved, it becomes a simple application of the similarity principle (Theorem 2.5.3) if we additionally asume that either $du(z)$ or $dv(z')$ is nonzero. We can choose holomorphic local coordinates near $z \in \Sigma$ and $z' \in \Sigma'$ and smooth coordinates near $u(z) = v(z') \in W$ so that without loss of generality, $(\Sigma, j) = (\Sigma', j') = (\mathbb{D}, i)$ with $z = z' = 0$, $W = \mathbb{C}^n$ and $u(0) = v(0) = 0$. If $du(0) \neq 0$, then we can also arrange these coordinates so that

$$u(z) = (z, 0) \quad \text{and} \quad J(z, 0) = i;$$

[8] It follows from the Cauchy-Riemann equation that if $u : (\Sigma, j) \to (W, J)$ is J-holomorphic, then at each point $z \in \Sigma$, its first derivative $du(z) : T_z\Sigma \to T_{u(z)}W$ is either injective or trivial. We are referring to points with $du(z) = 0$ as **non-immersed points** of u. The term "critical points" is also commonly used for this condition, but is slightly at odds with the usual definition of that term when $\dim W \geqslant 4$ since, strictly speaking, every point is critical in the sense that $du(z)$ can never be surjective.

indeed, this is a simple matter of restricting u to a smaller disk on which it is an embedding, rescaling to replace the smaller disk with $\mathbb{D}$, then extending the resulting embedding to an embedding $\mathbb{D} \times \mathbb{D}^{2n-2}_\epsilon \hookrightarrow \mathbb{C}^n$ with its derivatives in the normal direction along $\mathbb{D} \times \{0\}$ specified to be complex linear. In these coordinates, for each $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ we have

$$\begin{aligned} J(z,w) - i &= \int_0^1 \frac{d}{d\tau} J(z,\tau w)\, d\tau = \int_0^1 D_2 J(z,\tau w) w \, d\tau = \left(\int_0^1 D_2 J(z,\tau w)\, d\tau \right) w \\ &=: B(z,w) w, \end{aligned}$$

defining a smooth map $B : \mathbb{C}^n \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^{n-1}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$.

Now writing $v(z) = (\varphi(z), f(z)) \in \mathbb{C} \times \mathbb{C}^{n-1}$, the nonlinear Cauchy-Riemann equation for v gives

$$0 = \partial_s v + J(v) \partial_t v = \partial_s v + i\, \partial_t v + [B(\varphi, f) f] \partial_t v,$$

and applying the projection $\pi : \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ to this equation produces

$$0 = \bar{\partial} f + Af,$$

where $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^{n-1})$ is a smooth map defined by

$$A(z)w := \pi[B(\varphi(z), f(z))w] \partial_t v(z).$$

The similarity principle therefore implies that either f vanishes identically near 0 or its zero at the origin is isolated. □

LEMMA 2.6.3 (Branching). *Suppose $u : (\Sigma, j) \to (W, J)$ is a nonconstant pseudoholomorphic curve and $z_0 \in \Sigma$ is a non-immersed point of u. Then a neighborhood $\mathcal{U} \subset \Sigma$ of z_0 can be biholomorphically identified with the unit disk $\mathbb{D} \subset \mathbb{C}$ such that*

$$u(z) = v(z^k) \quad \text{for} \quad z \in \mathbb{D} = \mathcal{U},$$

where $k \in \mathbb{N}$, and $v : \mathbb{D} \to W$ is an injective J-holomorphic map with no non-immersed points except possibly at the origin. □

These two local results follow from a well-known formula of Micallef and White [**MW95**] describing the local behavior of J-holomorphic curves near non-immersed points and their intersections. The proof of that theorem is analytically quite involved, but one can also use an easier "approximate" version, which is proved in [**Wen20**, Appendix B.2] (see Remark 2.8.5 at the end of this chapter for further discussion of this). Since both are closely related to the phenomenon of unique continuation, you will not be surprised to learn that even beyond the "easy" case of Lemma 2.6.2 treated above, the similarity principle plays a role in the proof: the main idea is again to exploit the fact that locally J is always a small perturbation of i, hence the local behavior of J-holomorphic curves is also similar to the integrable case.

PROOF OF THEOREM 2.6.1. Let $\mathrm{Crit}(u) = \{z \in \Sigma \mid du(z) = 0\}$ denote the set of non-immersed points, and define $\Delta \subset \Sigma$ to be the set of all points $z \in \Sigma$ such that there exists $z' \in \Sigma$ and neighborhoods $z \in \mathcal{U} \subset \Sigma$ and $z' \in \mathcal{U}' \subset \Sigma$ with $u(z) = u(z')$ but $u(\mathcal{U} \backslash \{z\}) \cap u(\mathcal{U}' \backslash \{z'\}) = \emptyset$.

The lemmas quoted above imply that both of these sets are discrete. Both are therefore finite, and the set $\dot{\Sigma}' = u(\Sigma \backslash (\mathrm{Crit}(u) \cup \Delta)) \subset W$ is then a smooth submanifold of W with J-invariant tangent spaces, so it inherits a natural complex structure j' for which the inclusion $(\dot{\Sigma}', j') \hookrightarrow (W, J)$ is pseudoholomorphic. We shall now construct a new Riemann surface (Σ', j') from which $(\dot{\Sigma}', j')$ is obtained by removing a finite set of points. Let $\hat{\Delta} = (\mathrm{Crit}(u) \cup \Delta)/\sim$, where two points in $\mathrm{Crit}(u) \cup \Delta$ are defined to be equivalent whenever they have neighborhoods in Σ with identical images under u. Then for each $[z] \in \hat{\Delta}$, the branching lemma provides an injective J-holomorphic map $u_{[z]}$ from the unit disk $\mathbb{D}$ onto the image of a neighborhood of z under u. We define (Σ', j') by

$$\Sigma' = \dot{\Sigma}' \cup_{\Phi} \left(\coprod_{[z] \in \hat{\Delta}} \mathbb{D} \right),$$

where the gluing map Φ is the disjoint union of the maps $u_{[z]} : \mathbb{D} \backslash \{0\} \to \dot{\Sigma}'$ for each $[z] \in \hat{\Delta}$; since this map is holomorphic, the complex structure j' extends from $\dot{\Sigma}'$ to Σ'. Combining the maps $u_{[z]} : \mathbb{D} \to W$ with the inclusion $\dot{\Sigma}' \hookrightarrow W$ now defines a pseudoholomorphic map $v : (\Sigma', j') \to (W, J)$ which restricts to $\dot{\Sigma}'$ as an embedding and otherwise has at most finitely many non-immersed points and double points. Moreover, the restriction of u to $\Sigma \backslash (\mathrm{Crit}(u) \cup \Delta)$ defines a holomorphic map to $(\dot{\Sigma}', j')$ which extends by removal of singularities to a proper holomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ such that $u = v \circ \varphi$. Its holomorphicity implies that it has positive degree. □

2.7. Nonlinear local existence

Another consequence of the local regularity estimates for $\bar{\partial}$ is a nonlinear version of the local existence result in §2.5. One of its important consequences is the basic fact (originally a theorem of Gauss about conformal structures on surfaces) that all almost complex structures on a Riemann surface are integrable. In that context, we will sometimes also make use of the stability property written into Theorem 2.7.1 below: it implies that local holomorphic charts on sufficiently small regions can be perturbed smoothly under small perturbations of the complex structure.

For functions $f(s,t)$ on domains in $\mathbb{C}$ with complex coordinate $z = s + it$, it is often convenient to regard f formally as a function of the variables $z = s + it$ and $\bar{z} = s - it$, so that its partial derivatives are written as complex-linear combinations of

$$\frac{\partial f}{\partial z} := \frac{1}{2} (\partial_s - i \, \partial_t) \, f, \qquad \text{and} \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2} (\partial_s + i \, \partial_t) \, f.$$

Holomorphic functions are thus distinguished by the fact that since $\frac{\partial f}{\partial \bar{z}} \equiv 0$, their derivatives of all orders are fully determined by $\frac{\partial^k f}{\partial z^k}$ for $k \geqslant 0$. It is not hard to show that the latter also holds for J-holomorphic curves $u : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ at any point $z \in \mathbb{D}$ such that $J(u(z)) = i$; this follows by computing higher derivatives of the nonlinear Cauchy-Riemann equation at such a point.

THEOREM 2.7.1. *Assume J is a smooth almost complex structure on $\mathbb{C}^n$ with $J(0) = i$, and $a_1, \dots, a_m \in \mathbb{C}^n$ are constants for some $m \geqslant 0$. Then:*

(1) For any $\epsilon > 0$ sufficiently small, there exists a J-holomorphic map $u : (\mathbb{D}_\epsilon, i) \to (\mathbb{C}^n, J)$ satisfying $u(0) = 0$ and $\frac{\partial^k u}{\partial z^k}(0) = a_k$ for each $k = 1, \dots, m$.

(2) Given a J-holomorphic map $u : (\mathbb{D}_r, i) \to (\mathbb{C}^n, J)$ on a disk of some radius $r > 0$ satisfying $u(0) = 0$, if $x_\nu \in \mathbb{C}^n$ is a sequence converging to 0, $J_\nu \to J$ is a C^∞_{loc}-convergent sequence of almost complex structures on $\mathbb{C}^n$ and $\epsilon > 0$ is sufficiently small, then there also exists for ν sufficiently large a sequence of J_ν-holomorphic maps $u_\nu : (\mathbb{D}_\epsilon, i) \to (\mathbb{C}^n, J_\nu)$ that satisfy $u_\nu(0) = x_\nu$ and are C^∞-convergent to $u|_{\mathbb{D}_\epsilon}$.

REMARK 2.7.2. By an easy modification of the proof below, one could if desired also impose a converging sequence of constraints on finitely many derivatives of the sequence of maps u_ν in the second part of the statement.

REMARK 2.7.3. There is no uniqueness in Theorem 2.7.1, nor should one expect it: in the case $J \equiv i$, specifying $\frac{\partial^k u}{\partial z^k}(0)$ for all $k \geqslant 0$ up to some finite order still leaves an infinite-dimensional space of solutions to $\bar{\partial} u = 0$. On the other hand, specifying these derivatives for *all* $k \geqslant 0$ produces uniqueness but kills existence: there is a unique holomorphic Taylor series that has the correct derivatives, but it might have zero radius of convergence.

There are two main ingredients behind the proof of Theorem 2.7.1. One is the existence of a bounded right inverse to the operator $\bar{\partial} : W^{k,p}(\mathring{\mathbb{D}}) \to W^{k-1,p}(\mathring{\mathbb{D}})$ for every $k \in \mathbb{N}$ and $p \in (1, \infty)$, as provided by the fundamental elliptic estimates of §2.3 and Exercise 2.4.5. The other is the extension of standard differential calculus to functions defined on open subsets of Banach spaces, as presented e.g. in [**Lan93**, **Lan99**]. In particular, the surjectivity of $\bar{\partial}$ will be needed as a hypothesis for applying the implicit function theorem to a differentiable map between open subsets of infinite-dimensional Banach spaces, thereby proving that the zero-set of that map is a differentiable Banach submanifold. We will make considerably more use of that machinery in later chapters, typically in the context of smooth Banach manifolds and Banach space bundles (cf. §8.2). Since it is not entirely trivial in such settings to determine whether certain maps are differentiable, it will be useful to keep the following extension of the C^k-continuity property from Proposition 2.2.5 in mind:

PROPOSITION 2.7.4. *Under the same assumptions as in Proposition 2.2.5, if the open set $\Omega \subset \mathbb{R}^n$ is convex,*[9] *then the map*

$$\Phi : C^{k+r}(\Omega, \mathbb{R}^N) \times W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (f, u) \mapsto f \circ u$$

is of class C^r for each $r \in \mathbb{N}$, and its first partial derivatives are given by

$$D_1\Phi(f, u)g = g \circ u, \qquad D_2\Phi(f, u)v = (Df \circ u)v,$$

where the second expression makes sense due to Propositions 2.2.4 and 2.2.5 and should be understood as the pointwise product of the two $W^{k,p}$-functions $Df \circ u : \mathcal{U} \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and $v : \mathcal{U} \to \mathbb{R}^n$.

[9]The convexity assumption on $\Omega \subset \mathbb{R}^n$ is inessential, and can be relaxed at the cost of more cumbersome notation, cf. the setup for Theorem A.2.6.

PROOF. The main step is to prove that for any fixed $f \in C^{k+1}(\Omega, \mathbb{R}^N)$, the map

$$\Psi_f : W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : u \mapsto f \circ u$$

is differentiable, and its derivative is given by

$$D\Psi_f(u) = \Psi_{Df}(u). \tag{2.21}$$

The latter is a continuous function $W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ by Proposition 2.2.5 since Df is of class C^k. If (2.21) is established, then by induction, Ψ_f is of class C^r whenever f is of class C^{k+r} for some $r \in \mathbb{N}$. The partial derivatives of $\Phi(f, u)$ with respect to f are easier to handle since it is a linear function of f, so for instance the stated formula for $D_1\Phi(f, u)$ is obviously correct and Proposition 2.2.5 implies that it is continuous. In this way, one can proceed inductively to show that all partial derivatives of Φ up to order r exist and are continuous; we will leave the details of this inductive argument as an exercise (cf. [**Wenb**, Lemma 2.12.7]). By a standard theorem in differential calculus (see [**Lan93**, Chapter XIII, Theorem 7.1]), it will follow that Φ is of class C^r.

The proof of (2.21) proceeds as follows. For $\eta \in W^{k,p}(\mathcal{U}, \mathbb{R}^n)$ sufficiently small, we can exploit the convexity of Ω to write

$$\begin{aligned} \Psi_f(u + \eta) &= \Psi_f(u) + [f \circ (u + \eta) - f \circ u] = \Psi_f(u) + \int_0^1 \frac{d}{dt} f \circ (u + t\eta)\, dt \\ &= \Psi_f(u) + \left(\int_0^1 Df \circ (u + t\eta)\, dt \right) \eta \\ &=: \Psi_f(u) + (Df \circ u)\eta + [\theta \circ (u + \eta, u)]\, \eta, \end{aligned} \tag{2.22}$$

where for the last step we define a function $\theta : \Omega \times \Omega \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ by

$$\theta(x, y) := \int_0^1 [Df((1-t)y + tx) - Df(y)]\, dt.$$

This function is of class C^k since f is in C^{k+1}, and Proposition 2.2.5 thus implies that the map

$$\Psi_\theta : W^{k,p}(\mathcal{U}, \Omega \times \Omega) \to W^{k,p}(\mathcal{U}, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)) : (u, v) \mapsto \theta \circ (u, v)$$

is continuous, implying in particular that $\theta \circ (u + \eta, u) = \Psi_\theta(u + \eta, u)$ is $W^{k,p}$-convergent to $\Psi_\theta(u, u) = \theta \circ (u, u) = 0$ as $\eta \to 0$ in $W^{k,p}$. This allows us to rewrite (2.22) as

$$\Psi_f(u + \eta) = \Psi_f(u) + \Psi_{Df}(u)\eta + \Psi_\theta(u + \eta, u)\eta$$

and interpret it as the definition of the derivative of Ψ_f at u, with $\Psi_\theta(u + \eta, u)\eta$ as the remainder term. □

PROOF OF THEOREM 2.7.1. Assume without loss of generality $m \geqslant 1$. We can apply a rescaling trick as in §2.4.2 to zoom in on a neighborhood of the origin in $\mathbb{C}^n$, which has the effect of identifying any smooth almost complex structure J on $\mathbb{C}^n$ satisfying $J(0) = i$ with one that is arbitrarily C^∞-close to the constant complex structure i on any given compact subset. For existence, it therefore suffices to prove the following claim: given any $a_1, \ldots, a_m \in \mathbb{C}^n$, one can find a radius $R > 0$ and a C^∞-small neighborhood $\mathcal{U}$ of i in the space of all smooth almost complex structures on the disk $\mathbb{D}^{2n}_R \subset \mathbb{C}^n$ of radius R, such that for every $J \in \mathcal{U}$ there exists a J-holomorphic map

$$u : (\mathbb{D}, i) \to (\mathring{\mathbb{D}}^{2n}_R, J)$$

satisfying $u(0) = 0$ and $\frac{\partial^k u}{\partial z^k}(0) = a_k$ for all $k = 1, \ldots, m$. To start with, choose $R > 0$ large enough so that the unique holomorphic polynomial of degree m satisfying these conditions at the origin maps $\mathbb{D}$ into $\mathring{\mathbb{D}}^{2n}_R$; this polynomial is then a solution to the above problem for the case $J \equiv i$. Now pick any $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$ and consider the sets

$$\mathcal{M} := \left\{ (J, u) \in C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times W^{k+m,p}(\mathring{\mathbb{D}}, \mathring{\mathbb{D}}^{2n}_R) \;\middle|\; \partial_s u + J(u)\partial_t u = 0 \right\}$$

$$\mathcal{M}(J) := \left\{ u \in W^{k+m,p}(\mathring{\mathbb{D}}, \mathring{\mathbb{D}}^{2n}_R) \;\middle|\; (J, u) \in \mathcal{M} \right\} \qquad \text{for } J \in C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)).$$

We can present $\mathcal{M}$ as the zero set of the map

$$C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times W^{k+m,p}(\mathring{\mathbb{D}}, \mathring{\mathbb{D}}^{2n}_R) \xrightarrow{F} W^{k+m-1,p}(\mathring{\mathbb{D}}, \mathbb{C}^n),$$
$$(J, u) \mapsto \partial_s u + (J \circ u)\partial_t u,$$

which we claim is of class C^1. Indeed, the map $C^{k+m+1} \times W^{k+m,p} \to W^{k+m,p} : (J, u) \mapsto J \circ u$ is in C^1 by Proposition 2.7.4, $u \mapsto \partial_s u$ and $u \mapsto \partial_t u$ are bounded linear maps $W^{k+m,p} \to W^{k+m-1,p}$ and thus smooth, and $(J \circ u, \partial_t u) \mapsto (J \circ u)\partial_t u$ is the continuous bilinear product pairing $W^{k+m,p} \times W^{k+m-1,p} \to W^{k+m-1,p}$, thus also smooth. Whenever $J \equiv i$ and $F(i, u) = 0$, the partial derivative of F with respect to u is

$$D_2 F(i, u) = \bar{\partial} : W^{k+m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n) \to W^{k+m-1,p}(\mathring{\mathbb{D}}, \mathbb{C}^n),$$

which is surjective with a bounded right inverse by Exercise 2.4.5. It follows that $DF(i, u)$ is likewise surjective with a bounded right inverse, so that by the implicit function theorem, a neighborhood of $\{i\} \times \mathcal{M}(i)$ in $\mathcal{M}$ is a C^1-smooth Banach submanifold of $C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times W^{k+m,p}(\mathring{\mathbb{D}}, \mathring{\mathbb{D}}^{2n}_R)$. Since functions of class $W^{k+m,p}$ in $\mathring{\mathbb{D}}$ with $kp > 2$ have well-defined derivatives up to order m at every point, it follows that the map

$$\mathcal{M} \xrightarrow{\pi} C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times \mathbb{C}^{n(m+1)},$$
$$(J, u) \mapsto \left(J, u(0), \frac{\partial u}{\partial z}(0), \ldots, \frac{\partial^m u}{\partial z^m}(0) \right)$$

is of class C^1, and we claim that it is a submersion near $\{i\} \times \mathcal{M}(i)$. Indeed, the tangent space $T_{(i,u)}\mathcal{M}$ is $\ker DF(i, u)$, which contains $\{0\} \oplus \ker D_2 F(i, u) = \{0\} \oplus \ker \bar{\partial}$, i.e. the set of all pairs $(0, f)$ such that $f \in W^{k+m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$ is a holomorphic function. The map

$$\mathcal{M} \to C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) : (J, u) \mapsto J$$

is a submersion near $\{i\} \times \mathcal{M}(i)$ if and only if for every $(i, u) \in \mathcal{M}$ and $Y \in C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$, $\ker DF(i, u)$ contains an element of the form (Y, f) for some $f \in W^{k+m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$. This is true, since $DF(i,u)(Y,f) = D_1F(i,u)Y + D_2F(i,u)f$ and $D_2F(i,u) = \bar{\partial}$ is surjective. Now it suffices to observe that

$$\mathcal{M}(i) \to \mathbb{C}^{n(m+1)} : u \mapsto \left(u(0), \frac{\partial u}{\partial z}(0), \ldots, \frac{\partial^m u}{\partial z^m}(0) \right)$$

is likewise a submersion, because one can find holomorphic functions on $\mathbb{C}$ having arbitrary values for their first m derivatives with respect to z at 0.

Since $R > 0$ was chosen to ensure that $\pi^{-1}(i, 0, a_1, \ldots, a_m)$ is nonempty, the submersion property now enables us to find a neighborhood $\mathcal{U} \subset C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ of i and a C^1-smooth map

$$C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \supset \mathcal{U} \to \mathcal{M} : J \mapsto (J, u_J)$$

such that $\pi(J, u_J) = (J, 0, a_1, \ldots, a_m)$ for all $J \in \mathcal{U}$. For any $J \in \mathcal{U}$ that is also a smooth almost complex structure, the resulting map u_J will then be smooth by elliptic regularity, and the proof of existence is thus complete.

