PROBLEM SET 1

Suggested reading

Note: reading suggestions in Lee's "Introduction to Smooth Manifolds" refer to the 2003 edition—section and chapter numbers in the 2013 edition may differ.

- Agricola and Friedrich: §3.1 (Chapters 1 and 2 are not prerequisites for this)
- Lee: Chapter 1 (skip the section on "Topological Properties. . . ") and the first sections of Chapter 2 ("Smooth Functions. . . ") and Chapter 3 ("Tangent Vectors") respectively

Problems

1. (a) Let $\mathcal{U} \subset \mathbb{R}^n$ be an open subset, $f : \mathcal{U} \to \mathbb{R}^m$ a smooth map and $\mathbf{v} \in \mathbb{R}^n$ a vector. Recall that the *derivative* of f at $\mathbf{x} \in \mathcal{U}$ is the unique linear transformation $df(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + df(\mathbf{x})\mathbf{h} + \eta(\mathbf{h}) \cdot |\mathbf{h}|$$

for sufficiently small $\mathbf{h} \in \mathbb{R}^n$, where $\eta(\mathbf{h})$ is a function satisfying $\lim_{\mathbf{h}\to 0} \eta(\mathbf{h}) = 0$. A slightly simpler notion is the *directional derivative* in the direction $\mathbf{v}$, given by

$$\left.\frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\right|_{t=0} \in \mathbb{R}^m.$$

Use the chain rule to derive a simple expression for this directional derivative in terms of the linear transformation $df(\mathbf{x})$. (This is easy.)

   (b) Denoting $\mathbf{x} = (x^1, \ldots, x^n) \in \mathcal{U}$, $f(\mathbf{x}) = \mathbf{y} = (y^1, \ldots, y^m)$ and $\mathbf{v} = (v^1, \ldots, v^n)$, write out the components of the above directional derivative in terms of $v^1, \ldots, v^n$ and the partial derivatives $\frac{\partial y^i}{\partial x^j}$.

   (c) Show that the above directional derivative is also equal to

$$\left.\frac{d}{dt} f(\gamma(t))\right|_{t=0}$$

if $\gamma : (-1, 1) \to \mathcal{U}$ is any smooth path satisfying $\gamma(0) = \mathbf{x}$ and $\dot{\gamma}(0) = \mathbf{v}$. (This is also easy.)

2. An important definition: a *diffeomorphism* between open subsets of $\mathbb{R}^n$ is a homeomorphism which is both smooth and has a smooth inverse. To prove the latter, it's often useful to recall the *inverse function theorem*:

   If $\mathcal{U} \subset \mathbb{R}^n$ is an open subset and $f : \mathcal{U} \to \mathbb{R}^n$ is smooth, $f(x_0) = y_0$ and $df(x_0)$ is invertible, then f maps some open neighborhood of x_0 bijectively to an open neighborhood of y_0, the inverse f^{-1} is smooth and $df^{-1}(y_0)$ is the inverse matrix of $df(x_0)$.

   (a) Consider the definition of polar coordinates in the plane:

$$x = r \cos\theta, \quad y = r \sin\theta.$$

Show that the map $F(r, \theta) = (x, y)$ defines a diffeomorphism

$$F : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2 \setminus \mathbb{R}_+,$$

where $\mathbb{R}_+$ denotes the subset $\{(t, 0) \in \mathbb{R}^2 \mid t \geq 0\}$. *Note:* you have permission to say it's patently obvious that F is smooth, but it's not obvious that this is true for F^{-1}. Prove it without deriving an expression for F^{-1}; use the inverse function theorem instead.

(b) Again without writing down F^{-1} explicitly, derive the following expressions for the partial derivatives of r and θ with respect to x and y:

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r}$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$

3. Recall that the *n-dimensional sphere* is defined as the "unit sphere" in $\mathbb{R}^{n+1}$,

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}.$$

A related n-manifold is the *real projective n-space* $\mathbb{R}P^n$, which is most easily defined as the set of equivalence classes

$$\mathbb{R}P^n = S^n / \sim,$$

where we use an equivalence relation to identify antipodal points in S^n: $\mathbf{x} \sim -\mathbf{x}$. Find an explicit homeomorphism of S^1 to $\mathbb{R}P^1$. (*Beware:* this is not true in higher dimensions!)

4. Another important definition: a *diffeomorphism* between smooth manifolds is a homeomorphism which is smooth and has a smooth inverse. This idea is not always as simple as it sounds.

It's crucial to understand