For the result about convergent sequences, the same rescaling trick means that it suffices to prove the result is true with $r = \epsilon = 1$ under the assumption that $u(\mathbb{D}) \subset \mathring{\mathbb{D}}^{2n}_R$ and J is arbitrarily C^∞-close to i on $\mathbb{D}^{2n}_R$. Since rescaling can also be used to make u arbitrarily close to $0 \in W^{k,p}(\mathbb{D}^n, \mathring{\mathbb{D}}^{2n}_R)$, let us assume in particular that $(J, u) \in \mathcal{M}$ lies in the neighborhood of $\{i\} \times \mathcal{M}(i)$ on which π is a submersion. The submersion property then implies the existence of a neighborhood $\mathcal{V} \subset C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times \mathbb{D}^{2n}_R$ of $(J, 0)$ and a C^1-smooth map

$$C^{k+m+1}(\mathbb{D}^{2n}_R, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times \mathbb{D}^{2n}_R \supset \mathcal{V} \to \mathcal{M} : (J', x) \mapsto (J', u_{(J',x)})$$

such that $u_{(J,0)} = u$ and $u_{(J',x)}(0) = x$ for each $(J', x) \in \mathcal{V}$. Given the sequences $J_\nu \to J$ and $x_\nu \to 0$, we can then set $u_\nu := u_{(J_\nu, x_\nu)}$; a priori this converges to u in the $W^{k,p}$-topology on $\mathbb{D}$, so by elliptic regularity, it is also C^∞-convergent on $\mathbb{D}_r$ for every $r < 1$. □

EXERCISE 2.7.5. The standard complex structure i on the cylinder $\mathbb{R} \times S^1$ is defined by $i\partial_s = \partial_t$ and $i\partial_t = -\partial_s$ in the obvious coordinates (s,t). The first-order differential operator $\bar{\partial} = \partial_s + i\partial_t$ is thus defined for complex-valued functions on $\mathbb{R} \times S^1$ or any subset thereof. Consider a compact subset of the form

$$Z := [a,b] \times S^1 \subset \mathbb{R} \times S^1$$

for real numbers $a < b$.

(a) Show that for each $k \in \mathbb{N}$ and $p \in (1, \infty)$, the operator $\bar{\partial} : W^{k,p}(Z) \to W^{k-1,p}(Z)$ is surjective with a bounded right inverse. *Hint: (Z, i) is biholomorphically equivalent to a subset of the unit disk.*
(b) Prove the following cylindrical analogue of the stability statement in Theorem 2.7.1: for any C^∞_{loc}-convergent sequence $j_\nu \to i$ of complex structures on $\mathbb{R} \times S^1$, there exists for large ν a sequence of holomorphic embeddings $(Z, i) \hookrightarrow (\mathbb{R} \times S^1, j_\nu)$ that is C^∞-convergent to the obvious inclusion $Z \hookrightarrow \mathbb{R} \times S^1$.

2.8. The nonlinear equation on push-offs

In §2.1 we derived the linearized Cauchy-Riemann operator $\mathbf{D}_u : \Gamma(u^*TW) \to \Omega^{0,1}(\Sigma, u^*TW)$ for a J-holomorphic curve $u : (\Sigma, j) \to (W, J)$ by linearizing the nonlinear operator $\bar\partial_J u := du + J(u) \circ du \circ j$ at u, where $\bar\partial_J$ is imagined as a section of a vector bundle $\mathcal{E} \to \mathcal{B}$ over an infinite-dimensional manifold $\mathcal{B}$ consisting of maps $\Sigma \to W$. In the big picture, $\mathbf{D}_u$ is not just a first-order *approximation* of $\bar\partial_J$; we will show in this section that for nearby maps of the form $u' = \exp_u(\eta) : \Sigma \to W$ with $\eta \in \Gamma(u^*TW)$ sufficiently small, the nonlinear equation $\bar\partial_J u' = 0$ implies a corresponding linear equation $\mathbf{D}'_u\eta = 0$ for some Cauchy-Riemann type operator $\mathbf{D}'_u$ that is a small perturbation of $\mathbf{D}_u$. This is useful for the following reason: we will devote considerable effort in the next few chapters to studying the properties of linear Cauchy-Riemann type equations and their solutions. The ability to rewrite $\bar\partial_J u' = 0$ as $\mathbf{D}'_u\eta = 0$ means that many of the linear results we prove imply corresponding results for the nonlinear equation.

There is a basic idea from first-year analysis in the background: if $f : \mathcal{U} \to \mathbb{R}^m$ is a smooth map on some open domain $\mathcal{U} \subset \mathbb{R}^n$ and $f(x) = 0$ for some $x \in \mathcal{U}$, then writing $\mathbf{D}_x := df(x) : \mathbb{R}^n \to \mathbb{R}^m$ for its derivative and taking $h \in \mathbb{R}^n$ sufficiently small, one has

$$f(x+h) = \int_0^1 \frac{d}{dt} f(x+th)\,dt = \int_0^1 df(x+th)h\,dt = \left(\int_0^1 df(x+th)\,dt\right) h =: \mathbf{D}'_x h,$$

where the integral in parentheses defines a linear operator $\mathbf{D}'_x : \mathbb{R}^n \to \mathbb{R}^m$ that can be assumed arbitrarily close to $\mathbf{D}_x = df(x) = \int_0^1 df(x)\,dt$ if h is sufficiently small. A slightly subtle point here is that the definition of the map $\mathbf{D}'_x : \mathbb{R}^n \to \mathbb{R}^m$ also depends on h, but in most applications this is immaterial, because we are interested in drawing conclusions about nearby solutions $x + h$ to the nonlinear equation $f(x + h) = 0$ from general theorems about solutions to linear equations of the form $\mathbf{D}'_x h = 0$, and h is such a solution. But before we can carry out this type of computation for the nonlinear section $\bar\partial_J : \mathcal{B} \to \mathcal{E}$ and its linearization $\mathbf{D}_u$ at $u \in \bar\partial_J^{-1}(0)$, we have two problems: first, f in the computation above was a function valued in a single vector space, not a section of a vector bundle, and if it had been the latter, we would at least have needed to choose a connection to identify all the fibers in order for the computation to make sense. There is a more serious problem, however, that is unique to our infinite-dimensional setting: one could use a similarly general argument (with the aid of a connection) to rewrite $\bar\partial_J(\exp_u \eta)$ as $\mathbf{D}'_u\eta$ for some linear operator $\mathbf{D}'_u$, but from this perspective, it would not be obvious whether $\mathbf{D}'_u$ is also a *Cauchy-Riemann* type operator. That is something we will need to know, because the linear results proved in the next few chapters are valid specifically for Cauchy-Riemann type operators, and not necessarily for arbitrary small perturbations of them in the space of *all* bounded linear operators. The secret is to apply the integration trick used above to the finite-dimensional geometric data in the Cauchy-Riemann equation, rather than applying it directly to the infinite-dimensional section $\bar\partial_J$.

To set up the first result, suppose $u : (\Sigma, j) \to (W, J)$ is a J-holomorphic curve and write

$$E := u^*TW,$$

so E is a complex vector bundle over Σ with complex structure $J(u(z))$ at $z \in \Sigma$. It will be convenient in the following to write elements of the total space of vector bundles such as E as pairs (z, X) where $z \in \Sigma$ and X belongs to the fiber $E_z = T_{u(z)}W$, thus the zero-section in E consists of all points of the form $(z, 0) \in E$, and at any such point there is a canonical isomorphism

$$T_{(z,0)}E = T_z\Sigma \oplus E_z, \tag{2.23}$$

the first factor being the tangent space to the zero-section, and the second the vertical subspace of $T_{(z,0)}E$. Suppose $\mathcal{O} \subset u^*TW$ is a fiberwise-convex neighborhood of the zero-section, and

$$\Psi : \mathcal{O} \to W$$

is a smooth map whose restriction to the zero-section is u and whose derivative there restricts to the second factor in (2.23) as the identity map $E_z \to T_{u(z)}W$. One obvious way to define Ψ is as $\Psi(z, X) = \exp_{u(z)} X$ for a choice of connection on W, but the actual definition will be irrelevant in what follows. We will denote the set of sections of E with image in $\mathcal{O}$ by

$$\Gamma(\mathcal{O}) := \left\{ \eta \in \Gamma(E) \;\middle|\; (z, \eta(z)) \in \mathcal{O} \text{ for all } z \in \Sigma \right\}.$$

THEOREM 2.8.1. *Given a compact Riemann surface (Σ, j), a J-holomorphic curve $u : (\Sigma, j) \to (W, J)$ and a map $\Psi : \mathcal{O} \to W$ as described above, after possibly shrinking the neighborhood $\mathcal{O} \subset u^*TW$ of the zero-section, one can associate to each J-holomorphic curve $u' : (\Sigma, j) \to (W, J)$ of the form*

$$u'(z) = \Psi(z, \eta(z)), \qquad \eta \in \Gamma(\mathcal{O}) \subset \Gamma(u^*TW)$$

a linear Cauchy-Riemann type operator $\mathbf{D}$ on u^*TW such that $\mathbf{D}\eta = 0$. Moreover, if $\eta_k \to 0$ is a C^∞-convergent sequence of sections in $\Gamma(\mathcal{O})$ such that the maps $u'_k(z) := \Psi(z, \eta_k(z))$ are J-holomorphic curves $u'_k : (\Sigma, j) \to (W, J)$ for all k, then the associated linear Cauchy-Riemann type operators $\mathbf{D}_k$ satisfying $\mathbf{D}_k\eta_k = 0$ are also C^∞-convergent, with $\mathbf{D}_k \to \mathbf{D}_u$.

It is perhaps worth emphasizing what Theorem 2.8.1 does *not* say: there is no single linear operator $\mathbf{D}$ that makes the equations $\bar\partial_J u' = 0$ and $\mathbf{D}\eta = 0$ equivalent for *all* maps of the form $u'(z) = (z, \eta(z))$ with $\eta \in \Gamma(\mathcal{O})$. Instead, the operator $\mathbf{D}$ in this statement is determined by the specific solution u', and other nearby J-holomorphic curves of the form $u'_1(z) = (z, \eta_1(z))$ will not need to satisfy $\mathbf{D}\eta_1 = 0$. The point of this result is rather that we can deduce properties of the specific solution u' from the properties of solutions to the linear equation $\mathbf{D}\eta = 0$.

Before launching into the proof, we state a slightly more elaborate variant that will also come in useful. The idea is to consider a more general class of curves with images near that of u, written in the form

$$u' : (\Sigma', j') \to (W, J), \qquad u'(z) = \Psi(\varphi(z), \eta(z)),$$

where (Σ', j') is another Riemann surface, $\varphi : \Sigma' \to \Sigma$ is a smooth map that accounts for deviations of u' from u in directions tangential to u, and η is a vector field along $u \circ \varphi$ that points in directions normal to u. To make this more precise, we assume u^*TW is endowed with a splitting

$$u^*TW = T_u \oplus N_u$$

of complex vector bundles, where $T_u \subset u^*TW$ is a line bundle such that

$$\operatorname{im} du(z) \subset (T_u)_z \qquad \text{for all } z \in \Sigma.$$

If u is an immersion, then the bundle $T_u \subset u^*TW$ obviously exists and is uniquely determined by this condition; we will show in Chapter 15 that this is in fact true for *every* locally nonconstant J-holomorphic curve, even if $du(z)$ vanishes at isolated points. For now, we shall just assume the splitting exists, and refer to the complementary complex subbundle $N_u \subset u^*TW$ as the **normal bundle** of u. By Exercise 2.1.5, writing the linearized Cauchy-Riemann operator $\mathbf{D}_u$ in block form

$$\mathbf{D}_u = \begin{pmatrix} \mathbf{D}_u^T & \mathbf{D}_u^{TN} \\ \mathbf{D}_u^{NT} & \mathbf{D}_u^N \end{pmatrix} : \Gamma(T_u) \oplus \Gamma(N_u) \to \Omega^{0,1}(\Sigma, T_u) \oplus \Omega^{0,1}(\Sigma, N_u)$$

with respect to this splitting gives rise to linear Cauchy-Riemann type operators $\mathbf{D}_u^T$ and $\mathbf{D}_u^N$ on T_u and N_u respectively.

DEFINITION 2.8.2. The operator $\mathbf{D}_u^N : \Gamma(N_u) \to \Omega^{0,1}(\Sigma, N_u)$ described above is called the **normal Cauchy-Riemann operator** of u.

Given the neighborhood $\mathcal{O} \subset u^*TW$ of the zero-section in the statement of Theorem 2.8.1, let us denote the resulting neighborhood in the total space of the normal bundle by

$$\mathcal{O}^N := \mathcal{O} \cap N_u \subset N_u,$$

and also define pullbacks with respect to a smooth map $\varphi : \Sigma' \to \Sigma$ by

$$\varphi^*\mathcal{O} := \left\{ (z, X) \;\middle|\; z \in \Sigma' \text{ and } (\varphi(z), X) \in \mathcal{O} \right\} \subset \varphi^* u^* TW,$$
$$\varphi^*\mathcal{O}^N := \left\{ (z, X) \;\middle|\; z \in \Sigma' \text{ and } (\varphi(z), X) \in \mathcal{O}^N \right\} \subset \varphi^* N_u.$$

These give rise to corresponding sets of sections $\Gamma(\varphi^*\mathcal{O}) \subset \Gamma(\varphi^*u^*TW)$, $\Gamma(\varphi^*\mathcal{O}^N) \subset \Gamma(\varphi^*N_u)$. We recall from Exercise 2.1.4 the notion of the pullback $\varphi^*\mathbf{D}$ of a linear Cauchy-Riemann type operator $\mathbf{D}$ via a holomorphic map φ of Riemann surfaces.

THEOREM 2.8.3. *Given compact Riemann surfaces (Σ, j) and (Σ', j'), a nonconstant J-holomorphic curve $u : (\Sigma, j) \to (W, J)$, and a map $\Psi : \mathcal{O} \to W$ and splitting $u^*TW = T_u \oplus N_u$ as described above, after possibly shrinking the neighborhood $\mathcal{O} \subset u^*TW$ of the zero-section, one can associate to any J-holomorphic curve $u' : (\Sigma', j') \to (W, J)$ of the form*

$$u'(z) = \Psi(\varphi(z), \eta(z)), \qquad \varphi \in C^\infty(\Sigma', \Sigma), \;\; \eta \in \Gamma(\varphi^*\mathcal{O}^N) \subset \Gamma(\varphi^*N_u)$$

a linear Cauchy-Riemann type operator $\mathbf{D}^N$ on φ^*N_u such that $\mathbf{D}^N\eta = 0$. Moreover, this association has the following continuity property: suppose we are given C^∞-convergent sequences of

- *complex structures $j'_k \to j'$ on Σ',*

- *smooth maps* $\varphi_k \to \varphi$ *from* Σ' *to* Σ, *and*
- *sections* $\eta_k \to 0$ *of* $\varphi_k^* E$,

such that $\varphi : (\Sigma', j') \to (\Sigma, j)$ *is holomorphic and* $u'_k(z) := \Psi(\varphi_k(z), \eta_k(z))$ *defines a sequence of* J*-holomorphic curves* $u'_k : (\Sigma', j'_k) \to (W, J)$. *Then the associated linear Cauchy-Riemann type operators* $\mathbf{D}_k^N$ *on* $\varphi_k^* N_u$ *over* (Σ', j'_k) *satisfying* $\mathbf{D}_k^N \eta_k = 0$ *are also* C^∞*-convergent, with* $\mathbf{D}_k^N \to \varphi^* \mathbf{D}^N$.

REMARK 2.8.4. Theorems 2.8.1 and 2.8.3 can also be applied in some situations where Σ and Σ' are not compact; notably, we will later use them in cases where these are half-cylinders of the form $([R, \infty) \times S^1, i)$. A crucial detail in that setting is that the ambient almost complex structure is translation-invariant, and therefore satisfies global C^∞-bounds along the image of u, so that any quantitative convergence estimate on a domain of the form $[N - r, N + r] \times S^1 \subset [R, \infty) \times S^1$ becomes equally valid for any N, and is therefore valid on the entire half-cylinder.

There is a common setup for the proofs of Theorems 2.8.1 and 2.8.3. We continue to abbreviate $E := u^*TW$ where $u : (\Sigma, j) \to (W, J)$ is a J-holomorphic curve, and we assume $\mathcal{O} \subset E$ is a fiberwise-convex neighborhood of the zero-section $\Sigma \subset E$, $\Psi : \mathcal{O} \to W$ is a smooth map satisfying $\Psi|_\Sigma = u$ as described in the paragraph preceding Theorem 2.8.1, (Σ', j') is a Riemann surface, $\varphi : \Sigma' \to \Sigma$ is a smooth map, η is a section of $\varphi^* E$ such that $(\varphi(z), \eta(z)) \in \mathcal{O}$ for every $z \in \Sigma'$, and $u' : \Sigma' \to W$ is the map $u'(z) = \Psi(\varphi(z), \eta(z))$. Choose a linear connection ∇ on E; this extends (2.23) to a splitting

$$T_{(z,v)} E \cong T_z \Sigma \oplus E_z = T_z \Sigma \oplus T_{u(z)} W \tag{2.24}$$

at every point $(z, v) \in E$, where the factor $T_z\Sigma$ corresponds to the horizontal subspace and E_z to the vertical subspace. We shall always use this splitting when talking about tangent vectors to the total space of E, so for instance, the derivative of a path $\gamma(t) = (x(t), v(t)) \in E$ is now given by

$$\dot{\gamma}(t) = (\dot{x}(t), \nabla_t v(t)) \in T_{x(t)} \Sigma \oplus E_{x(t)} = T_{\gamma(t)} E.$$

We can now write the derivative of $\Psi : \mathcal{O} \to W$ at a point (z, X) in block form as

$$d\Psi(z, X) = \begin{pmatrix} \alpha(z, X) & \beta(z, X) \end{pmatrix} : T_z \Sigma \oplus E_z \to T_{\Psi(z,X)} W,$$

and observe that the stated assumptions on Ψ imply

$$\alpha(z, 0) = du(z) : T_z \Sigma \to E_z, \qquad \text{and} \qquad \beta(z, 0) = \mathbb{1} : E_z \to E_z.$$

We shall assume for the rest of the argument that the neighborhood $\mathcal{O} \subset E$ is small enough so that $\beta(z, X)$ is invertible for every $(z, X) \in \mathcal{O}$. In this case, we can define another smooth function F on $\mathcal{O}$ by

$$F(z, X) := \beta(z, X)^{-1} \circ \alpha(z, X) \in \operatorname{Hom}_{\mathbb{R}}(T_z \Sigma, E_z),$$

and it satisfies $F(z, 0) = du(z)$. This function—strictly speaking, it is a section of some vector bundle—is one of several we shall encounter with the property that for each fixed $z \in \Sigma$, the function on $\mathcal{O}_z := \mathcal{O} \cap E_z$ defined by $v \mapsto F(z, v)$ takes values in a fixed vector space, in this particular case $\operatorname{Hom}_{\mathbb{R}}(T_z \Sigma, E_z)$. Whenever this happens, the convexity of $\mathcal{O}_z$ allows us to apply the fundamental theorem of calculus and write

$$\begin{aligned} F(z,X) &= F(z,0) + \int_0^1 \frac{d}{d\tau} F(z,\tau X)\,d\tau = F(z,0) + \int_0^1 D_2 F(z,\tau X) X\,d\tau \\ &=: F(z,0) + F'(z,X)X, \end{aligned}$$

where we have defined a new smooth function F' on $\mathcal{O}$ by

$$F'(z,X) := \int_0^1 D_2 F(z,\tau X)\,d\tau \in \operatorname{Hom}_{\mathbb{R}}\big(E_z, \operatorname{Hom}_{\mathbb{R}}(T_z\Sigma, E_z)\big),$$

so in particular, $F'(z,0) = D_2F(z,0)$ is the derivative of F in vertical directions at a point in the zero-section. In this particular example, the result is the formula

$$F(z,X) = du(z) + F'(z,X)X.$$

Here is another important example in which this trick can be applied: we can smoothly associate to each $(z,X) \in \mathcal{O}$ a complex structure on E_z defined by

$$\widehat{J}(z,X) := \beta(z,X)^{-1} \circ J(\Psi(z,X)) \circ \beta(z,X) \in \operatorname{End}_{\mathbb{R}}(E_z),$$

and since $\widehat{J}(z,0) = J(u(z))$, we then have

$$\widehat{J}(z,X) = J(u(z)) + \widehat{J}'(z,X)X,$$

where $\widehat{J}'(z,0)$ is the vertical derivative of $\widehat{J}$ at a point in the zero-section. We will use analogous notation in some other examples below.

Applying the nonlinear Cauchy-Riemann operator to the map $u'(z) = (\varphi(z), \eta(z))$ now gives

(2.25)

$$\begin{aligned} \bar\partial_J u' &= du' + J(u') \circ du' \circ j' \\ &= d\Psi(\varphi,\eta) \circ d(\varphi,\eta) + J(u') \circ d\Psi(\varphi,\eta) \circ d(\varphi,\eta) \circ j' \\ &= \begin{pmatrix} \alpha(\varphi,\eta) & \beta(\varphi,\eta) \end{pmatrix} \begin{pmatrix} d\varphi \\ \nabla\eta \end{pmatrix} + J(u') \begin{pmatrix} \alpha(\varphi,\eta) & \beta(\varphi,\eta) \end{pmatrix} \begin{pmatrix} d\varphi \circ j' \\ \nabla\eta \circ j' \end{pmatrix} \\ &= \alpha(\varphi,\eta) \circ d\varphi + \beta(\varphi,\eta) \circ \nabla\eta + J(u') \circ \alpha(\varphi,\eta) \circ d\varphi \circ j' \\ &\qquad + J(u') \circ \beta(\varphi,\eta) \circ \nabla\eta \circ j' \\ &= \beta(\varphi,\eta) \circ \Big[\nabla\eta + \widehat{J}(\varphi,\eta) \circ \nabla\eta \circ j' \\ &\qquad + F(\varphi,\eta) \circ d\varphi + \widehat{J}(\varphi,\eta) \circ F(\varphi,\eta) \circ d\varphi \circ j' \Big], \\ &= \beta(\varphi,\eta) \circ \Big[\nabla\eta + J(u \circ \varphi) \circ \nabla\eta \circ j' \\ &\qquad + F(\varphi,\eta) \circ d\varphi + \widehat{J}(\varphi,\eta) \circ F(\varphi,\eta) \circ d\varphi \circ j' + \Big(\widehat{J}'(\varphi,\eta)\eta\Big) \circ \nabla\eta \circ j' \Big]. \end{aligned}$$

Since $\beta(\varphi,\eta)$ is everywhere invertible, this implies that u' is J-holomorphic if and only if the expression in square brackets vanishes. That expression is a section of the fixed vector bundle $\operatorname{Hom}_{\mathbb{R}}(T\Sigma', \varphi^*E)$ for every choice of section η. If we now pick $(\Sigma', j') = (\Sigma, j)$ and $\varphi = \operatorname{Id}$ and choose a section $\eta \in \Gamma(E)$, then since $\beta(z,0) \equiv \mathbb{1}$ for all $z \in \Sigma$, plugging in the maps $u_\rho(z) := (z, \rho\eta(z))$ for $\rho \in (-\epsilon, \epsilon)$ and differentiating with respect to ρ at $\rho = 0$ gives a formula for $\mathbf{D}_u\eta$, namely

(2.26)
$$\begin{aligned}\mathbf{D}_u\eta &= \partial_\rho\Big[\nabla(\rho\eta) + J(u)\circ\nabla(\rho\eta)\circ j + F(\cdot,\rho\eta) + \widehat{J}(\cdot,\rho\eta)\circ F(\cdot,\rho\eta)\circ j \\ &\qquad + \Big(\widehat{J}'(\cdot,\rho\eta)\rho\eta\Big)\circ\nabla(\rho\eta)\circ j\Big]\Big|_{\rho=0} \\ &= \nabla\eta + J(u)\circ\nabla\eta\circ j + F'(\cdot,0)\eta + J(u)\circ\big(F'(\cdot,0)\eta\big)\circ j + \Big(\widehat{J}'(\cdot,0)\eta\Big)\circ du\circ j.\end{aligned}$$

We would now like to interpret the expression in square brackets at the end of (2.25) as a linear Cauchy-Riemann type operator acting on $\eta \in \Gamma(\varphi^*E)$. We can abbreviate the whole expression as

$$\mathbf{D}_0\eta + \widetilde{C}(\cdot,\eta)$$

by defining the Cauchy-Riemann type operator $\mathbf{D}_0\eta := \nabla\eta + J(u\circ\varphi)\circ\nabla\eta\circ j'$ and the function on $\varphi^*\mathcal{O} \subset \varphi^*E$ given by

$$\begin{aligned}\widetilde{C}(z,X) &:= F(\varphi(z),X)\circ d\varphi(z) + \widehat{J}(\varphi(z),X)\circ F(\varphi(z),X)\circ d\varphi(z)\circ j'(z) \\ &\qquad + \Big(\widehat{J}'(\varphi(z),X)X\Big)\circ\nabla\eta(z)\circ j'(z) \in \operatorname{Hom}_{\mathbb{R}}(T_z\Sigma',(\varphi^*E)_z).\end{aligned} \tag{2.27}$$

While the values of $\widetilde{C}(z,X)$ according to this definition are real-linear maps in general, the particular values $\widetilde{C}(z,\eta(z))$ are guaranteed to be complex antilinear if $\bar\partial_J u' = 0$, because in this case $\mathbf{D}_0\eta + \widetilde{C}(\cdot,\eta) = 0$, where $\mathbf{D}_0\eta$ is a section of $\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma',\varphi^*E)$. It follows that if we now replace $\widetilde{C}$ with its complex-antilinear part

$$C(z,X) := \frac{1}{2}\Big[\widetilde{C}(z,X) + J(u(\varphi(z)))\circ\widetilde{C}(z,X)\circ j'(z)\Big] \in \overline{\operatorname{Hom}}_{\mathbb{C}}(T_z\Sigma',(\varphi^*E)_z),$$

then it is still true that $\bar\partial_J u' = 0$ if and only if $\mathbf{D}_0\eta + C(\cdot,\eta) = 0$. If we could now prove $C(z,0) = 0$ for all $z \in \Sigma'$, then the usual integration trick would allow us to write $C(z,X) = C'(z,X)X$ and thus define $A(z)X := C'(z,\eta(z))X$ as a linear zeroth-order term making $\mathbf{D}_0 + A$ into a linear Cauchy-Riemann type operator that annihilates η. This will not always work, but it works in two special cases that are relevant for Theorems 2.8.1 and 2.8.3.

Focusing for the moment on Theorem 2.8.1, let us assume $(\Sigma', j') = (\Sigma, j)$ and φ is the identity map, so $\widetilde{C}$ is now a function on $\mathcal{O}$, and its definition simplifies to

$$\widetilde{C}(z,X) = F(z,X) + \widehat{J}(z,X)\circ F(z,X)\circ j(z) + \Big(\widehat{J}'(z,X)X\Big)\circ\nabla\eta(z)\circ j(z).$$

Since $F(z,0) = du(z)$ is complex linear and $\widehat{J}(z,0) = J(u(z))$, we have

$$\widetilde{C}(z,0) = du(z) + J(u(z))\circ du(z)\circ j(z) = 0,$$

implying that $C(z,0)$ vanishes as well, thus we can write

$$C(z,X) = C'(z,X)X \qquad \text{for} \qquad C'(z,X) := \int_0^1 D_2C(z,\tau X)\,d\tau$$

and define $A(z) := C'(z, \eta(z))$, so that whenever $\bar{\partial}_J u' = 0$, the section $\eta \in \Gamma(\mathcal{O})$ must satisfy the linear Cauchy-Riemann type equation

$$\nabla\eta + J(u) \circ \nabla\eta \circ j + A\eta = 0.$$

Suppose now that $\eta_k \in \Gamma(\mathcal{O})$ is a sequence of sections converging in C^∞ to 0 such that $\bar{\partial} u'_k = 0$ for $u'_k(z) := (z, \eta_k(z))$. Carrying out the construction above then gives a sequence of operators of the form $\mathbf{D}_k := \mathbf{D}_0 + A_k$, where $A_k(z) = C'_k(z, \eta_k(z))$, $C'_k(z, X) = \int_0^1 D_2 C_k(z, \tau X)\, d\tau$, C_k is the complex-antilinear part of $\widetilde{C}_k$, and $\widetilde{C}_k$ is given by

$$\widetilde{C}_k(z, X) = F(z, X) + \widehat{J}(z, X) \circ F(z, X) \circ j(z) + \left(\widehat{J}'(z, X)X\right) \circ \nabla\eta_k(z) \circ j(z).$$

As $\eta_k \to 0$, the latter converges to

$$\widetilde{C}_\infty(z, X) := F(z, X) + \widehat{J}(z, X) \circ F(z, X) \circ j(z),$$

so C_k converges to the complex-antilinear part C_∞ or $\widetilde{C}_\infty$ and A_k converges to $C'_\infty(\cdot, 0)$, which is just the complex-antilinear part of $\widetilde{C}'_\infty(\cdot, 0)$. For the latter, we have

$$\begin{aligned}\widetilde{C}'_\infty(z, 0)X &= F'(z, 0)X + \widehat{J}(z, 0) \circ \left(F'(z, 0)X\right) \circ j(z) + \left(\widehat{J}'(z, 0)X\right) \circ F(z, 0) \circ j(z)\\ &= F'(z, 0)X + J(u(z)) \circ \left(F'(z, 0)X\right) \circ j(z) + \left(\widehat{J}'(z, 0)X\right) \circ du(z) \circ j(z),\end{aligned}$$

which is precisely the zeroth-order term that appears in our formula (2.26) for $\mathbf{D}_u$. This proves that $\widetilde{C}'_\infty(z, 0)X$ is already complex antilinear, and thus matches $C'_\infty(z, 0)X$, and it follows that our sequence of Cauchy-Riemann type operators $\mathbf{D}_k$ converges to $\mathbf{D}_u$. The proof of Theorem 2.8.1 is thus complete.

For the situation in Theorem 2.8.3, we are given a splitting $u^*TW = T_u \oplus N_u$ and can choose the connection ∇ to respect it, in which case the operator $\mathbf{D}_0\eta = \nabla\eta + J(u\circ\varphi)\circ\nabla\eta\circ j'$ splits into a direct sum of two linear Cauchy-Riemann type operators $\mathbf{D}_0^T$ and $\mathbf{D}_0^N$ on φ^*T_u and φ^*N_u respectively. Taking $\eta \in \Gamma(\varphi^*\mathcal{O}^N) \subset \Gamma(\varphi^*N_u)$ and writing $\pi_N : u^*TW \to N_u$ for the fiberwise-linear projection along T_u, the vanishing of the expression in square brackets at the end of (2.25) then implies that

$$\mathbf{D}_0^N + \pi_N C(\cdot, \eta) = 0.$$

Using the relation $d\varphi \circ j' = j(\varphi) \circ (d\varphi - \bar{\partial}_j \varphi)$, (2.27) becomes

$$\begin{aligned}\widetilde{C}(z, X) = F(\varphi(z), X) \circ d\varphi(z) + \widehat{J}(\varphi(z), X) \circ F(\varphi(z), X) \circ j(\varphi(z)) \circ d\varphi(z)\\ - \widehat{J}(\varphi(z), X) \circ F(\varphi(z), X) \circ j(\varphi(z)) \circ \bar{\partial}_j \varphi(z)\\ + \left(\widehat{J}'(\varphi(z), X)X\right) \circ \nabla\eta(z) \circ j'(z),\end{aligned} \tag{2.28}$$

which satisfies

$$\begin{aligned}\widetilde{C}(z, 0) &= du(\varphi(z)) \circ d\varphi(z) + J(u(\varphi(z))) \circ du(\varphi(z)) \circ j(\varphi(z)) \circ d\varphi(z)\\ &\qquad - J(u(\varphi(z))) \circ du(\varphi(z)) \circ j(\varphi(z)) \circ \bar{\partial}_j \varphi(z)\\ &= du(\varphi(z)) \circ \bar{\partial}_j \varphi(z).\end{aligned}$$

The expression is complex antilinear, so $C(z,0)$ is exactly the same, and as luck would have it, its image is contained in the subbundle $\varphi^* T_u$, thus composing it with the projection π_N kills it. We conclude that $\pi_N C(\cdot, \eta)$ can be written in the form $\pi_N C'(\cdot,\eta)\eta =: A^N \eta$, defining a linear Cauchy-Riemann type operator $\mathbf{D}^N := \mathbf{D}^N_0 + A^N$ on $\varphi^* N_u$ with $A^N(z) := \pi_N C'(z, \eta(z))$ such that $\mathbf{D}^N \eta = 0$.

Now suppose we have C^∞-convergent sequences $j'_k \to j'$, $\varphi_k \to \varphi$ and $\eta_k \to 0$, where $\varphi_k : \Sigma' \to \Sigma$ are smooth maps, $\varphi : (\Sigma', j') \to (\Sigma, j)$ is holomorphic, $\eta_k \in \Gamma(\varphi_k^* \mathcal{O}^N) \subset \Gamma(\varphi_k^* N_u)$ and $u'_k(z) := \Psi(\varphi_k(z), \eta_k(z))$ defines a J-holomorphic curve $(\Sigma', j'_k) \to (W, J)$ for every k. The fact that φ is holomorphic implies

$$\bar{\partial}_{j'_k, j} \varphi_k := d\varphi_k + j(\varphi_k) \circ d\varphi_k \circ j'_k \to 0$$

in C^∞. For each k, the construction above now gives a linear Cauchy-Riemann type operator $\mathbf{D}^N_k$ on $\varphi_k^* N_u$ that annihilates η_k, given by the formula

$$\mathbf{D}^N_k \eta = \nabla\eta + J(u(\varphi_k)) \circ \nabla\eta \circ j'_k + A^N_k \eta,$$

where $A^N_k(z) = \pi_N C'_k(z, \eta_k)$ and C_k is the complex-antilinear part of $\widetilde{C}_k$, which according to (2.28) satisfies

$$\begin{aligned}\widetilde{C}_k(z,X) = {} & F(\varphi_k(z), X) \circ d\varphi_k(z) + \widehat{J}(\varphi_k(z), X) \circ F(\varphi_k(z), X) \circ j(\varphi_k(z)) \circ d\varphi_k(z) \\ & - \widehat{J}(\varphi_k(z), X) \circ F(\varphi_k(z), X) \circ j(\varphi_k(z)) \circ \bar{\partial}_{j'_k, j} \varphi_k(z) \\ & + \left(\widehat{J}'(\varphi_k(z), X) X \right) \circ \nabla \eta_k(z) \circ j'_k(z).\end{aligned}$$

The parts involving $\bar{\partial}_{j'_k, j}\varphi_k$ and $\nabla\eta_k$ disappear as $k \to \infty$, leaving

$$\widetilde{C}_\infty(z,X) := F(\varphi(z), X) \circ d\varphi(z) + \widehat{J}(\varphi(z), X) \circ F(\varphi(z), X) \circ j(\varphi(z)) \circ d\varphi(z),$$

and thus

$$\begin{aligned}\widetilde{C}'_\infty(z,0)X = \Big[& F'(\varphi(z),0)X + J(u(\varphi(z))) \circ \left(F'(\varphi(z),0)X \right) \circ j(\varphi(z)) \\ & + \left(\widehat{J}'(\varphi(z),0)X \right) \circ du(\varphi(z)) \circ j(\varphi(z)) \Big] \circ d\varphi(z).\end{aligned}$$

If you apply the pullback operator φ^* to the zeroth-order term in our formula (2.26) for $\mathbf{D}_u$, the expression you end up with is precisely this one, which implies that $\mathbf{D}^N_k$ converges to $\varphi^* \mathbf{D}^N_u$ as $k \to \infty$, thus completing the proof of Theorem 2.8.3.

REMARK 2.8.5. Various more technical versions of Theorems 2.8.1 and 2.8.3 are possible under weaker regularity hypotheses. For example, if the map $\varphi : \Sigma' \to \Sigma$ in Theorem 2.8.3 is assumed to be of class C^m for some finite $m \geqslant 1$, then $\varphi^* N_u \to \Sigma'$ is no longer a smooth vector bundle, but is instead a bundle of class C^m. On these, one can define the notion of a connection or linear Cauchy-Riemann type operator *of class* C^{m-1}; the latter looks locally like $\bar{\partial} + A$ for a zeroth-order term that is a function of class C^{m-1}. (For the reason why are saying C^{m-1} here instead of C^m, see Exercise 4.1.1.) Inspecting the proof of Theorem 2.8.3, one finds that it still works if φ is only of class C^m, and the resulting linear Cauchy-Riemann type operator $\mathbf{D}^N$ on $\varphi^* N_u$ is of class C^{m-1}; moreover, $\mathbf{D}^N$ can be assumed arbitrarily C^{m-1}-close to $\varphi_0^* \mathbf{D}^N_u$ if η is sufficiently C^m-small and φ is sufficiently C^m-close to a holomorphic map $\varphi_0 : (\Sigma', j') \to (\Sigma, j)$.

One application of this generalization is to prove the "approximate" version of the theorem of Micallef-White [**MW95**] mentioned in §2.6, which lies in the background of the dichotomy between simple and multiply covered holomorphic curves. To show for instance that two connected J-holomorphic curves u and v with non-identical images have only *isolated* intersections, the hardest step is to understand the following local picture: for some almost complex structure J on $\mathbb{C}^n$ that matches i at the origin, suppose u and v are both J-holomorphic maps $\mathbb{D} \to \mathbb{C} \times \mathbb{C}^{n-1}$ of the form

$$u(z) = (f(z), \hat{u}(z)), \qquad v(z) = (g(z), \hat{v}(z))$$

with $u(0) = v(0) = 0$ such that $f, g : \mathbb{D} \to \mathbb{C}$ vanish to the same order $k \in \mathbb{N}$ at 0, while $\hat{u}$ and $\hat{v}$ each vanish to some strictly higher order. The assumption about f and g means that after suitable reparametrizations of u and v near the origin, they can be rewritten in the form

$$u(z) = (z^k, \hat{u}'(z)), \qquad v(z) = (z^k, \hat{u}'(z) + \eta(z)), \tag{2.29}$$

for functions $\hat{u}'$ and η that also vanish to order greater than k at 0. The intersections of u and v in this neighborhood of the origin are thus in bijective correspondence with the zeroes of η and its reparametrizations $\eta_j(z) := \eta(e^{2\pi i j/k} z)$ for $j \in \mathbb{Z}$. A variant of Theorem 2.8.3 then shows that each of the functions η_j is annihilated by some Cauchy-Riemann type operator, and is therefore subject to the similarity principle, so its zero set is discrete unless $\eta_j \equiv 0$. This requires weakened regularity in Theorem 2.8.3, however, because the reparametrizations leading to (2.29) can be shown to be of class C^1, but need not be smooth. The Cauchy-Riemann type operators that annihilate the η_j will therefore be only of class C^0 in general; fortunately, the hypotheses of the similarity principle (Theorem 2.5.3) only require them to be of class L^p for some $p > 2$.

For a detailed version of the argument just outlined, see [**Wen20**, Appendix B.2].

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APPENDIX A

Sobolev spaces

Contents

In this appendix, we review some of the standard properties of Sobolev spaces, in particular using them to prove Propositions 2.2.4, 2.2.5 and 2.2.8 from §2.2, and elucidating the construction of Sobolev spaces of sections on vector bundles. A good reference for the necessary background material is [**AF03**].

A.1. Approximation, extension and embedding theorems

Unless otherwise noted, all functions in the following are assumed to be defined on a nonempty open subset

$$\mathcal{U} \subset \mathbb{R}^n$$

with its standard Lebesgue measure, and taking values in a finite-dimensional normed vector space that will usually not need to be specified, though occasionally we will assume it is $\mathbb{R}$ or $\mathbb{C}$ so that one can define products of functions. The domain $\mathcal{U}$ will also sometimes have additional conditions specified such as boundedness or regularity at the boundary, though we will try not to add too many more restrictions than are really needed. The most useful assumption to impose on $\mathcal{U}$ is known as the **strong local Lipschitz condition**: if $\mathcal{U}$ is bounded, then it means simply that near every boundary point of $\mathcal{U}$, one can find smooth local coordinates in which $\mathcal{U}$ looks like the region bounded by the graph of a Lipschitz-continuous function, and in this case we call $\mathcal{U}$ a **bounded Lipschitz domain**. If $\mathcal{U}$ is unbounded, then one needs to impose extra conditions guaranteeing e.g. uniformity of Lipschitz constants, and the precise definition becomes a bit lengthy (see [**AF03**, §4.9]). For our purposes, all we really need to know about the strong local Lipschitz condition is that that it is satisfied both by bounded Lipschitz domains and by relatively tame unbounded domains such as $(0,1) \times (0,\infty) \subset \mathbb{R}^2$ which have smooth boundary with finitely many corners. We will repeatedly need to use the generalized version of **Hölder's inequality**, which states that for any finite collection of measurable functions $f_1, \dots, f_m$,

$$\left\| \prod_{i=1}^m |f_i| \right\|_{L^p} \leqslant \prod_{i=1}^m \|f_i\|_{L^{p_i}} \quad \text{for } 1 \leqslant p \leqslant p_1, \dots, p_m \leqslant \infty \text{ with } \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}. \tag{A.1}$$

This is an easy corollary of the standard version,

$$\big\| |f| \cdot |g| \big\|_{L^1} \leqslant \|f\|_{L^p} \cdot \|g\|_{L^q} \quad \text{whenever } 1 \leqslant p, q \leqslant \infty \text{ and } 1 = \frac{1}{p} + \frac{1}{q}.$$

For an integer $k \geqslant 0$ and real number $p \in [1, \infty]$, we define $W^{k,p}(\mathcal{U})$ as in §2.2 to be the Banach space of all $f \in L^p(\mathcal{U})$ which have weak partial derivatives $\partial^\alpha f \in L^p(\mathcal{U})$ for all $|\alpha| \leqslant k$. For $p = 2$, these spaces are also often denoted by

$$H^k(\mathcal{U}) := W^{k,2}(\mathcal{U}),$$

and they admit Hilbert space structures with inner product

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \leqslant k} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2}.$$

We denote by

$$W_0^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U}), \qquad H_0^k(\mathcal{U}) \subset H^k(\mathcal{U})$$

the closed subspaces defined as the closures of $C_0^\infty(\mathcal{U})$ with respect to the relevant norms. Since $C_0^\infty(\mathcal{U})$ is dense in $L^p(\mathcal{U})$ for $1 \leqslant p < \infty$ (see e.g. [**LL01**, §2.19]), there is no difference between $W^{0,p}(\mathcal{U})$ and $W_0^{0,p}(\mathcal{U})$ for $p < \infty$, but in general $W_0^{k,p}(\mathcal{U}) \neq W^{k,p}(\mathcal{U})$ for $k \geqslant 1$, with a few notable exceptions such as the case $\mathcal{U} = \mathbb{R}^n$ (cf. Corollary A.1.2 below). Let

$$W_{\text{loc}}^{k,p}(\mathcal{U}) := \big\{ \text{functions } f \text{ on } \mathcal{U} \;\big|\; f \in W^{k,p}(\mathcal{V}) \text{ for all open subsets } \mathcal{V} \subset \mathcal{U} \text{ with compact closure } \overline{\mathcal{V}} \subset \mathcal{U} \big\},$$

and say that a sequence $f_j \in W_{\text{loc}}^{k,p}(\mathcal{U})$ converges in $W_{\text{loc}}^{k,p}$ to $f \in W_{\text{loc}}^{k,p}(\mathcal{U})$ if the restrictions to all precompact open subsets $\mathcal{V} \subset \overline{\mathcal{V}} \subset \mathcal{U}$ converge in $W^{k,p}(\mathcal{V})$. Recall that for $k \in \{0, 1, 2, \dots, \infty\}$, $C^k(\mathcal{U})$ denotes the space of functions on $\mathcal{U}$ with continuous derivatives up to order k, while

$$C^k(\overline{\mathcal{U}}) \subset C^k(\mathcal{U})$$

is the space of $f \in C^k(\mathcal{U})$ such that for all $|\alpha| \leqslant k$, $\partial^\alpha f$ is bounded and uniformly continuous.

THEOREM A.1.1 ([**AF03**, §3.17, 3.22]). *For any open subset $\mathcal{U} \subset \mathbb{R}^n$, and any $k \geqslant 0$, $1 \leqslant p < \infty$, the subspace*

$$C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. Moreover, if $\mathcal{U} \subset \mathbb{R}^n$ satisfies the strong local Lipschitz condition, then the space

$$\left\{ f \in C^\infty(\mathcal{U}) \;\middle|\; f = \tilde{f}|_{\mathcal{U}} \text{ for some } \tilde{f} \in C_0^\infty(\mathbb{R}^n) \right\}$$

is also dense in $W^{k,p}(\mathcal{U})$, so in particular,

$$C^\infty(\overline{\mathcal{U}}) \cap W^{k,p}(\mathcal{U}) \subset W^{k,p}(\mathcal{U})$$

is dense. □

Corollary A.1.2. *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for every $k \geqslant 0$ and $p \in [1,\infty)$.* □

Here is another useful characterization of $W_0^{k,p}(\mathcal{U})$:

Theorem A.1.3 ([**AF03**, §5.29]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition. Then a function $f \in W^{k,p}(\mathcal{U})$ belongs to $W_0^{k,p}(\mathcal{U})$ if and only if the function $\widetilde{f}$ on $\mathbb{R}^n$ defined to match f on $\mathcal{U}$ and 0 everywhere else belongs to $W^{k,p}(\mathbb{R}^n)$.* □

While it is obvious from the definitions that functions in $W_0^{k,p}(\mathcal{U})$ always admit extensions of class $W^{k,p}$ over $\mathbb{R}^n$, this is much less obvious for functions in $W^{k,p}(\mathcal{U})$ in general, and it is not true without sufficient assumptions about the regularity of $\partial\mathcal{U}$. For our purposes it suffices to consider the following case.

Theorem A.1.4 ([**AF03**, §5.22]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open subset such that $\partial\overline{\mathcal{U}}$ is a submanifold of class C^m for some $m \in \{1,2,3,\ldots,\infty\}$. Then there exists a linear operator E that maps functions defined almost everywhere on $\mathcal{U}$ to functions defined almost everywhere on $\mathbb{R}^n$ and has the following properties:*

- *For every function f on $\mathcal{U}$, $Ef|_{\mathcal{U}} \equiv f$ almost everywhere;*
- *For every nonnegative integer $k \leqslant m$ and every $p \in [1,\infty)$, E defines a bounded linear operator $W^{k,p}(\mathcal{U}) \to W^{k,p}(\mathbb{R}^n)$.*

□

Corollary A.1.5. *Suppose $\mathcal{U},\mathcal{U}' \subset \mathbb{R}^n$ are open subsets such that $\mathcal{U}$ has compact closure contained in $\mathcal{U}'$. If $\mathcal{U}$ satisfies the hypothesis of Theorem A.1.4, then the resulting extension operator E can be chosen such that it maps each $W^{k,p}(\mathcal{U})$ for $k \leqslant m$ and $1 \leqslant p < \infty$ into $W_0^{k,p}(\mathcal{U}')$.*

Proof. Choose a smooth function $\rho : \mathcal{U}' \to [0,1]$ that has compact support and equals 1 on $\overline{\mathcal{U}}$, then replace the operator E given by Theorem A.1.4 with the operator $f \mapsto \rho \cdot Ef$. □

To state the Sobolev embedding theorem in its proper generality, recall that for $0 < \alpha \leqslant 1$, the **Hölder seminorm** of a function f on $\mathcal{U}$ is defined by

$$|f|_{C^\alpha} := |f|_{C^\alpha(\mathcal{U})} := \sup_{x \neq y \in \mathcal{U}} \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

and $C^{k,\alpha}(\mathcal{U})$ is then defined as the Banach space of functions $f \in C^k(\overline{\mathcal{U}})$ for which the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \max_{|\beta|=k} |\partial^\beta f|_{C^\alpha}$$

is finite. In reading the following statement, it is important to remember that elements of $W^{k,p}(\mathcal{U})$ are technically not functions, but rather *equivalence classes* of functions defined almost everywhere. Thus when we say e.g. that there is an inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow C^{m,\alpha}(\mathcal{U})$, the literal meaning is that for every function f representing an element of $W^{k,p}(\mathcal{U})$, one can change the values of f in a unique way on some set of measure zero in $\mathcal{U}$ so that after this change, $f \in C^{m,\alpha}(\mathcal{U})$. Continuity of the inclusion means that there is a bound of the form

$$\|f\|_{C^{m,\alpha}} \leqslant c\|f\|_{W^{k,p}}$$

for all $f \in W^{k,p}(\mathcal{U})$, where $c > 0$ is a constant which may in general depend on m, α, k, p and $\mathcal{U}$, but not on f.

THEOREM A.1.6 ([AF03, §4.12]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $k \geqslant 1$ is an integer and $1 \leqslant p < \infty$.*

(1) If $0 < k - n/p \leqslant 1$, then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\mathcal{U}) \qquad \text{for each } \alpha \in (0,1) \text{ with } \alpha \leqslant k - n/p,$$
$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}) \qquad \text{for each } q \in [p,\infty].$$

(2) If $kp < n$ and $p^* > p$ is defined by the condition

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n},$$

then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p,p^*].$$

(3) If $kp = n$, then there exist continuous inclusions

$$W^{k,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U}), \qquad \text{for each } q \in [p,\infty).$$

Moreover, the spaces $W_0^{k,p}(\mathcal{U})$ admit similar inclusions under no assumption on the open subset $\mathcal{U} \subset \mathbb{R}^n$. □

Under the same assumption on the domain $\mathcal{U}$, one can apply Theorem A.1.6 to successive derivatives of functions in $W^{k,p}(\mathcal{U})$ and thus obtain the following inclusions for any integer $d \geqslant 0$:

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\mathcal{U}) \quad \text{if } 0 < k - n/p \leqslant 1,\ 0 < \alpha < 1 \text{ and } \alpha \leqslant k - n/p, \tag{A.2}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp > n \text{ and } p \leqslant q \leqslant \infty, \tag{A.3}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp < n \text{ and } p \leqslant q \leqslant p^*, \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}, \tag{A.4}$$

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{if } kp = n \text{ and } p \leqslant q < \infty. \tag{A.5}$$

REMARK A.1.7. The embedding theorem suggests that one should intuitively think of $W^{k,p}(\mathcal{U})$ as consisting of functions with "$k - n/p$ continuous derivatives," where the number $k - n/p$ may in general be a non-integer and/or negative. This provides a useful mnemonic for results about embeddings of one Sobolev space into another, such as the following.

COROLLARY A.1.8. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $1 \leqslant p,q < \infty$, and $k,m \geqslant 0$ are integers satisfying*

$$k \geqslant m, \qquad p \leqslant q, \qquad \text{and} \qquad k - \frac{n}{p} \geqslant m - \frac{n}{q}.$$

Then there exists a continuous inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$. □

EXERCISE A.1.9. Derive Corollary A.1.8 from Theorem A.1.6 by checking that under the stated conditions, there is a continuous inclusion $W^{k-m,p}(\mathcal{U}) \hookrightarrow L^q(\mathcal{U})$. Show also that the hypothesis $p \leqslant q$ is unnecessary if $\mathcal{U} \subset \mathbb{R}^n$ has finite measure.

By the Arzelà-Ascoli theorem, the natural inclusion

$$C^{k,\alpha'}(\mathcal{U}) \hookrightarrow C^{k,\alpha}(\mathcal{U})$$

for $\alpha < \alpha'$ is a compact operator whenever $\mathcal{U} \subset \mathbb{R}^n$ is bounded. It follows that if $\mathcal{U} \subset \mathbb{R}^n$ in (A.2) is bounded and α is *strictly* less than the extremal value $k-n/p$, then the inclusion (A.2) is also compact. A similar statement holds for the inclusion (A.4) when $p \leqslant q < p^*$, and this is known as the **Rellich-Kondrachov compactness theorem**. We summarize these as follows:

THEOREM A.1.10 ([**AF03**, §6.3]). *Assume $\mathcal{U} \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $k \geqslant 1$ and $d \geqslant 0$ are integers and $1 \leqslant p < \infty$.*

(1) If $kp > n$ and $k - n/p < 1$, then the inclusions

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\mathcal{U}) \quad \text{for } \alpha \in (0, k-n/p),$$
$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, \infty)$$

are compact.

(2) If $kp \leqslant n$ and $p^* \in (p, \infty]$ is defined by the condition $1/p^* = 1/p - k/n$, then the inclusions

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow W^{d,q}(\mathcal{U}) \quad \text{for } q \in [p, p^*)$$

are compact.

In particular, the continuous inclusion $W^{k,p}(\mathcal{U}) \hookrightarrow W^{m,q}(\mathcal{U})$ in Corollary A.1.8 is compact whenever the inequality $k - n/p \geqslant m - n/q$ is strict. □

On connected 1-dimensional domains $\mathcal{U} \subset \mathbb{R}$, the spaces $W^{1,p}(\mathcal{U})$ admit an alternative characterization in terms of classical derivatives defined almost everywhere:

PROPOSITION A.1.11. *For $-\infty < a < b < \infty$, every absolutely continuous function on $[a, b]$ belongs to $W^{1,1}((a, b))$ and has a weak derivative that is equal to its classical derivative almost everywhere. Conversely, every function in $W^{1,1}((a, b))$ is equal almost everywhere to an absolutely continuous function defined on $[a, b]$.*

PROOF. Let us denote the classical derivative of a function f by f'_c and the weak derivative by f'_w whenever there is danger of confusion. If f is absolutely continuous on $[a, b]$, then for every test function $\varphi \in C_0^\infty((a, b))$, $f\varphi$ defines an absolutely continuous function on $[a, b]$ that vanishes at the end points, so the fundamental theorem of calculus implies $\int_{[a,b]}(f\varphi)'_c = \int_{[a,b]} f'_c\varphi + \int_{[a,b]} f\varphi' = 0$, proving that the almost everywhere defined function $f'_c \in L^1([a, b])$ is also the weak derivative f'_w, and thus $f \in W^{1,1}((a, b))$.

Conversely, suppose $f \in W^{1,1}((a, b))$, so it has a weak derivative $f'_w \in L^1((a, b))$. We can then define an absolutely continuous function g on $[a, b]$ by $g(x) := \int_a^x f'_w$, which is differentiable almost everywhere and satisfies $g'_c = f'_w$. By the argument of the previous paragraph, g'_c is also a weak derivative g'_w, thus $g - f$ is a function on

(a, b) with vanishing weak derivative, implying via [LL01, Theorem 6.11] that $g - f$ is equal almost everywhere to a constant. ☐

COROLLARY A.1.12. *For $-\infty < a < b < \infty$ and $1 \leqslant p \leqslant \infty$, $W^{1,p}((a, b))$ has a canonical identification with the space of absolutely continuous functions on $[a, b]$ whose classical derivatives belong to $L^p([a, b])$.* ☐

A.2. Products, compositions, and rescaling

We now restate and prove Propositions 2.2.4, 2.2.5 and 2.2.8 from §2.2. These are all corollaries of the Sobolev embedding theorem, so in particular they hold for the same class of domains $\mathcal{U} \subset \mathbb{R}^n$, and the restrictions on $\mathcal{U}$ can be dropped at the cost of replacing each space $W^{k,p}$ by $W^{k,p}_0$.

We begin by generalizing Prop. 2.2.4, hence we consider Sobolev spaces of functions valued in $\mathbb{R}$ or $\mathbb{C}$ so that pointwise products of functions are well defined almost everywhere. We say that there is a **continuous product map**,

$$W^{k_1,p_1}(\mathcal{U}) \times \ldots \times W^{k_m,p_m}(\mathcal{U}) \to W^{k,p}(\mathcal{U}),$$

or a continuous product **pairing** in the case $m = 2$, if for every set of functions $f_i \in W^{k_i,p_i}(\mathcal{U})$ with $i = 1, \ldots, m$, the pointwise product function $f_1 \cdot \ldots \cdot f_m$ is in $W^{k,p}(\mathcal{U})$ and there is an estimate of the form

$$\|f_1 \cdot \ldots \cdot f_m\|_{W^{k,p}} \leqslant c\|f_1\|_{W^{k_1,p_1}} \cdot \ldots \cdot \|f_m\|_{W^{k_m,p_m}}$$

for some constant $c > 0$ not depending on $f_1, \ldots, f_m$. The case $m = 2$, $k_1 = k_2 = k$ and $p_1 = p_2 = p$ is especially interesting, as the space $W^{k,p}(\mathcal{U})$ is then a **Banach algebra**. More generally, one can ask under what circumstances multiplication by functions of class $W^{k,p}$ defines a bounded linear operator on functions of class $W^{m,q}$. A hint about this comes from the world of classically differentiable functions: multiplication by C^k-smooth functions defines a continuous map $C^m \to C^m$ if and only if $k \geqslant m$. The corresponding answer in Sobolev spaces turns out to be that functions of class $W^{k,p}$ need to have strictly more than zero derivatives in the sense of Remark A.1.7, and at least as many derivatives as functions of class $W^{m,q}$.

THEOREM A.2.1. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $1 \leqslant p, q < \infty$, and $k, m \geqslant 0$ are integers satisfying*

$$k \geqslant m, \qquad kp > n, \qquad \text{and} \qquad k - \frac{n}{p} \geqslant m - \frac{n}{q}.$$

Then there exists a continuous product pairing

$$W^{k,p}(\mathcal{U}, \mathbb{C}) \times W^{m,q}(\mathcal{U}, \mathbb{C}) \to W^{m,q}(\mathcal{U}, \mathbb{C}) : (f, g) \mapsto fg.$$

The following preparatory lemma will be useful both for proving the product estimate and for further results below. It is an easy consequence of Theorem A.1.6 and Hölder's inequality.

LEMMA A.2.2. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $m \geqslant 2$ is an integer, and we are given positive numbers*

$p_1, \ldots, p_m \geqslant 1$ and integers $k_1, \ldots, k_m \geqslant 0$. Let $I := \{i \in \{1, \ldots, m\} \mid k_i p_i \leqslant n\}$. Then for any $q \geqslant 1$ satisfying

$$\sum_{i \in I} \left(\frac{1}{p_i} - \frac{k_i}{n} \right) < \frac{1}{q} \leqslant \sum_{i=1}^{m} \frac{1}{p_i},$$

there is a continuous product map

$$W^{k_1,p_1}(\mathcal{U}) \times \ldots \times W^{k_m,p_m}(\mathcal{U}) \to L^q(\mathcal{U}).$$

PROOF. By the generalized Hölder inequality (A.1), it suffices to show that for any $q \geqslant 1$ in the stated range, one can find numbers $q_1, \ldots, q_m \in [q, \infty]$ satisfying $1/q = 1/q_1 + \ldots + 1/q_m$ for which Theorem A.1.6 provides continuous inclusions

$$W^{k_i,p_i}(\mathcal{U}) \hookrightarrow L^{q_i}(\mathcal{U})$$

for each $i = 1, \ldots, m$. Whenever $k_i p_i > n$, this inclusion is valid with q_i chosen freely from the interval $[p_i, \infty]$, so $1/q_i$ can then take any value subject to the constraint

$$0 \leqslant \frac{1}{q_i} \leqslant \frac{1}{p_i}.$$

If on the other hand $k_i p_i \leqslant n$, then we can arrange $1/q_i$ to take any value in the range

$$\frac{1}{p_i} - \frac{k_i}{n} < \frac{1}{q_i} \leqslant \frac{1}{p_i}.$$

Adding these up, the range of values for $\sum_i \frac{1}{q_i}$ that we can achieve in this way covers the stated interval. □

PROOF OF THEOREM A.2.1. By density of smooth functions, it suffices to prove that an estimate of the form

$$\|fg\|_{W^{m,q}} \leqslant c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}$$

holds for all $f \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ and $g \in C^\infty(\mathcal{U}) \cap W^{m,q}(\mathcal{U})$. Equivalently, we need to show that for all f and g of this type and every multiindex α of degree $|\alpha| \leqslant m$, there is a constant $c > 0$ independent of f and g such that

$$\|\partial^\alpha(fg)\|_{L^q} \leqslant c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}.$$

Since f and g are smooth, we are free to use the product rule in computing $\partial^\alpha(fg)$, which will then be a linear combination of terms of the form $\partial^\beta f \cdot \partial^\gamma g$ where $|\alpha| = |\beta| + |\gamma|$, hence we have reduced the problem to proving a bound

$$\|\partial^\beta f \cdot \partial^\gamma g\|_{L^q} \leqslant c\|f\|_{W^{k,p}}\|g\|_{W^{m,q}}$$

for every pair of multiindices β, γ with $|\beta| + |\gamma| \leqslant m$. Since $\partial^\beta f \in W^{k-|\beta|,p}(\mathcal{U})$ and $\partial^\gamma f \in W^{m-|\gamma|,q}(\mathcal{U})$, the result follows if we can assume that for every pair of integers $a, b \geqslant 0$ satisfying $a + b \leqslant m$, there exists a continuous product pairing

$$W^{k-a,p}(\mathcal{U}) \times W^{m-b,q}(\mathcal{U}) \to L^q(\mathcal{U}). \tag{A.6}$$

If $(k-a)p > n$, then $W^{k-a,p} \hookrightarrow L^\infty$ and (A.6) is immediate since $W^{m-b,q} \hookrightarrow L^q(\mathcal{U})$. For the remaining cases, we shall apply Lemma A.2.2, noting that the condition $1/q \leqslant 1/p + 1/q$ is trivially satisfied.

If $(m-b)q > n$ but $(k-a)p \leqslant n$, then the hypotheses of the lemma are satisfied if and only if

$$\frac{1}{p} - \frac{k-a}{n} < \frac{1}{q}.$$

Since $\frac{1}{p} - \frac{k}{n} \leqslant \frac{1}{q} - \frac{m}{n}$ by assumption, we have

$$\frac{1}{p} - \frac{k-a}{n} = \frac{1}{p} - \frac{k}{n} + \frac{a}{n} \leqslant \frac{1}{q} - \frac{m}{n} + \frac{a}{n} \leqslant \frac{1}{q}$$

since $a \leqslant m$, and equality holds only if $a = m$, $b = 0$ and $k - n/p = m - n/q$, which implies $mq > n$. In this case $W^{m-b,q} = W^{m,q} \hookrightarrow L^\infty$, and the pairing (A.6) follows because $W^{k-a,p} = W^{k-m,p}$ embeds continuously into L^q: the latter follows from Theorem A.1.6 since $\frac{1}{p} - \frac{k-m}{n} = \frac{1}{q}$.

Finally, when $(k-a)p \leqslant n$ and $(m-b)q \leqslant n$, the hypotheses of the lemma are satisfied since

$$\left(\frac{1}{p} - \frac{k-a}{n}\right) + \left(\frac{1}{q} - \frac{m-b}{n}\right) \leqslant \frac{1}{p} - \frac{k}{n} + \frac{1}{q} - \frac{m}{n} + \frac{m}{n} = \left(\frac{1}{p} - \frac{k}{n}\right) + \frac{1}{q} < \frac{1}{q},$$

where we've used the assumption $kp > n$ and the fact that $a + b \leqslant m$. □

REMARK A.2.3. A much simpler argument shows similarly that for any open domain $\mathcal{U} \subset \mathbb{R}^n$, any integer $k \geqslant 1$ and any $p \in [1, \infty)$, there is a continuous product pairing

$$C^k(\overline{\mathcal{U}}, \mathbb{C}) \times W^{k,p}(\mathcal{U}, \mathbb{C}) \times W^{k,p}(\mathcal{U}, \mathbb{C}).$$

As in Theorem A.2.1, this follows from the density of $C^\infty \cap W^{k,p} \subset W^{k,p}$ after showing that all $f \in C^k(\overline{\mathcal{U}})$ and $g \in C^\infty(\mathcal{U}) \cap W^{k,p}(\mathcal{U})$ satisfy an estimate of the form $\|fg\|_{W^{k,p}} \leqslant c\|f\|_{C^k}\|g\|_{W^{k,p}}$. The latter follows easily from the definition of the $W^{k,p}$-norm.

In general it is not straightforward to say when the usual product rule $\partial_i(fg) = \partial_i f \cdot g + f \cdot \partial_i g$ does or does not hold in the sense of weak derivatives. If g and $\partial_i g$ are locally integrable and f is smooth, then there is no trouble: the formula can be derived in this case directly from the definition of weak derivatives, using the observation that for any test function $\varphi \in C_0^\infty(\mathcal{U})$, φf is also in $C_0^\infty(\mathcal{U})$ and satisfies the product rule. If on the other hand f and g are not continuous but have well-defined weak derivatives and a locally integrable product, then there is no guarantee in general that any of $\partial_i(fg)$, $\partial_i f \cdot g$ or $f \cdot \partial_i g$ should be well-defined locally integrable functions. Theorem A.2.1 provides a means of resolving this question whenever f and g belong to suitable Sobolev spaces.

PROPOSITION A.2.4. *Suppose k, m, p, q and $\mathcal{U} \subset \mathbb{R}^n$ satisfy the same conditions as in Theorem A.2.1, and $m \geqslant 1$. Then for every $f \in W^{k,p}(\mathcal{U}, \mathbb{C})$ and $g \in W^{m,q}(\mathcal{U}, \mathbb{C})$, the weak partial derivatives of $fg \in W^{m,q}(\mathcal{U}, \mathbb{C})$ are given almost everywhere by the usual Leibniz rule $\partial_i(fg) = \partial_i f \cdot g + f \cdot \partial_i g$.*

PROOF. Choose sequences of smooth functions f_j, g_j with $f_j \to f$ in $W^{k,p}$ and $g_j \to g$ in $W^{m,q}$. Then since $k \geqslant m \geqslant 1$, there is also L^p-convergence $\partial_i f_j \to \partial_i f$ and L^q-convergence $\partial_i g_j \to \partial_i g$, so after restricting to a subsequence, we may assume that all four of the sequences f_j, $\partial_i f_j$, g_j and $\partial_i g_j$ converge pointwise almost everywhere. The continuity of the product pairing $W^{k,p} \times W^{m,q} \to W^{m,q}$ now implies $W^{m,q}$-convergence $f_j g_j \to fg$ and thus L^q-convergence

$$\partial_i(f_j g_j) = \partial_i f_j \cdot g_j + f_j \cdot \partial_i g_j \to \partial_i(fg).$$

The result follows since $\partial_i f_j \cdot g_j + f_j \cdot \partial_i g_j$ also converges pointwise almost everywhere to $\partial_i f \cdot g + f \cdot \partial_i g$. □

REMARK A.2.5. A slight simplification of the same argument as in Proposition A.2.4 shows that the product rule also holds (without any assumption on the open domain $\mathcal{U} \subset \mathbb{R}^n$) for $f \in C^m(\overline{\mathcal{U}}, \mathbb{C})$ and $g \in W^{m,p}(\mathcal{U}, \mathbb{C})$ for any $p \in [1, \infty)$ if $m \geqslant 1$. The key facts here are the continuity of the product pairing $C^m \times W^{m,p} \to W^{m,p}$ and the density of C^1 in $W^{m,p}$, so that f and g can be approximated by pairs for which the classical product rule holds. Both results can also be extended in a similar manner to prove the expected formula for $\partial^\alpha(fg)$ for any multiindex α of order $|\alpha| \leqslant m$.

The next result generalizes Proposition 2.2.5 and concerns the following question: if $f : \mathcal{U} \to \mathbb{R}^m$ is a function of class $W^{k,p}$ whose graph lies in some open subset $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$, and $\Psi : \mathcal{V} \to \mathbb{R}^N$ is another function, under what conditions can we conclude that the function

$$\mathcal{U} \to \mathbb{R}^N : x \mapsto \Psi(x, f(x))$$

is in $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$? We will abbreviate this function in the following by $\Psi \circ (\mathrm{Id} \times f)$, and we would also like to know whether it depends continuously (in the $W^{k,p}$-topology) on f and Ψ. The following theorem is stated rather generally, but on first reading you may prefer to assume $\mathcal{U} \subset \mathbb{R}^n$ is bounded, in which case some of the hypotheses become vacuous. We will say that an open subset $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$ is a **star-shaped neighborhood of** $f : \mathcal{U} \to \mathbb{R}^m$ if it contains the graph of f and

$$(x, v) \in \mathcal{V} \quad \Rightarrow \quad (x, tv + (1-t)f(x)) \in \mathcal{V} \text{ for all } t \in [0, 1].$$

THEOREM A.2.6. *Assume $\mathcal{U} \subset \mathbb{R}^n$ is an open subset satisfying the strong local Lipschitz condition, $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, and $\mathcal{V} \subset \mathcal{U} \times \mathbb{R}^m$ is a star-shaped neighborhood of some function $f_0 \in W^{k,p}(\mathcal{U}, \mathbb{R}^m)$. Assume also $\mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}) \subset W^{k,p}(\mathcal{U}, \mathbb{R}^m)$ is an open neighborhood of f_0 such that*

$$(x, f(x)) \in \mathcal{V} \quad \text{for all } x \in \mathcal{U} \text{ and } f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}),$$

and $C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N) \subset C^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ is a closed linear subspace such that all $\Psi \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^n)$ have the following properties:[1]

(1) There exists a bounded subset $\mathcal{K} \subset \mathcal{U}$ such that $\Psi(x, v)$ is independent of x for all $x \in \mathcal{U} \backslash \mathcal{K}$;

(2) $\Psi \circ (\mathrm{Id} \times f_0) \in L^p(\mathcal{U}, \mathbb{R}^N)$.

Then there is a well-defined and continuous map

$$\mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}) \xrightarrow{T} \mathscr{L}\big(C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N), W^{k,p}(\mathcal{U}, \mathbb{R}^N)\big),$$
$$T(f)\Psi := \Psi \circ (\mathrm{Id} \times f),$$

[1]Both of the conditions on $\Psi \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^n)$ are vacuous if $\mathcal{U} \subset \mathbb{R}^n$ is bounded.

so in particular the map

$$C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N) \times \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V}) \to W^{k,p}(\mathcal{U}, \mathbb{R}^N) : (\Psi, f) \mapsto \Psi \circ (\mathrm{Id} \times f),$$

is well defined and continuous. Moreover, for each $\Psi \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ *and* $f \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$, *the weak partial derivatives of* $\Psi \circ (\mathrm{Id} \times f)$ *are given almost everywhere by the classical formula*

$$\partial_j \left[\Psi \circ (\mathrm{Id} \times f) \right](x) = \partial_j \Psi(x, f(x)) + D_2 \Psi(x, f(x)) \partial_j f(x),$$

where $\partial_j \Psi$ *denotes the partial derivative of* $\Psi(x, v)$ *with respect to the* j*th coordinate in* $x \in \mathbb{R}^n$, *and* $D_2 \Psi$ *is its differential with respect to* $v \in \mathbb{R}^m$.

PROOF. We will show first that if $f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is smooth, then $\Psi \circ (\mathrm{Id} \times f)$ belongs to $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ for every $\Psi \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N)$. Since $\mathcal{V}$ is a star-shaped neighborhood of f_0, we have

$$\begin{aligned} |\Psi(x, f(x)) - \Psi(x, f_0(x))| &= \left| \int_0^1 \frac{d}{dt} \Psi\big(x, t f(x) + (1-t) f_0(x)\big)\, dt \right| \\ &\leqslant \left(\int_0^1 \left| D_2 \Psi\big(x, t f(x) + (1-t) f_0(x)\big) \right| dt \right) \cdot |f(x) - f_0(x)| \\ &\leqslant \|\Psi\|_{C^1(\mathcal{V})} \cdot |f(x) - f_0(x)| \end{aligned}$$

for all $x \in \mathcal{U}$, implying

$$\|\Psi \circ (\mathrm{Id} \times f) - \Psi \circ (\mathrm{Id} \times f_0)\|_{L^p} \leqslant \|\Psi\|_{C^1(\mathcal{V})} \cdot \|f - f_0\|_{L^p}, \tag{A.7}$$

hence $\Psi \circ (\mathrm{Id} \times f) \in L^p(\mathcal{U}, \mathbb{R}^N)$.

For $\ell = 1, \ldots, k$, we can regard the ℓth derivative of Ψ with respect to variables in $\mathbb{R}^m$ as a bounded and uniformly continuous map from $\mathcal{V}$ into the vector space of symmetric ℓ-multilinear maps from $\mathbb{R}^m$ to $\mathbb{R}^N$, denoting this by

$$D_2^\ell \Psi : \mathcal{V} \to \mathrm{Hom}((\mathbb{R}^m)^{\otimes \ell}, \mathbb{R}^N).$$

Denote the partial derivatives with respect to variables in $\mathcal{U} \subset \mathbb{R}^n$ by

$$D_1^\beta \Psi : \mathcal{V} \to \mathbb{R}^N,$$

where β is a multiindex in n variables. Now for any multiindex α with $|\alpha| \leqslant k$, the derivative $\partial^\alpha(\Psi \circ (\mathrm{Id} \times f))$ is a linear combination of product functions of the form

$$(D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f))(\partial^{\beta_1} f, \ldots, \partial^{\beta_\ell} f) : \mathcal{U} \to \mathbb{R}^N, \tag{A.8}$$

where $\ell + |\gamma| \in \{1, \ldots, |\alpha|\}$ and $|\beta_1| + \ldots + |\beta_\ell| = |\alpha| - |\gamma|$. If $\ell = 0$ but $|\gamma| > 0$, then this expression is clearly in $L^p(\mathcal{U}, \mathbb{R}^N)$ since it is continuous and $D_1^\gamma \Psi(x, v) = 0$ for $x \in \mathcal{U} \backslash \mathcal{K}$, where $\mathcal{K}$ is bounded. For $\ell \geqslant 1$, it satisfies

$$\left\| (D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f))(\partial^{\beta_1} f, \ldots, \partial^{\beta_\ell} f) \right\|_{L^p(\mathcal{U})} \leqslant \|D_1^\gamma D_2^\ell \Psi\|_{C^0(\mathcal{V})} \cdot \left\| \prod_{j=1}^{\ell} |\partial^{\beta_j} f| \right\|_{L^p(\mathcal{U})} \tag{A.9}$$

if the product on the right hand side has finite L^p-norm. The latter is trivially true if $\ell = 1$. To deal with the $\ell \geqslant 2$ case, note that $\partial^{\beta_j} f \in W^{k - |\beta_j|, p}(\mathcal{U})$ for each $j = 1, \ldots, \ell$, so the necessary bound will follow from the existence of a continuous product map

$$W^{k - m_1, p}(\mathcal{U}) \times \ldots \times W^{k - m_\ell, p}(\mathcal{U}) \to L^p(\mathcal{U})$$

for $m_j := |\beta_j|$, and we claim that such a product map does exist whenever $kp > n$ and $m_1, \dots, m_\ell \geqslant 0$ are integers satisfying $m_1 + \dots + m_\ell \leqslant k$. To see this, note first that since $W^{k-m_j,p} \hookrightarrow L^\infty$ whenever $(k - m_j)p > n$, it suffices to prove the claim under the assumption that $(k - m_j)p \leqslant n$ for every $j = 1, \dots, \ell$. In this case, Lemma A.2.2 provides the desired product map if the condition

$$\sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) < \frac{1}{p} \leqslant \sum_{j=1}^{\ell} \frac{1}{p}$$

is satisfied. And it is: using $kp > n$, $\ell \geqslant 2$ and $m_1 + \dots + m_\ell \leqslant k$, we find

$$\begin{aligned} \sum_{j=1}^{\ell} \left(\frac{1}{p} - \frac{k - m_j}{n} \right) &= \ell \left(\frac{1}{p} - \frac{k}{n} \right) + \frac{m_1 + \dots + m_\ell}{n} \\ &\leqslant \frac{1}{p} + (\ell - 1) \left(\frac{1}{p} - \frac{k}{n} \right) < \frac{1}{p}. \end{aligned}$$

This proves that $\Psi \circ (\mathrm{Id} \times f) \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$.

An important detail in both of the estimates (A.7) and (A.9) is that on the right hand side, the term depending on Ψ is bounded by something linearly proportional to $\|\Psi\|_{C^k(\mathcal{V})}$, and the same is true of other estimates mentioned below that can be derived in a similar manner. We will not comment on this point any further, but it is the reason why rather than just proving that the map $(\Psi, f) \mapsto \Psi \circ (\mathrm{Id} \times f)$ is continuous, we will obtain the stronger result that the map sending f to the linear operator $\Psi \mapsto \Psi \circ (\mathrm{Id} \times f)$ is continuous with respect to the operator norm.

Next, suppose $f \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is not necessarily smooth but $f_i \in \mathcal{O}^{k,p}(\mathcal{U}; \mathcal{V})$ is a sequence of smooth functions converging to f in $W^{k,p}$, while $\Psi_i \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ converges to $\Psi \in C_0^k(\overline{\mathcal{V}}, \mathbb{R}^N)$ in C^k. Then the same argument we used to estimate $\|\Psi \circ (\mathrm{Id} \times f) - \Psi \circ (\mathrm{Id} \times f_0)\|_{L^p}$ shows that $\Psi_i \circ (\mathrm{Id} \times f_i) \to \Psi \circ (\mathrm{Id} \times f)$ in L^p, and since f_i is also C^0-convergent, the compactly supported functions $D_1^\gamma \Psi_i \circ (\mathrm{Id} \times f_i)$ converge to $D_1^\gamma \Psi \circ (\mathrm{Id} \times f)$ in L^p for each multiindex with $1 \leqslant |\gamma| \leqslant k$. For $\ell \geqslant 1$ and $|\gamma| + \ell \leqslant k$, $D_1^\gamma D_2^\ell \Psi_i \circ (\mathrm{Id} \times f_i)$ converges to $D_1^\gamma D_2^\ell \Psi \circ (\mathrm{Id} \times f)$ in $C^0(\overline{\mathcal{U}}, \mathbb{R}^N)$, and each of the derivatives $\partial^{\beta_j} f_i$ appearing in (A.8) also converges in $L^p(\mathcal{U})$. In light of the continuous product maps discussed above, it follows that each derivative $\partial^\alpha(\Psi_i \circ (\mathrm{Id} \times f_i))$ for $|\alpha| \leqslant k$ is L^p-convergent, and its limit is necessarily (by Exercise A.2.7 below) the corresponding weak derivative $\partial^\alpha(\Psi \circ (\mathrm{Id} \times f))$, hence $\Psi \circ (\mathrm{Id} \times f) \in W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ and $\Psi_i \circ (\mathrm{Id} \times f_i) \xrightarrow{W^{k,p}} \Psi \circ (\mathrm{Id} \times f)$. Since all sequences in this discussion can also be replaced with subsequences that are pointwise almost everywhere convergent, this also proves that the classical formula for $\partial^\alpha(\Psi_i \circ (\mathrm{Id} \times f_i))$ for each $|\alpha| \leqslant k$ remains valid for computing the corresponding weak derivative $\partial^\alpha(\Psi \circ (\mathrm{Id} \times f))$. With this understood, one can now repeat the arguments of this paragraph for an arbitrary $W^{k,p}$-convergent sequence $f_i \to f$ without assuming the f_i are smooth, thus proving the continuity of the map $(\Psi, f) \mapsto \Psi \circ (\mathrm{Id} \times f)$. □

EXERCISE A.2.7. Show that if f_i is a sequence of smooth functions on an open set $\mathcal{U} \subset \mathbb{R}^n$ with $f_i \xrightarrow{L^p} f$ and $\partial^\alpha f_i \xrightarrow{L^p} g$ for some multiindex α and functions $f, g \in L^p(\mathcal{U})$, then $\partial^\alpha f = g$ in the sense of distributions.

The following result on coordinate transformations of the domain can be proved in an analogous way to Theorem A.2.6, though it is considerably easier since there is no need to worry about Sobolev product maps (and thus no need to assume $kp > n$ or impose regularity conditions on the domain).

THEOREM A.2.8 ([AF03, §3.41]). *Assume $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, and $\mathcal{U}, \mathcal{U}' \subset \mathbb{R}^n$ are open subsets with a C^k-smooth diffeomorphism $\varphi : \mathcal{U} \to \mathcal{U}'$ such that all derivatives of φ and φ^{-1} up to order k are bounded and uniformly continuous. Then there is a well-defined Banach space isomorphism*

$$W^{k,p}(\mathcal{U}') \to W^{k,p}(\mathcal{U}) : f \mapsto f \circ \varphi.$$

□

Next, we restate and prove Proposition 2.2.8. Denote by $\mathring{\mathbb{D}}^n$ and $\mathring{\mathbb{D}}^n_\epsilon(x_0)$ the open balls of radius 1 and ϵ about the origin and a point x_0 respectively in $\mathbb{R}^n$.

THEOREM A.2.9. *Assume $p \in [1, \infty)$ and $k \in \mathbb{N}$ satisfy $kp > n$, and for a given point $x_0 \in \mathring{\mathbb{D}}^n$ with $\epsilon_0 := \operatorname{dist}(x_0, \partial\mathbb{D}^n)$, associate to each $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ and $\epsilon \in (0, \epsilon_0)$ the function $f_\epsilon \in W^{k,p}(\mathring{\mathbb{D}}^n)$ defined by*

$$f_\epsilon(x) := f(x_0 + \epsilon x).$$

Then for each $\alpha \in (0, 1)$ satisfying $\alpha \leqslant k - \frac{n}{p}$, there exists a constant $C > 0$ such that the estimate

$$\|f_\epsilon - f_\epsilon(0)\|_{W^{k,p}} \leqslant C\epsilon^\alpha \|f - f(x_0)\|_{W^{k,p}}$$

holds for all $f \in W^{k,p}(\mathring{\mathbb{D}}^n)$ and $\epsilon \in (0, \epsilon_0)$.

PROOF. To estimate $\|f_\epsilon - f_\epsilon(0)\|_{L^p}$, we use the fact that $f - f(x_0) \in W^{k,p}$ is Hölder continuous, i.e. Theorem A.1.6 embeds $W^{k,p}$ continuously into $C^{0,\alpha}$ for any $\alpha \in (0, 1)$ with $\alpha \leqslant k - n/p$, thus f satisfies

$$|f(x) - f(x_0)| \leqslant c\|f - f(x_0)\|_{W^{k,p}(\mathring{\mathbb{D}}^n)} \cdot |x - x_0|^\alpha \qquad \text{for all } x \in \mathring{\mathbb{D}}^n_{\epsilon_0}(x_0)$$

for some constant $c > 0$. We therefore have

$$\begin{aligned} \|f_\epsilon - f_\epsilon(0)\|^p_{L^p} &= \int_{\mathbb{D}^n} |f(x_0 + \epsilon x) - f(x_0)|^p \leqslant c^p \|f - f(x_0)\|^p_{W^{k,p}} \int_{\mathbb{D}^n} |\epsilon x|^{\alpha p} \\ &= c^p \|f - f(x_0)\|^p_{W^{k,p}} \cdot \epsilon^{\alpha p} \int_{\mathbb{D}^n} |x|^{\alpha p} =: C^p \epsilon^{\alpha p} \|f - f(x_0)\|^p_{W^{k,p}} \end{aligned}$$

for a suitable constant $C > 0$, implying $\|f_\epsilon - f_\epsilon(0)\|_{L^p} \leqslant C\epsilon^\alpha \|f - f(x_0)\|_{W^{k,p}}$.

Next, consider a multiindex β of order $|\beta| = m \in \{1, \ldots, k\}$. The functions $\partial^\beta(f - f(x_0)) = \partial^\beta f$ and $\partial^\beta(f_\epsilon - f_\epsilon(0)) = \partial^\beta f_\epsilon$ for each $\epsilon \in (0, \epsilon_0)$ are then in $W^{k-m,p}(\mathring{\mathbb{D}}^n)$, and we need to establish bounds on $\|\partial^\beta f_\epsilon\|_{L^p}$ in terms of the $W^{k,p}$-norm of $f - f(x_0)$. If $m < k$, then Theorem A.1.6 gives a continuous inclusion

$$W^{k-m,p}(\mathring{\mathbb{D}}^n) \hookrightarrow L^q(\mathring{\mathbb{D}}^n) \tag{A.10}$$

for any $q \in [p, \infty)$ satisfying $1/q \geqslant 1/p - (k - m)/n$. The same is also trivially true in the case $m = k$, since q and p must then be equal. Notice that if $(k - m)p \geqslant n$, then q is allowed to be arbitrarily large. We will therefore assume in general that (A.10) holds with $q \in [p, \infty)$ satisfying

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{p},$$

where $r = \frac{n}{k-m} \in (0, \infty]$ if $(k-m)p < n$ and otherwise $r = p + \delta$ for some $\delta > 0$ which may be chosen arbitrarily small. Given this, we apply change of variables and Hölder's inequality to find

$$\begin{aligned}
\|\partial^\beta f_\epsilon\|^p_{L^p(\mathring{\mathbb{D}}^n)} &= \epsilon^{mp} \int_{\mathbb{D}^n} |\partial^\beta f(x_0 + \epsilon x)|^p = \epsilon^{mp-n} \int_{\mathbb{D}^n_\epsilon(x_0)} |\partial^\beta f(x)|^p \\
&\leqslant \epsilon^{mp-n} \|\partial^\beta f\|^p_{L^q(\mathring{\mathbb{D}}^n_\epsilon)} \|1\|^p_{L^r(\mathring{\mathbb{D}}^n_\epsilon)} \\
&\leqslant \epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon(x_0))\right]^{p/r} \|\partial^\beta f\|^p_{L^q(\mathring{\mathbb{D}}^n)} \\
&\leqslant c\epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon(x_0))\right]^{p/r} \|\partial^\beta f\|^p_{W^{k-m,p}(\mathring{\mathbb{D}}^n)} \\
&\leqslant c\epsilon^{mp-n} \left[\mathrm{Vol}(\mathbb{D}^n_\epsilon(x_0))\right]^{p/r} \|f - f(x_0)\|^p_{W^{k,p}(\mathring{\mathbb{D}}^n)}
\end{aligned}$$

for some constant $c > 0$. Writing $\mathrm{Vol}(\mathbb{D}^n_\epsilon(x_0)) = C\epsilon^n$ for a suitable constant $C > 0$, the exponent on ϵ in this expression becomes $mp - n + \frac{np}{r}$. If $(k-m)p < 0$, this is exactly $kp - n = (k - n/p)p$, and otherwise, taking $r - p > 0$ to be arbitrarily small makes it less than but arbitrarily close to mp. Since $\alpha \leqslant k - n/p$ and $\alpha < 1 \leqslant m$, we are now free to replace this exponent with αp and rewrite the established estimate as $\|\partial^\beta f_\epsilon\|_{L^p} \leqslant C\epsilon^\alpha \|f - f(x_0)\|_{W^{k,p}}$. □

A.3. Difference quotients

If f is a function on $\mathbb{R}^n$, then for every $i = 1, \ldots, n$ and $h \in \mathbb{R}\backslash\{0\}$, the **difference quotient**

$$D^h_i f(x_1, \ldots, x_n) := \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

defines a function $D^h_i f$ on $\mathbb{R}^n$. The **total difference quotient** of f is then the n-tuple of functions

$$D^h f := (D^h_1 f, \ldots, D^h_n f),$$

so for example if $f : \mathbb{R}^n \to \mathbb{R}^m$, then $D^h f : \mathbb{R}^n \to \mathbb{R}^{mn}$. The transformation $f \mapsto D^h_i f$ is obviously linear for any fixed number h, and it satisfies a Leibniz rule

$$D^h_i(fg) = D^h_i f \cdot g + f \cdot D^h_i g$$

whenever pointwise products of f and g can be defined (e.g. if both are real or complex valued). It also commutes with differentiation

$$D^h_i(\partial_j f) = \partial_j(D^h_i f)$$

on any function f for which $\partial_j f$ can be defined (weakly or strongly). Clearly if $f \in W^{k,p}(\mathbb{R}^n)$, then $D^h f \in W^{k,p}(\mathbb{R}^n)$ for every $h \in \mathbb{R}\backslash 0$, and if f is supported in an open subset $\mathcal{U} \subset \mathbb{R}^n$, then $D^h f$ is supported in an arbitrarily small neighborhood of $\overline{\mathcal{U}}$ for sufficiently small $|h|$. Moreover, if f is a function defined only on $\mathcal{U} \subset \mathbb{R}^n$, then on any open subset $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$, $D^h f$ can be defined on $\mathcal{V}$ for any $h \in \mathbb{R}\backslash\{0\}$ satisfying

$$|h| < \operatorname{dist}(\mathcal{V}, \mathbb{R}^n \backslash \mathcal{U}) := \inf \left\{ |x - y| \;\middle|\; x \in \mathcal{V} \text{ and } y \in \mathbb{R}^n \backslash \mathcal{U} \right\}.$$

The following result about difference quotients is useful for proving local regularity of solutions to PDEs, as in §2.4.

THEOREM A.3.1. *Assume $\mathcal{V} \subset \mathcal{U} \subset \mathbb{R}^n$ are open subsets with $\mathcal{V}$ having compact closure contained in $\mathcal{U}$, $1 \leqslant p < \infty$, and $k \in \mathbb{N}$.*

(1) If $f \in W^{k,p}(\mathcal{U})$, then $D^h f$ converges to ∇f in $W^{k-1,p}$ on $\mathcal{V}$ as $h \to 0$, and

$$\|D^h f\|_{W^{k-1,p}(\mathcal{V})} \leqslant \|\nabla f\|_{W^{k-1,p}(\mathcal{U})}$$

for all $h \neq 0$ with $|h| < \operatorname{dist}(\mathcal{V}, \mathbb{R}^n \backslash \mathcal{U})$.

(2) Suppose $p > 1$, $f \in W^{k-1,p}(\mathcal{U})$ and the difference quotients $D^h f$ satisfy a uniform bound

$$\|D^h f\|_{W^{k-1,p}(\mathcal{V})} \leqslant C$$

for all $h \neq 0$ with $|h|$ sufficiently small. Then $f|_{\mathcal{V}} \in W^{k,p}(\mathcal{V})$ and its first derivative satisfies $\|\nabla f\|_{W^{k-1,p}(\mathcal{V})} \leqslant m_{k,p} C$, where $m_{k,p} \in \mathbb{N}$ is a constant depending only on the definition of the $W^{k-1,p}$-norm.

The next few results are intended as preparation for the proof of Theorem A.3.1.

LEMMA A.3.2. *For any open subset $\mathcal{U} \subset \mathbb{R}^n$ and continuously differentiable function f on $\mathcal{U}$, the difference quotients $D_i^h f$ converge to $\partial_i f$ uniformly on compact subsets as $h \to 0$.*

PROOF. Fix a compact subset $\mathcal{K} \subset \mathcal{U}$. Then for every $x \in \mathcal{K}$ and $h \in \mathbb{R}\backslash\{0\}$ sufficiently small, the mean value theorem gives

$$D_i^h f(x) = \partial_i f(x')$$

where

$$x' := (x_1, \ldots, x_{i-1}, x_i + th, x_{i+1}, \ldots, x_n) \in \mathcal{U}$$

for some $t \in [0, 1]$, so in particular, $|x' - x| \leqslant |h|$. We then have $|\partial_i f(x) - D_i^h f(x)| = |\partial_i f(x) - \partial_i f(x')|$, and the result follows since both x and x' may be assumed to lie in a compact subset of $\mathcal{U}$, on which $\partial_i f$ is uniformly continuous. □

PROPOSITION A.3.3. *Suppose $1 \leqslant p < \infty$, $\mathcal{U} \subset \mathbb{R}^n$ is an open subset and $f \in W^{1,p}(\mathcal{U})$. Then for any open subset $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$, $\|D^h f\|_{L^p(\mathcal{V})} \leqslant \|\nabla f\|_{L^p(\mathcal{U})}$ for every $h \neq 0$ with $|h| < \operatorname{dist}(\mathcal{V}, \mathbb{R}^n \backslash \mathcal{U})$, and $D^h f \to \nabla f$ in L^p on $\mathcal{V}$ as $h \to 0$.*

PROOF. We show first that for any $f \in W^{1,p}(\mathcal{U})$,

$$\|D_i^h f\|_{L^p(\mathcal{V})} \leqslant \|\partial_i f\|_{L^p(\mathcal{U})}, \qquad i = 1, \ldots, n \tag{A.11}$$

for every $\mathcal{V} \subset \mathcal{U}$ with compact closure in $\mathcal{U}$ and every $h \neq 0$ with $|h| < \operatorname{dist}(\mathcal{V}, \mathbb{R}^n \backslash \mathcal{U})$. Indeed, if $f \in W^{1,p}(\mathcal{U}) \cap C^\infty(\mathcal{U})$, then denoting the standard basis of $\mathbb{R}^n$ by $(e_1, \ldots, e_n)$, we have

$$\left|D_i^h f(x)\right| = \left|\frac{f(x+he_i)-f(x)}{h}\right| = \left|\frac{1}{h}\int_0^1 \frac{d}{dt} f(x+the_i)\,dt\right|$$
$$= \left|\int_0^1 \partial_i f(x+the_i)\,dt\right| \leqslant \int_0^1 |\partial_i f(x+the_i)|\,dt.$$

Then since any measurable function $g : [0,1] \to \mathbb{R}$ satisfies

$$\left(\int_0^1 |g(t)|\,dt\right)^p \leqslant \int_0^1 |g(t)|^p\,dt$$

by Jensen's inequality, this gives

$$\|D_i^h f\|^p_{L^p(\mathcal{V})} = \int_{\mathcal{V}} \left|D_i^h f(x)\right|^p\,d\mu(x) \leqslant \int_{\mathcal{V}} \left(\int_0^1 |\partial_i f(x+the_i)|\,dt\right)^p d\mu(x)$$
$$\leqslant \int_{\mathcal{V}}\int_0^1 |\partial_i f(x+the_i)|^p\,dt\,d\mu(x) = \int_0^1 \int_{\mathcal{V}} |\partial_i f(x+the_i)|^p\,d\mu(x)\,dt$$
$$\leqslant \int_0^1 \|\partial_i f\|^p_{L^p(\mathcal{U})}\,dt = \|\partial_i f\|^p_{L^p(\mathcal{U})}.$$

This estimate extends to every $f \in W^{1,p}(\mathcal{U})$ by density of smooth functions.

Next, suppose $f \in W^{1,p}(\mathcal{U})$ and $\epsilon > 0$ is given. Choose a smooth approximation $f_\epsilon \in W^{1,p}(\mathcal{U}) \cap C^\infty(\mathcal{U})$ with $\|f - f_\epsilon\|_{W^{1,p}(\mathcal{U})} < \epsilon/3$. By Lemma A.3.2, $D_i^h f_\epsilon \to \partial_i f_\epsilon$ in C^0_{loc} on $\mathcal{U}$ as $h \to 0$, and since $\mathcal{V}$ has finite measure, this implies we can find $\delta > 0$ such that $|h| < \delta$ implies $\|D_i^h f_\epsilon - \partial_i f_\epsilon\|_{L^p(\mathcal{V})} < \epsilon/3$. Now by (A.11),

$$\|D_i^h f_\epsilon - D_i^h f\|_{L^p(\mathcal{V})} \leqslant \|\partial_i f_\epsilon - \partial_i f\|_{L^p(\mathcal{U})} \leqslant \|f_\epsilon - f\|_{W^{1,p}(\mathcal{U})} < \epsilon/3,$$

so combining these estimates gives $\|D_i^h f - \partial_i f\|_{L^p(\mathcal{V})} < \epsilon$ whenever $|h| < \delta$. ☐

The proof of the next proposition will require the following standard result from real analysis, known as the **Banach-Alaoglu theorem**. It follows easily from the separability of L^p-spaces for $p < \infty$ together with the duality of L^p and L^q for $1/p + 1/q = 1$; see for instance [**LL01**, §2.18].

THEOREM A.3.4 (Banach-Alaoglu). *For any measurable subset $\mathcal{U} \subset \mathbb{R}^n$, if $1 < p < \infty$, then every bounded sequence $f_j \in L^p(\mathcal{U})$ has a weakly convergent subsequence, i.e. after passing to a subsequence, one can find a function $f_\infty \in L^p(\mathcal{U})$ such that for every $\varphi \in L^q(\mathcal{U})$ with $1/p + 1/q = 1$, $\int_{\mathcal{U}} f_j\varphi \to \int_{\mathcal{U}} f_\infty \varphi$.* ☐

REMARK A.3.5. One popular way of summarizing the Banach-Alaoglu theorem is the statement that "closed balls in L^p are weakly compact"; indeed, if $f_j \in L^p(\mathcal{U})$ satisfies the bound $\|f_j\|_{L^p} \leqslant C$, then the weak limit f_∞ provided by Theorem A.3.4 also satisfies $\|f_\infty\|_{L^p} \leqslant C$. The latter follows from the general fact that for any sequence $f_j \in L^p(\mathcal{U})$ converging weakly to some $f_\infty \in L^p(\mathcal{U})$,

$$\|f_\infty\|_{L^p(\mathcal{U})} \leqslant \liminf \|f_j\|_{L^p(\mathcal{U})}.$$

The proof of this is not hard; see e.g. [**LL01**, §2.11].

PROPOSITION A.3.6. *Suppose $\mathcal{V} \subset \mathcal{U} \subset \mathbb{R}^n$ are open subsets such that $\mathcal{V}$ has compact closure contained in $\mathcal{U}$, $1 < p < \infty$, f is a measurable function on $\mathcal{U}$ with $\|f\|_{L^p(\mathcal{V})} < \infty$, and there exist constants $C > 0$ and $\delta > 0$ such that*

$$\|D_i^h f\|_{L^p(\mathcal{V})} \leqslant C \qquad \text{whenever } 0 < |h| < \delta.$$

Then $f|_{\mathcal{V}}$ has a weak partial derivative $\partial_i f \in L^p(\mathcal{V})$ satisfying $\|\partial_i f\|_{L^p(\mathcal{V})} \leqslant C$.

PROOF. For any sequence $h_j \to 0$ of sufficiently small nonzero real numbers, the sequence $D_i^{h_j} f$ satisfies $\|D_i^{h_j} f\|_{L^p(\mathcal{V})} \leqslant C$, thus the Banach-Alaoglu theorem implies that after passing to a subsequence, one finds a function $g \in L^p(\mathcal{V})$ with $\|g\|_{L^p(\mathcal{V})} \leqslant C$ such that

$$\int_{\mathcal{V}} (D_i^{h_j} f)\varphi \to \int_{\mathcal{V}} g\varphi$$

for all $\varphi \in L^q(\mathcal{V})$, where $1/p + 1/q = 1$. In particular, this is true for all test functions $\varphi \in C_0^\infty(\mathcal{V})$, and in this case there is an "integration by parts" relation

$$\begin{aligned}\int_{\mathcal{V}} (D_i^{h_j} f)\varphi &= \int_{\mathcal{V}} \frac{f(x + h_j e_i) - f(x)}{h_j} \varphi(x)\, d\mu(x) \\ &= -\int_{\mathcal{V}} f(x) \frac{\varphi(x - h_j e_i) - \varphi(x)}{-h_j}\, d\mu(x) = -\int_{\mathcal{V}} f D_i^{-h_j}\varphi.\end{aligned}$$

By Lemma A.3.2, $D_i^{-h_j}\varphi \to \partial_i\varphi$ uniformly on $\mathcal{V}$ and thus also in $L^q(\mathcal{V})$, so taking the limit of the integrals, we've shown

$$\int_{\mathcal{V}} g\varphi = -\int_{\mathcal{V}} f\partial_i\varphi \qquad \text{for all } \varphi \in C_0^\infty(\mathcal{V}),$$

or in other words, $\partial_i f = g \in L^p(\mathcal{V})$. □

PROOF OF THEOREM A.3.1. The two statements in the theorem follow by applying Propositions A.3.3 and A.3.6 respectively to $\partial^\alpha f$ for every multiindex α with $|\alpha| \leqslant k-1$, using the fact that $D^h(\partial^\alpha f) = \partial^\alpha(D^h f)$. For the bound on $\|\nabla f\|_{W^{k-1,p}(\mathcal{V})}$, we observe that by assumption,

$$\|D^h f\|_{W^{k-1,p}(\mathcal{V})} = \sum_{|\alpha| \leqslant k-1} \|\partial^\alpha(D^h f)\|_{L^p(\mathcal{V})} = \sum_{|\alpha| \leqslant k-1} \|D^h(\partial^\alpha f)\|_{L^p(\mathcal{V})} \leqslant C,$$

thus each individual term in this sum satisfies $\|D^h(\partial^\alpha f)\|_{L^p(\mathcal{V})} \leqslant C$, implying $\|\nabla(\partial^\alpha f)\|_{L^p(\mathcal{V})} \leqslant C$ and thus

$$\begin{aligned}\|\nabla f\|_{W^{k-1,p}(\mathcal{V})} &= \sum_{|\alpha| \leqslant k-1} \|\partial^\alpha(\nabla f)\|_{L^p(\mathcal{V})} = \sum_{|\alpha| \leqslant k-1} \|\nabla(\partial^\alpha f)\|_{L^p(\mathcal{V})} \\ &\leqslant \sum_{|\alpha| \leqslant k-1} C =: m_{k,p} C.\end{aligned}$$

□

A.4. Spaces of sections of vector bundles

In this section, fix a field

$$\mathbb{F} := \mathbb{R} \text{ or } \mathbb{C},$$

assume M is a smooth n-dimensional manifold, possibly with boundary, and $\pi : E \to M$ is a smooth vector bundle of rank m over $\mathbb{F}$. This comes with a "bundle atlas" $\mathcal{A}(\pi)$, a set whose elements $\alpha \in \mathcal{A}(\pi)$ each consist of the following data:

(1) An open subset $\mathcal{U}_\alpha \subset M$;
(2) A smooth local coordinate chart $\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha$, where Ω_α is an open subset of $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geqslant 0\}$;
(3) A smooth local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{F}^m$.

Smoothness of φ_α and Φ_α means as usual that for every pair $\alpha, \beta \in \mathcal{A}(\pi)$, the coordinate transformations

$$\varphi_{\beta\alpha} := \varphi_\beta^{-1} \circ \varphi_\alpha : \Omega_{\alpha\beta} \xrightarrow{\cong} \Omega_{\beta\alpha}, \qquad \Omega_{\alpha\beta} := \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

and transition maps

$$g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F}) \quad \text{such that} \quad \Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, g_{\beta\alpha}(x)v) \quad \text{for } x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta,\ v \in \mathbb{F}^m$$

are smooth, and we shall assume the bundle atlas is maximal in the sense that any triple $(\mathcal{U}, \varphi, \Phi)$ that is smoothly compatible with every $\alpha \in \mathcal{A}(\pi)$ also belongs to $\mathcal{A}(\pi)$.

Any $\alpha \in \mathcal{A}(\pi)$ now associates to sections $\eta : M \to E$ their local coordinate representatives

$$\eta^\alpha := \mathrm{pr}_2 \circ \Phi_\alpha \circ \eta \circ \varphi_\alpha^{-1} : \Omega_\alpha \to \mathbb{F}^m,$$

where $\mathrm{pr}_2 : \mathcal{U}_\alpha \times \mathbb{F}^m \to \mathbb{F}^m$ is the projection, and the representatives with respect to two distinct $\alpha, \beta \in \mathcal{A}(\pi)$ are related by

$$\eta^\beta = (g_{\beta\alpha} \circ \varphi_\beta^{-1})(\eta^\alpha \circ \varphi_{\alpha\beta}) \quad \text{on } \Omega_{\beta\alpha} \subset \Omega_\beta.$$

For $p \in [1, \infty]$ and each integer $k \geqslant 0$, we then define the topological vector space of sections of class $W^{k,p}_{\mathrm{loc}}$ by

$$W^{k,p}_{\mathrm{loc}}(E) := \left\{ \eta : M \to E \;\middle|\; \text{sections such that } \eta^\alpha \in W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{F}^m) \text{ for all } \alpha \in \mathcal{A}(\pi) \right\},$$

where convergence $\eta_j \to \eta$ in $W^{k,p}_{\mathrm{loc}}(E)$ means that $\eta_j^\alpha \to \eta^\alpha$ in $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha, \mathbb{F}^m)$ for all $\alpha \in \mathcal{A}(\pi)$. Note that Ω_α is not necessarily an open subset of $\mathbb{R}^n$ since it may contain points in $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$, but its interior $\mathring{\Omega}_\alpha$ is open in $\mathbb{R}^n$, and $W^{k,p}_{\mathrm{loc}}(\mathring{\Omega}_\alpha)$ is thus defined as in §A.1. Strictly speaking, elements of $\eta \in W^{k,p}_{\mathrm{loc}}(E)$ are not sections but *equivalence classes* of sections defined almost everywhere—the latter notion is defined with respect to any measure arising from a smooth volume element on M, and it does not depend on this choice.

It turns out that $W^{k,p}_{\mathrm{loc}}(E)$ can be given the structure of a Banach space if M is compact. This follows from the fact that M can then be covered by a finite subset of the atlas $\mathcal{A}(\pi)$, but we must be a little bit careful: not all charts in $\mathcal{A}(\pi)$ are equally suitable for defining $W^{k,p}$-norms on sections, because e.g. even a nice smooth section $\eta \in \Gamma(E)$ may have $\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)} = \infty$ if $\Omega_\alpha \subset \mathbb{R}^n_+$ is unbounded. One way to deal with this is as follows: we will say that $\alpha \in \mathcal{A}(\pi)$ is a **precompact chart** if there exists $\alpha' \in \mathcal{A}(\pi)$ and a compact subset $\mathcal{K} \subset M$ such that

$$\mathcal{U}_\alpha \subset \mathcal{K} \subset \mathcal{U}_{\alpha'}.$$

When this is the case, $\Omega_\alpha \subset \mathbb{R}^n_+$ is necessarily bounded, and the transition maps between two precompact charts necessarily have bounded derivatives of all orders, as they are restrictions to precompact subsets of maps that are smooth on larger domains. If M is compact, then one can always find a finite subset $I \subset \mathcal{A}(\pi)$ consisting of precompact charts such that $M = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$.

DEFINITION A.4.1. Suppose $E \to M$ is a smooth vector bundle over a compact manifold M, and $I \subset \mathcal{A}(\pi)$ is a finite set of precompact charts such that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is an open cover of M. We then define $W^{k,p}(E)$ as the vector space of all sections $\eta : M \to E$ for which the norm

$$\|\eta\|_{W^{k,p}} := \|\eta\|_{W^{k,p}(E)} := \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}$$

is finite.

The norm in the above definition depends on auxiliary choices, but it is easy to see that the resulting definition of the space $W^{k,p}(E)$ and its topology do not. In fact:

PROPOSITION A.4.2. *If M is compact, then $W^{k,p}(E) = W^{k,p}_{\mathrm{loc}}(E)$, and a sequence η_j converges to η in $W^{k,p}_{\mathrm{loc}}(E)$ if and only if the norm given in Definition A.4.1 satisfies $\|\eta_j - \eta\|_{W^{k,p}(E)} \to 0$.*

The proposition is an immediate consequence of the following.

LEMMA A.4.3. *Suppose M is a smooth manifold, $\pi : E \to M$ is a smooth vector bundle, $\{\beta\} \cup J \subset \mathcal{A}(\pi)$ is a finite collection of charts such that $M = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$ and all coordinate transformations and transition maps relating any two charts in the collection $\{\beta\} \cup J$ have bounded derivatives of all orders (e.g. it suffices to assume all are precompact). Then there exists a constant $c > 0$ such that*

$$\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} \leqslant c \sum_{\alpha \in J} \|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}$$

for all sections $\eta : M \to E$ with $\eta^\alpha \in W^{k,p}(\mathring{\Omega}_\alpha)$ for every $\alpha \in J$.

PROOF. Choose a partition of unity $\{\rho_\alpha : M \to [0,1]\}_{\alpha \in J}$ subordinate to the finite open cover $\{\mathcal{U}_\alpha\}_{\alpha \in J}$. Now $\eta = \sum_{\alpha \in J} \rho_\alpha \eta$, and each $\rho_\alpha \eta$ is supported in $\mathcal{U}_\alpha$, so $(\rho_\alpha \eta)^\beta$ has support in $\Omega_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Thus using Theorem A.2.8 with the fact that $g_{\beta\alpha}$, φ_β^{-1}, $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$ are all smooth functions with bounded derivatives of all orders on the domains in question, we find

$$\begin{aligned}\|\eta^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} &= \left\|\sum_{\alpha\in J}(\rho_\alpha\eta)^\beta\right\|_{W^{k,p}(\mathring{\Omega}_\beta)} \leqslant \sum_{\alpha\in J}\|(\rho_\alpha\eta)^\beta\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})}\\ &= \sum_{\alpha\in J}\|(\rho_\alpha\circ\varphi_\beta^{-1})(g_{\beta\alpha}\circ\varphi_\beta^{-1})(\eta^\alpha\circ\varphi_{\alpha\beta})\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})}\\ &\leqslant c\sum_{\alpha\in J}\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_{\alpha\beta})} \leqslant c\sum_{\alpha\in J}\|\eta^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)}.\end{aligned}$$

□

COROLLARY A.4.4. *If M is compact, then the norm on $W^{k,p}(E)$ given by Definition A.4.1 is independent of all auxiliary choices up to equivalence of norms.* □

THEOREM A.4.5. *For any smooth vector bundle $\pi : E \to M$ over a compact manifold M, $W^{k,p}(E)$ is a Banach space.*

PROOF. If $\eta_j \in W^{k,p}(E)$ is a Cauchy sequence, then for some chosen finite collection $I \subset \mathcal{A}(\pi)$ of precompact charts covering M, the sequences η_j^α for $\alpha \in I$ are Cauchy in $W^{k,p}(\mathring{\Omega}_\alpha)$ and thus have limits $\xi^{(\alpha)} \in W^{k,p}(\mathring{\Omega}_\alpha, \mathbb{F}^m)$. Choosing a partition of unity $\{\rho_\alpha : M \to [0,1]\}_{\alpha\in I}$ subordinate to $\{\mathcal{U}_\alpha\}_{\alpha\in I}$, we can now associate to each $\alpha \in I$ a section $\eta_{\infty,\alpha} \in W^{k,p}(E)$ characterized uniquely by the condition that it vanishes outside of $\mathcal{U}_\alpha$ and is represented in the trivialization on $\mathcal{U}_\alpha$ by

$$\eta_{\infty,\alpha}^\alpha = (\rho_\alpha \circ \varphi_\alpha^{-1})\xi^{(\alpha)}.$$

We claim that $\rho_\alpha\eta_j \to \eta_{\infty,\alpha}$ in $W^{k,p}(E)$ for each $\alpha \in I$. Indeed, we have

$$(\rho_\alpha\eta_j)^\alpha = (\rho_\alpha\circ\varphi_\alpha^{-1})\eta_j^\alpha \to (\rho_\alpha\circ\varphi_\alpha^{-1})\xi^{(\alpha)} = \eta_{\infty,\alpha}^\alpha \quad \text{in } W^{k,p}(\mathring{\Omega}_\alpha)$$

since $\eta_j^\alpha \to \xi^{(\alpha)}$. For all other $\beta \in I$ not equal to α, $(\rho_\alpha\eta_j)^\beta - \eta_{\infty,\alpha}^\beta \in W^{k,p}(\mathring{\Omega}_\beta, \mathbb{F}^m)$ has support in $\Omega_{\beta\alpha} = \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, thus

$$\|(\rho_\alpha\eta_j)^\beta - \eta_{\infty,\alpha}^\beta\|_{W^{k,p}(\mathring{\Omega}_\beta)} = \|(\rho_\alpha\eta_j)^\beta - \eta_{\infty,\alpha}^\beta\|_{W^{k,p}(\mathring{\Omega}_{\beta\alpha})} \leqslant c\|(\rho_\alpha\eta_j)^\alpha - \eta_{\infty,\alpha}^\alpha\|_{W^{k,p}(\mathring{\Omega}_\alpha)},$$

where the inequality comes from Lemma A.4.3 after replacing M with $\mathcal{U}_\alpha$, and $\mathcal{U}_\beta$ with $\mathcal{U}_\beta \cap \mathcal{U}_\alpha$ (note that the lemma does not require M to be compact). With the claim established, we have

$$\eta_j = \sum_{\alpha\in I}\rho_\alpha\eta_j \to \sum_{\alpha\in I}\eta_{\infty,\alpha} \quad \text{in } W^{k,p}(E).$$

□

REMARK A.4.6. One can use exactly the same approach to show that when M is compact, the space $C^k(E)$ of C^k-smooth sections $\eta : M \to E$ has a canonical (up to equivalence of norms) Banach space structure for each finite integer $k \geqslant 0$ such that convergence in the C^k-norm is equivalent to uniform convergence of all derivatives up to order k.

EXERCISE A.4.7. For $\mathcal{U} \subset \mathbb{R}^n$ an open subset, the space $W^{k,p}_{\text{loc}}(\mathcal{U})$ was defined in §A.1, but one can give it an alternative definition in the present context by viewing functions on $\mathcal{U}$ as sections of a trivial vector bundle over $\mathcal{U}$, with the latter viewed as a noncompact smooth n-manifold. Show that these two definitions of $W^{k,p}_{\text{loc}}(\mathcal{U})$ are equivalent.

EXERCISE A.4.8. Suppose $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, so its closure $\overline{\mathcal{U}} \subset \mathbb{R}^n$ is a smooth compact submanifold with boundary, and let $E \to \overline{\mathcal{U}}$ be a trivial vector bundle. Show that there is a canonical Banach space isomorphism between $W^{k,p}(\mathcal{U})$ as defined in §A.1 and $W^{k,p}(E)$ as defined in the present section. *Hint: Recall that sections in $W^{k,p}(E)$ are only required to be defined almost everywhere, so in particular if the domain M is a manifold with boundary, they need not be well defined on ∂M.*

In light of Exercise A.4.8, the natural generalization of $W^{k,p}_0(\mathcal{U})$ in the present setting is

$$W^{k,p}_0(E) := \overline{C^\infty_0(E|_{M\setminus\partial M})},$$

i.e. it is the closure in the $W^{k,p}$-norm of the space of smooth sections that vanish near the boundary. Density of smooth sections will imply that this is the same as $W^{k,p}(E)$ if M is closed, but in general $W^{k,p}_0(E)$ is a closed subspace of $W^{k,p}(E)$.

The partition of unity argument in Theorem A.4.5 contains all the essential ideas needed to generalize results about Sobolev spaces on domains in $\mathbb{R}^n$ to compact manifolds. We now state the essential results, leaving the proofs as exercises.

THEOREM A.4.9. *Assume M is a smooth compact n-manifold, possibly with boundary, $\pi : E \to M$ is a smooth vector bundle of finite rank, $k \geqslant 0$ is an integer and $1 \leqslant p < \infty$. Then the Banach space $W^{k,p}(E)$ has the following properties.*

(1) The space $\Gamma(E)$ of smooth sections is dense in $W^{k,p}(E)$.

(2) If $kp > n$, then for each integer $d \geqslant 0$, there exists a continuous and compact inclusion

$$W^{k+d,p}(E) \hookrightarrow C^d(E).$$

(3) The natural inclusion

$$W^{k+1,p}(E) \hookrightarrow W^{k,p}(E)$$

is compact.

(4) Suppose $F, G \to M$ are smooth vector bundles such that there exists a smooth bundle map

$$E \otimes F \to G : \eta \otimes \xi \mapsto \eta \cdot \xi.$$

Then if $kp > n$ and $0 \leqslant m \leqslant k$, there exists a continuous product pairing

$$W^{k,p}(E) \times W^{m,p}(F) \to W^{m,p}(G) : (\eta, \xi) \mapsto \eta \cdot \xi.$$

In particular, products of $W^{k,p}$ sections give $W^{k,p}$ sections whenever $kp > n$.

(5) Suppose $F \to M$ is another smooth vector bundle, $\mathcal{V} \subset E$ is an open subset that intersects every fiber of E, and we consider the spaces

$$W^{k,p}(\mathcal{V}) := \{\eta \in W^{k,p}(E) \mid \eta(M) \subset \mathcal{V}\}$$

and

$$C^k_M(\mathcal{V}, F) := \left\{\Phi : \mathcal{V} \to F \mid \text{fiber-preserving maps of class } C^k\right\},$$

where the latter is assigned the topology of C^k-convergence on compact subsets. If $kp > n$, then $W^{k,p}(\mathcal{V})$ is an open subset of $W^{k,p}(E)$, and the map

$$C^k_M(\mathcal{V}, F) \times W^{k,p}(\mathcal{V}) \to W^{k,p}(F) : (\Phi, \eta) \mapsto \Phi \circ \eta$$

is well defined and continuous.

(6) If N is another smooth compact manifold and $\varphi : N \to M$ is a smooth diffeomorphism, then there is a Banach space isomorphism

$$W^{k,p}(E) \to W^{k,p}(\varphi^* E) : \eta \mapsto \eta \circ \varphi.$$

□

REMARK A.4.10. It is sometimes useful to extend the definitions and results of this section to vector bundles that are not smooth, e.g. vector bundles of class C^k or $W^{k,p}$, for which all transition maps are required to be of class C^k or $W^{k,p}$ respectively. The latter makes sense in general only if $kp > n$, so that transition maps are at least continuous. Given a bundle of this type, one can enhance the arguments of this section with the aid of Theorem A.2.1 to show that $W^{m,p}(E)$ is a well-defined Banach space for every $m \leqslant k$, though it would not be well defined if $m > k$. Such spaces arise frequently in global analysis, e.g. if f is a non-smooth element in the Banach manifold $\mathcal{B}$ of $W^{k,p}$-smooth maps of M into another manifold N, then $f^*TN \to M$ is in general a vector bundle of class $W^{k,p}$, and $T_f\mathcal{B} = W^{k,p}(f^*TN)$.

A.5. Some remarks on domains with cylindrical ends

For bundles $\pi : E \to M$ with M noncompact, $W^{k,p}(E)$ is not generally well defined without making additional choices. When $M = \dot{\Sigma} = \Sigma \setminus \Gamma$ is a punctured Riemann surface and $\pi : E \to \dot{\Sigma}$ is equipped with an asymptotically Hermitian structure $\{(E_z, J_z, \omega_z)\}_{z \in \Gamma}$ as defined in Chapter 4, one nice way to define $W^{k,p}(E)$ was introduced in §4.1: one takes it to be the space of sections in $W^{k,p}_{\text{loc}}(E)$ whose $W^{k,p}$-norms on each cylindrical end are finite with respect to a choice of asymptotic trivialization. This definition requires the convenient fact that complex vector bundles over S^1 are always trivial, though one can also do without this by using the ideas in the previous section. Indeed, any collection of local trivializations on the asymptotic bundle $E_z \to S^1$ covering S^1 gives rise via the asymptotically Hermitian structure to a collection of trivializations on E covering the corresponding cylindrical end $\dot{\mathcal{U}}_z$. The key fact is then that S^1 is compact, hence one can always choose such a covering to be finite: combining this with a finite covering of $\dot{\Sigma}$ in the complement of its cylindrical ends by precompact charts, we obtain a covering of $\dot{\Sigma}$ by a finite collection of bundle charts that are not all precompact, but nonetheless have the property that all transition maps have bounded derivatives of all orders. This is enough to define a $W^{k,p}$-norm for sections of $E \to \dot{\Sigma}$ as in Definition A.4.1 and to prove that it does not depend on the choices of charts or local trivializations, though it does depend on the asymptotically Hermitian structure.

With this definition understood, one can easily generalize the Sobolev embedding theorem and other important statements in Theorem A.4.9 to the setting of an asymptotically Hermitian bundle over a punctured Riemann surface. We shall leave the details of this generalization as an exercise, but take the opportunity to point out a few important differences from the compact case.

First, since $\dot{\Sigma}$ is not compact, neither are the inclusions

$$W^{k+d,p}(E) \hookrightarrow C^d(E), \qquad W^{k+1,p}(E) \hookrightarrow W^{k,p}(E).$$

The proof of compactness fails due to the fact that cylindrical ends require local trivializations over unbounded domains of the form $(0,\infty) \times (0,1) \subset \mathbb{R}^2$, for which Theorem A.1.10 does not hold. And indeed, considering unbounded shifts on the infinite cylinder $\dot{\Sigma} = \mathbb{R} \times S^1$, it is easy to find a sequence of $W^{k,p}$-bounded functions with $kp > 2$ that do not have a C^0-convergent subsequence. That is the bad news.

The good news is that if $\eta \in W^{k+d,p}(E)$ for $kp > 2$, then one can say considerably more about η than just that it is C^d-smooth. Indeed, restricting to one of the cylindrical ends $[0,\infty) \times S^1 \subset \dot{\Sigma}$, notice that the finiteness of the $W^{k+d,p}$-norm over $\dot{\Sigma}$ implies

$$\|\eta\|_{W^{k+d,p}((R,\infty)\times S^1)} \to 0 \quad \text{as} \quad R \to \infty.$$

Since these domains are all naturally diffeomorphic for different values of R, the C^d-norm of η over $(R,\infty) \times S^1$ is bounded by the $W^{k+d,p}$-norm via a constant that does not depend on R, so this implies an asymptotic decay condition

$$\|\eta\|_{C^d([R,\infty)\times S^1)} \to 0 \quad \text{as} \quad R \to \infty$$

for every $\eta \in W^{k+d,p}(E)$.

Here is another useful piece of good news: since $\dot{\Sigma}$ does not have boundary, $W^{k,p}(E) = W^{k,p}_0(E)$.

THEOREM A.5.1. *Given an asymptotically Hermitian bundle E over a punctured Riemann surface $\dot{\Sigma}$, the space $C^\infty_0(E)$ of smooth sections with compact support is dense in $W^{k,p}(E)$ for all $k \geqslant 0$ and $1 \leqslant p < \infty$.*

PROOF. We can assume as in Definition A.4.1 that the $W^{k,p}$-norm for sections η of E is given by

$$\|\eta\|_{W^{k,p}} = \sum_{\alpha \in I} \|\eta^\alpha\|_{W^{k,p}(\Omega_\alpha)},$$

where $I \subset \mathcal{A}(\pi)$ is a finite collection of bundle charts

$$\alpha = \left(\varphi_\alpha : \mathcal{U}_\alpha \xrightarrow{\cong} \Omega_\alpha \,,\; \Phi_\alpha : E|_{\mathcal{U}_\alpha} \xrightarrow{\cong} \mathcal{U}_\alpha \times \mathbb{C}^n\right)$$

such that each of the open sets $\Omega_\alpha \subset \mathbb{C}$ is either bounded or (for charts over the cylindrical ends) of the form

$$\Omega_\alpha = (0,\infty) \times \omega_\alpha \subset \mathbb{R}^2 = \mathbb{C}$$

for some bounded open subset $\omega_\alpha \subset \mathbb{R}$. Now given $\eta \in W^{k,p}(E)$, Theorem A.1.1 provides for each $\alpha \in I$ a sequence $\eta^\alpha_j \in W^{k,p}(\Omega_\alpha)$ of smooth functions with bounded support such that $\eta_j^\alpha \to \eta^\alpha$ in $W^{k,p}(\Omega_\alpha)$. Choose a partition of unity $\{\rho_\alpha : \dot{\Sigma} \to [0,1]\}_{\alpha \in I}$ subordinate to the open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ and let

$$\eta_j := \sum_{\alpha \in I} \rho_\alpha (\eta_j^\alpha \circ \varphi_\alpha) \in W^{k,p}(E).$$

These sections are smooth and have compact support since the η_j^α have bounded support in Ω_α, and they converge in $W^{k,p}$ to η. □

APPENDIX B

The Floer C_ϵ space

The C_ϵ-topology for functions was introduced by Floer [**Flo88b**] to provide a Banach manifold of perturbed geometric structures without departing from the smooth category: it is a way to circumvent the annoying fact that spaces of smooth functions which arise naturally in geometric settings are not Banach spaces. The construction of C_ϵ spaces generally depends on several arbitrary choices and is thus far from canonical, but this detail is unimportant since the C_ϵ space itself is never the main object of interest. What is important is merely the properties that it has, namely that it not only embeds continuously into C^∞ and contains an abundance of nontrivial functions, but also is a separable Banach space and can therefore be used in the Sard-Smale theorem for genericity arguments. We shall prove these facts in this appendix.

Fix a smooth finite-rank vector bundle $\pi : E \to M$ over a finite-dimensional compact manifold M, possibly with boundary. For each integer $k \geqslant 0$, we denote by $C^k(E)$ the Banach space of C^k-smooth sections of E; note that the norm on $C^k(E)$ depends on various auxiliary choices but is well defined up to equivalence of norms since M is compact. Now if $\boldsymbol{\epsilon} = (\epsilon_k)_{k=0}^\infty$ is a sequence of positive numbers with $\epsilon_k \to 0$, set

$$C_{\boldsymbol{\epsilon}}(E) = \left\{ \eta \in \Gamma(E) \;\middle|\; \|\eta\|_{C_{\boldsymbol{\epsilon}}} < \infty \right\},$$

where the $C_{\boldsymbol{\epsilon}}$-norm is defined by

$$\|\eta\|_{C_{\boldsymbol{\epsilon}}} = \sum_{k=0}^{\infty} \epsilon_k \|\eta\|_{C^k}. \tag{B.1}$$

The norm for $C_{\boldsymbol{\epsilon}}(E)$ is somewhat more delicate than for $C^k(E)$, e.g. its equivalence class is not obviously independent of auxiliary choices. This remark is meant as a sanity check, but it should not cause extra concern since, in practice, the space $C_{\boldsymbol{\epsilon}}(E)$ is typically regarded as an auxiliary choice in itself. In many applications, one fixes an open subset $\mathcal{U} \subset M$ and considers the closed subspace

$$C_{\boldsymbol{\epsilon}}(E;\mathcal{U}) = \left\{ \eta \in C_{\boldsymbol{\epsilon}}(E) \;\middle|\; \eta|_{M \setminus \mathcal{U}} \equiv 0 \right\}.$$

Remark B.0.1. The requirement for M to be compact can be relaxed as long as $\mathcal{U} \subset M$ has compact closure: e.g. in one situation of frequent interest in this book, we take M to be the noncompact completion of a symplectic cobordism. In this case $C_{\boldsymbol{\epsilon}}(E;\mathcal{U})$ can be defined as a closed subspace of $C_{\boldsymbol{\epsilon}}(E|_{M_0})$ where $M_0 \subset M$ is any compact manifold with boundary that contains the closure of $\mathcal{U}$. For this reason, we lose no generality in continuing under the assumption that M is compact.

In order to prove things about $C_\epsilon(E)$, we will need to specify a more precise definition of the C^k-norms. To this end, define a sequence of vector bundles $E^{(k)} \to M$ for integers $k \geqslant 0$ inductively by

$$E^{(0)} := E, \qquad E^{(k+1)} := \mathrm{Hom}(TM, E^{(k)}).$$

Choose connections and bundle metrics on both TM and E; these induce connections and bundle metrics on each of the $E^{(k)}$, so that for any section $\xi \in \Gamma(E^{(k)})$, the covariant derivative $\nabla\xi$ is now a section of $E^{(k+1)}$. In particular for $\eta \in \Gamma(E)$, we can define the "kth covariant derivative" of η as a section

$$\nabla^k \eta \in \Gamma(E^{(k)}).$$

Using the bundle metrics to define C^0-norms for sections of $E^{(k)}$, we can then define

$$\|\eta\|_{C^k(E)} = \sum_{m=0}^{k} \|\nabla^m \eta\|_{C^0(E^{(m)})},$$

where by convention $\nabla^0\eta := \eta$. We will assume throughout the following that the C^k-norms appearing in (B.1) are defined in this way.

THEOREM B.0.2. *$C_\epsilon(E)$ is a Banach space.*

PROOF. We need to show that C_ϵ-Cauchy sequences converge in the C_ϵ-norm. It is clear from the definitions that if $\eta_j \in C_\epsilon(E)$ is Cauchy, then η_j is also C^k-Cauchy for every $k \geqslant 0$, hence its derivatives $\nabla^k\eta_j$ for every k are C^0-convergent to continuous sections ξ^k of $E^{(k)}$. This convergence implies that $\xi^{k+1} = \nabla\xi^k$ in the sense of distributions, hence by the equivalence of classical and distributional derivatives (see e.g. [**LL01**, §6.10]), $\eta_\infty := \xi^0$ is smooth with $\nabla^k\eta_\infty = \xi^k$, so that $\nabla^k\eta_j \to \nabla^k\eta_\infty$ in $C^0(E^{(k)})$ for all k.

We claim $\eta_\infty \in C_\epsilon(E)$. Choose $N > 0$ such that $\|\eta_i - \eta_j\|_{C_\epsilon} < 1$ for all $i, j \geqslant N$. Then for every $m \in \mathbb{N}$ and every $i \geqslant N$,

$$\begin{aligned}\sum_{k=0}^{m} \epsilon_k \|\eta_i\|_{C^k} &\leqslant \sum_{k=0}^{m} \epsilon_k \|\eta_i - \eta_N\|_{C^k} + \sum_{k=0}^{m} \epsilon_k \|\eta_N\|_{C^k} \\ &\leqslant \|\eta_i - \eta_N\|_{C_\epsilon} + \|\eta_N\|_{C_\epsilon} < 1 + \|\eta_N\|_{C_\epsilon}.\end{aligned}$$

Fixing m and letting $i \to \infty$, we then have

$$\sum_{k=0}^{m} \epsilon_k \|\eta_\infty\|_{C^k} \leqslant 1 + \|\eta_N\|_{C_\epsilon}$$

for all m, so we can now let $m \to \infty$ and conclude $\|\eta_\infty\|_{C_\epsilon} \leqslant 1 + \|\eta_N\|_{C_\epsilon} < \infty$.

The argument that $\|\eta_j - \eta_\infty\|_{C_\epsilon} \to 0$ as $j \to \infty$ is similar: pick $\epsilon > 0$ and N such that $\|\eta_i - \eta_j\|_{C_\epsilon} < \epsilon$ for all $i, j \geqslant N$. Then for a fixed $m \in \mathbb{N}$, we can let $i \to \infty$ in the expression $\sum_{k=0}^{m} \epsilon_k \|\eta_i - \eta_j\|_{C^k} < \epsilon$, giving

$$\sum_{k=0}^{m} \epsilon_k \|\eta_\infty - \eta_j\|_{C^k} \leqslant \epsilon.$$

This is true for every m, so we can take $m \to \infty$ and conclude $\|\eta_\infty - \eta_j\|_{C_\epsilon} \leqslant \epsilon$ for all $j \geqslant N$. □

To show that $C_\epsilon(E)$ is also separable, we will follow a hint[1] from [HS95] and embed it isometrically into another Banach space that can be more easily shown to be separable. For each integer $k \geqslant 0$, define the vector bundle

$$F^{(k)} = E^{(0)} \oplus \ldots \oplus E^{(k)},$$

and let X_ϵ denote the vector space of all sequences

$$\xi := (\xi^0, \xi^1, \xi^2, \ldots) \in \prod_{k=0}^{\infty} C^0(F^{(k)})$$

such that

$$\|\xi\|_{X_\epsilon} := \sum_{k=0}^{\infty} \epsilon_k \|\xi^k\|_{C^0} < \infty.$$

EXERCISE B.0.3. Adapt the proof of Theorem B.0.2 to show that X_ϵ is also a Banach space.

LEMMA B.0.4. *X_ϵ is separable.*

PROOF. Since $C^0(F^{(k)})$ is separable for each $k \geqslant 0$, we can fix countable dense subsets $P^k \subset C^0(F^{(k)})$. The set

$$P := \left\{ (\xi^0, \ldots, \xi^N, 0, 0, \ldots) \in X_\epsilon \;\middle|\; N \geqslant 0 \text{ and } \xi^k \in P^k \text{ for all } k = 0, \ldots, N \right\}$$

is then countable and dense in X_ϵ. □

THEOREM B.0.5. *$C_\epsilon(E)$ is separable.*

PROOF. Consider the injective linear map

$$C_\epsilon(E) \hookrightarrow X_\epsilon : \eta \mapsto \left(\eta, (\eta, \nabla\eta), (\eta, \nabla\eta, \nabla^2\eta), \ldots \right).$$

This is an isometric embedding and thus presents $C_\epsilon(E)$ as a closed linear subspace of X_ϵ, hence the theorem follows from Lemma B.0.4 and the fact that subspaces of separable metric spaces are always separable. □

Note that given any open subset $\mathcal{U} \subset M$, Theorems B.0.2 and B.0.5 also hold for $C_\epsilon(E;\mathcal{U})$, as a closed subspace of $C_\epsilon(E)$. So far in this discussion, however, there has been no guarantee that $C_\epsilon(E)$ or $C_\epsilon(E;\mathcal{U})$ contains anything other than the zero-section, though it is clear that in theory, one should always be able to enlarge the space by choosing new sequences ϵ_k that converge to zero faster. The following result says that $C_\epsilon(E;\mathcal{U})$ can always be made large enough to be useful in applications.

THEOREM B.0.6. *Given an open subset $\mathcal{U} \subset M$, the sequence ϵ_k can be chosen to have the following properties:*

(1) $C_\epsilon(E;\mathcal{U})$ is dense in the space of continuous sections vanishing outside $\mathcal{U}$.
(2) Given any point $p \in \mathcal{U}$, a neighborhood $\mathcal{N}_p \subset \mathcal{U}$ of p, a number $\delta > 0$ and a continuous section η_0 of E, there exists a section $\eta \in \Gamma(E)$ and a smooth compactly supported function $\beta : \mathcal{N}_p \to [0,1]$ such that

$$\beta\eta \in C_\epsilon(E;\mathcal{U}), \qquad \beta(p)\eta(p) = \eta_0(p), \qquad \text{and} \qquad \|\eta - \eta_0\|_{C^0} < \delta.$$

[1] Thanks to Sam Lisi for explaining to me what the hint in [HS95] was referring to.

PROOF. Note first that it suffices to find two separate sequences ϵ_k and ϵ'_k that have the first and second property respectively, as the sequence of minima $\min(\epsilon_k, \epsilon'_k)$ will then have both properties.

The following construction for the first property is based on a suggestion by Barney Bramham. Observe first that the space $C^0(E;\mathcal{U})$ of continuous sections vanishing outside $\mathcal{U}$ is a closed subspace of $C^0(E)$ and is thus separable, so we can choose a countable C^0-dense subset $P \subset C^0(E;\mathcal{U})$. Moreover, the space of *smooth* sections vanishing outside $\mathcal{U}$ is dense in $C^0(E;\mathcal{U})$, hence we can assume without loss of generality that the sections in P are smooth. Now write $P = \{\eta_1, \eta_2, \eta_3, \ldots\}$ and define $\epsilon_k > 0$ for every integer $k \geqslant 0$ to have the property

$$\epsilon_k < \frac{1}{2^k} \min\left\{\frac{1}{\|\eta_1\|_{C^k}}, \ldots, \frac{1}{\|\eta_k\|_{C^k}}\right\}.$$

Then every η_j is in $C_\epsilon(E;\mathcal{U})$, as

$$\|\eta_j\|_{C_\epsilon} < \sum_{k=0}^{j-1} \epsilon_k \|\eta_j\|_{C^k} + \sum_{k=j}^{\infty} \frac{1}{2^k} < \infty.$$

The second property is essentially local, so it can be deduced from Lemma B.0.7 below. □

LEMMA B.0.7. *Suppose $\beta : \mathring{\mathbb{D}}^n \to [0,1]$ is a smooth function with compact support on the open unit ball $\mathring{\mathbb{D}}^n \subset \mathbb{R}^n$ and $\beta(0) = 1$. One can choose a sequence of positive numbers $\epsilon_k \to 0$ such that for every $\eta_0 \in \mathbb{R}^m$ and $r > 0$, the function $\eta : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$\eta(p) := \beta(p/r)\eta_0$$

satisfies $\sum_{k=0}^{\infty} \epsilon_k \|\eta\|_{C^k} < \infty$.

PROOF. Define $\epsilon_k > 0$ so that for $k \geqslant 1$,

$$\epsilon_k = \frac{1}{k^k \|\beta\|_{C^k}}.$$

Then

$$\sum_{k=1}^{\infty} \epsilon_k \|\eta\|_{C^k} \leqslant \sum_{k=1}^{\infty} \frac{1}{k^k \|\beta\|_{C^k}} \frac{\|\beta\|_{C^k}}{r^k} = \sum_{k=1}^{\infty} \left(\frac{1/r}{k}\right)^k < \infty.$$

□

APPENDIX C

Genericity in the space of asymptotic operators

The purpose of this appendix is to prove Lemma 3.4.3, which was needed for our definition of spectral flow in §3.4. The proof combines some ideas from that section with the technique used in Chapter 9 to prove generic transversality of moduli spaces via the Sard-Smale theorem. Some knowledge of that technique should thus be considered a prerequisite for this appendix; if you have never seen it before and were directed here after reading the statement of Lemma 3.4.3, you might want to skip this for now and come back after you've read as far as Chapter 9.

Recalling the notation from Chapter 3, we fix the real Hilbert spaces

$$\mathcal{H} = L^2(S^1, \mathbb{R}^{2n}), \qquad \mathcal{D} = H^1(S^1, \mathbb{R}^{2n}),$$

the symmetric index 0 Fredholm operator

$$\mathbf{T}_{\mathrm{ref}} = -J_0\,\partial_t : \mathcal{D} \to \mathcal{H}$$

and, given a bounded family of symmetric matrices $S \in L^\infty(S^1, \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n}))$, refer to any operator of the form

$$\mathbf{A} = -J_0\,\partial_t - S : \mathcal{D} \to \mathcal{H}$$

as an **asymptotic operator**. Such operators belong to the space of symmetric compact perturbations of $\mathbf{T}_{\mathrm{ref}}$,

$$\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) = \left\{ \mathbf{T}_{\mathrm{ref}} + \mathbf{K} : \mathcal{D} \to \mathcal{H} \;\middle|\; \mathbf{K} \in \mathscr{L}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{H}) \right\},$$

which we regard as a smooth Banach manifold via its obvious identification with the space $\mathscr{L}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{H})$ of symmetric bounded linear operators on $\mathcal{H}$. For $k \in \mathbb{N}$, we denote by

$$\mathrm{Fred}^{\mathrm{sym},k}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}}) \subset \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$$

the finite-codimensional submanifold determined by the condition $\dim_{\mathbb{R}} \ker \mathbf{A} = \dim_{\mathbb{R}} \operatorname{coker} \mathbf{A} = k$.

Here is the statement of Lemma 3.4.3 again.

LEMMA. *Fix a smooth path $[-1, 1] \to L^\infty(S^1, \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})) : s \mapsto S_s$ and consider the 1-parameter family of symmetric index 0 Fredholm operators*

$$\mathbf{A}_s := -J_0\,\partial_t - S_s : H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

for $s \in [-1, 1]$, assuming $\mathbf{A}_{\pm 1}$ are isomorphisms. Then after replacing S_s by a family of the form $\widetilde{S}_s(t) := S_s(t) + B(s, t)$ for some smooth function $B : [-1, 1] \to \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ that vanishes for $s = \pm 1$ and may be assumed arbitrarily C^∞-small, one can arrange that the following conditions hold:

(1) For each $s \in (-1,1)$*, all eigenvalues of* $\mathbf{A}_s$ *are simple.*
(2) All intersections of the smooth path

$$(-1,1) \to \mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}}) : s \mapsto \mathbf{A}_s$$

with $\mathrm{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ *are transverse.*

We shall now prove this by constructing a Floer-type space of $C_{\boldsymbol{\epsilon}}$-smooth (see Appendix B) perturbed families of asymptotic operators, and using the Sard-Smale theorem to find a countable collection of comeager subsets whose intersection contains perturbations achieving the desired conditions.

Choose a sequence of positive numbers $\boldsymbol{\epsilon} = (\epsilon_k)_{k=0}^{\infty}$ with $\epsilon_k \to 0$ to define a separable Banach space

$$\mathcal{A}_{\boldsymbol{\epsilon}} := \left\{ B \in C^\infty([-1,1]\times S^1, \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})) \;\middle|\; \|B\|_{C_{\boldsymbol{\epsilon}}} < \infty \text{ and } B(\pm 1,\cdot) \equiv 0 \right\},$$

and assume via Theorem B.0.6 that $\mathcal{A}_{\boldsymbol{\epsilon}}$ is dense in the Banach space of continuous functions $[-1,1]\times S^1 \to \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ vanishing at $\{\pm 1\}\times S^1$. We then consider perturbed 1-parameter families of asymptotic operators of the form

$$\mathbf{A}^B_s := \mathbf{A}_s + B(s,\cdot) : \mathcal{D} \to \mathcal{H}$$

for $B \in \mathcal{A}_{\boldsymbol{\epsilon}}$, $s \in [-1,1]$. Remarks 3.4.1 and 3.4.2 imply that the perturbed family defines a smooth path in $\mathrm{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H},\mathbf{T}_{\mathrm{ref}})$ as long as the original path $s \mapsto S_s$ is smooth in $L^\infty(S^1, \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n}))$. For each $k \in \mathbb{N}$ and $B \in \mathcal{A}_{\boldsymbol{\epsilon}}$, define the set

$$\mathcal{V}^k(B) = \left\{ (s,\lambda) \in (-1,1)\times\mathbb{R} \;\middle|\; \dim_{\mathbb{R}} \ker\left(\mathbf{A}^B_s - \lambda\right) = k \right\}.$$

To show that eigenvalues are generically simple, we need to show that for a comeager set of choices of $B \in \mathcal{A}_{\boldsymbol{\epsilon}}$, $\mathcal{V}^k(B)$ is empty for all $k \geqslant 2$. Given $(s_0,\lambda_0) \in \mathcal{V}^k(B)$, recall from §3.4 that there exist decompositions

$$\mathcal{D} = V \oplus K, \qquad \mathcal{H} = W \oplus K$$

where $K = \ker\left(\mathbf{A}^B_{s_0} - \lambda_0\right)$, $W = \mathrm{im}\left(\mathbf{A}^B_{s_0} - \lambda_0\right)$ is the L^2-orthogonal complement of K, and $V = W \cap \mathcal{D}$, so that any symmetric bounded linear operator $\mathbf{T}$ in a sufficiently small neighborhood $\mathcal{O} \subset \mathscr{L}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D},\mathcal{H})$ of $\mathbf{A}^B_{s_0} - \lambda_0$ can be written in block form

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$$

with $\mathbf{A} : V \to W$ invertible. This gives rise to a smooth map

$$\Phi : \mathcal{O} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K) : \mathbf{T} \mapsto \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$$

whose zero set is precisely the set of nearby symmetric operators with k-dimensional kernel. A neighborhood of (s_0,λ_0) in $\mathcal{V}^k(B)$ can thus be identified with the zero set of the map

$$\Psi_B(s,\lambda) := \Phi(\mathbf{A}^B_s - \lambda) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K),$$

defined for $(s,\lambda) \in (-1,1)\times\mathbb{R}$ sufficiently close to (s_0,λ_0). Notice that the derivative $d\Psi_B(s,\lambda) : \mathbb{R}\oplus\mathbb{R} \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$ is Fredholm since its domain and target are both finite dimensional, and it can only ever be surjective when $k = \dim_{\mathbb{R}} K = 1$.

The following space will now play the role of a "universal moduli space" as in Chapter 9: let

$$\mathcal{V}^k = \left\{(s,\lambda,B) \in (-1,1) \times \mathbb{R} \times \mathcal{A}_\epsilon \;\middle|\; (s,\lambda) \in \mathcal{V}^k(B)\right\}.$$

The proof that this is a smooth Banach manifold depends on the following algebraic lemma.

LEMMA C.0.1. *Fix an asymptotic operator* $\mathbf{A} = -J_0\,\partial_t - S$ *and a linear transformation*

$$\Upsilon : \ker \mathbf{A} \to \ker \mathbf{A}$$

that is symmetric with respect to the L^2*-product. Then there exists a continuous loop* $B : S^1 \to \mathrm{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ *such that*

$$\langle \eta, B\xi\rangle_{L^2} = \langle \eta, \Upsilon\xi\rangle_{L^2}$$

for all $\eta, \xi \in \ker \mathbf{A}$.

PROOF. Note first that every nontrivial loop $\eta \in \ker \mathbf{A} \subset H^1(S^1,\mathbb{R}^{2n})$ is continuous and nowhere zero due to the generalized existence/uniqueness result for solutions to linear ODEs in Exercise 3.3.1. It follows that if we fix a basis $(\eta_1,\ldots,\eta_k)$ for $\ker \mathbf{A}$, then the vectors $\eta_1(t),\ldots,\eta_k(t) \in \mathbb{R}^{2n}$ are also linearly independent for all $t \in S^1$ and thus span a continuous S^1-family of k-dimensional subspaces $V_t \subset \mathbb{R}^{2n}$, each equipped with a distinguished basis. There is therefore a unique continuous S^1-family of linear transformations $\widehat{B}(t) : V_t \to V_t$ such that for every $\eta \in \ker \mathbf{A}$, $\widehat{B}(t)\eta(t) = (\Upsilon\eta)(t)$ for all t. Extend $\widehat{B}(t)$ arbitrarily to a continuous family of linear maps on $\mathbb{R}^{2n}$.

The matrices $\widehat{B}(t) \in \mathrm{End}(\mathbb{R}^{2n})$ need not be symmetric, but they do satisfy

$$\langle \eta, \widehat{B}\xi\rangle_{L^2} = \langle \eta, \Upsilon\xi\rangle_{L^2} \qquad \text{for all } \eta, \xi \in \ker \mathbf{A}.$$

Since Υ is symmetric, this implies moreover that for all $\eta, \xi \in \ker \mathbf{A}$,

$$\langle \eta, \Upsilon\xi\rangle_{L^2} = \langle \xi, \Upsilon\eta\rangle_{L^2} = \langle \xi, \widehat{B}\eta\rangle_{L^2} = \langle \eta, \widehat{B}^{\mathrm{T}}\xi\rangle_{L^2}.$$

The loop $B := \frac{1}{2}(\widehat{B} + \widehat{B}^{\mathrm{T}})$ thus has the desired properties. □

Now using the previously described construction in the space of symmetric Fredholm operators, a neighborhood of any point (s_0,λ_0,B_0) in $\mathcal{V}^k$ can be identified with the zero set of a smooth map of the form

$$\Psi(s,\lambda,B) := \Psi_B(s,\lambda) \in \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K),$$

defined for all (s,λ,B) sufficiently close to (s_0,λ_0,B_0) in $(-1,1) \times \mathbb{R} \times \mathcal{A}_\epsilon$, where $K = \ker\left(\mathbf{A}^{B_0}_{s_0} - \lambda_0\right)$. The partial derivative of Ψ with respect to the third variable at (s_0,λ_0,B_0) is then a linear map

$$\mathbf{L} := D_3\Psi(s_0,\lambda_0,B_0) : \mathcal{A}_\epsilon \to \mathrm{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$$

of the form

$$\mathbf{L}B : K \to K : \eta \mapsto \pi_K(B(s_0,\cdot)\eta), \tag{C.1}$$

where $\pi_K : W \oplus K \to K$ is the orthogonal projection. We claim that $\mathbf{L}$ is surjective. Indeed, for any $\Upsilon \in \operatorname{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$, Lemma C.0.1 provides a continuous loop $C_0 : S^1 \to \operatorname{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ such that

$$\pi_K(C_0\eta) = \Upsilon\eta \quad \text{for all } \eta \in K,$$

and this can be extended to a continuous function $C : [-1,1] \times S^1 \to \operatorname{End}^{\mathrm{sym}}(\mathbb{R}^{2n})$ satisfying $C(s_0, \cdot) \equiv C_0$ and $C(\pm 1, \cdot) \equiv 0$ since $s_0 \neq \pm 1$. The function C might fail to be of class $C_{\boldsymbol{\epsilon}}$, but since it can be approximated arbitrarily well in the C^0-norm by functions in $\mathcal{A}_{\boldsymbol{\epsilon}}$, we conclude that the image of $\mathbf{L}$ is dense in $\operatorname{End}^{\mathrm{sym}}_{\mathbb{R}}(K)$. Since the latter is finite dimensional, the claim follows.

The implicit function theorem now gives $\mathcal{V}^k$ the structure of a smooth Banach submanifold of $(-1,1) \times \mathbb{R} \times \mathcal{A}_{\boldsymbol{\epsilon}}$, and it is separable since the latter is also separable. Consider the projection

$$\pi : \mathcal{V}^k \to \mathcal{A}_{\boldsymbol{\epsilon}} : (s, \lambda, B) \mapsto B, \tag{C.2}$$

which is a smooth map of separable Banach manifolds whose fibers $\pi^{-1}(B)$ are the spaces $\mathcal{V}^k(B)$. Using Lemma 9.1.1, the fact that each map Ψ_B is Fredholm implies that π is also a Fredholm map, so the Sard-Smale theorem implies that the regular values of π form a comeager subset

$$\mathcal{A}^{\mathrm{reg},k}_{\boldsymbol{\epsilon}} \subset \mathcal{A}_{\boldsymbol{\epsilon}}.$$

The intersection

$$\mathcal{A}^{\mathrm{reg}}_{\boldsymbol{\epsilon}} := \bigcap_{k \in \mathbb{N}} \mathcal{A}^{\mathrm{reg},k}_{\boldsymbol{\epsilon}}$$

is then another comeager subset of $\mathcal{A}_{\boldsymbol{\epsilon}}$, with the property that for each $B \in \mathcal{A}^{\mathrm{reg}}_{\boldsymbol{\epsilon}}$ and every $k \in \mathbb{N}$ and $(s, \lambda) \in \mathcal{V}^k(B)$, $d\Psi_B(s, \lambda)$ is (by Lemma 9.1.1) surjective. As was observed previously, this is impossible for dimensional reasons if $k \geqslant 2$, implying that $\mathcal{V}^k(B)$ is then empty.

To find perturbations that also achieve the transversality condition, we use a similar argument: define for each $B \in \mathcal{A}_{\boldsymbol{\epsilon}}$ the subset

$$\mathcal{V}^0(B) = \left\{ s \in (-1,1) \;\middle|\; \dim_{\mathbb{R}} \ker \mathbf{A}^B_s = 1 \right\},$$

along with the corresponding universal set

$$\mathcal{V}^0 = \left\{ (s, B) \in (-1,1) \times \mathcal{A}_{\boldsymbol{\epsilon}} \;\middle|\; s \in \mathcal{V}^0(B) \right\}.$$

A neighborhood of any (s_0, B_0) in $\mathcal{V}^0$ is then the zero set of a smooth map of the form

$$\Psi(s, B) = \Phi(\mathbf{A}^B_s) \in \operatorname{End}^{\mathrm{sym}}_{\mathbb{R}}(\ker \mathbf{A}^{B_0}_{s_0}),$$

defined for all $(s, B) \in (-1,1) \times \mathcal{A}_{\boldsymbol{\epsilon}}$ close enough to (s_0, B_0). For a fixed $B \in \mathcal{A}_{\boldsymbol{\epsilon}}$ near B_0 and $s_1 \in \mathcal{V}^0(B)$ near s_0, a neighborhood of s_1 in $\mathcal{V}^0(B)$ is then the zero set of $\Psi_B(s) := \Psi(s, B)$, and the intersection of the path $s \mapsto \mathbf{A}^B_s \in \operatorname{Fred}^{\mathrm{sym}}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ with $\operatorname{Fred}^{\mathrm{sym},1}_{\mathbb{R}}(\mathcal{D}, \mathcal{H}, \mathbf{T}_{\mathrm{ref}})$ at $s = s_1$ is transverse if and only if

$$d\Psi_B(s_1) : \mathbb{R} \to \operatorname{End}^{\mathrm{sym}}_{\mathbb{R}}(\ker \mathbf{A}^{B_0}_{s_0})$$

is surjective. At (s_0, B_0), the partial derivative of Ψ with respect to B is again the same operator

$$\mathbf{L} = D_2\Psi(s_0, B_0) : \mathcal{A}_{\boldsymbol{\epsilon}} \to \mathrm{End}_{\mathbb{R}}^{\mathrm{sym}}(\ker \mathbf{A}_{s_0}^{B_0})$$

as in (C.1), which we've already seen is surjective due to Lemma C.0.1. Thus one can apply the Sard-Smale theorem to the projection

$$\mathcal{V}^0 \to \mathcal{A}_{\boldsymbol{\epsilon}} : (s, B) \mapsto B,$$

obtaining a comeager subset $\mathcal{A}_{\boldsymbol{\epsilon}}^{\mathrm{reg},0} \subset \mathcal{A}_{\boldsymbol{\epsilon}}$ such that all paths $\mathbf{A}_s + B(s, \cdot)$ for $B \in \mathcal{A}_{\boldsymbol{\epsilon}}^{\mathrm{reg},0}$ satisfy the required transversality condition. The comeager subset $\mathcal{A}_{\boldsymbol{\epsilon}}^{\mathrm{reg},0} \cap \mathcal{A}_{\boldsymbol{\epsilon}}^{\mathrm{reg}} \subset \mathcal{A}_{\boldsymbol{\epsilon}}$ thus consists of perturbed families of operators for which all desired conditions are satisfied, and it contains a sequence converging in the C^∞-topology to 0. This concludes the proof of Lemma 3.4.3.

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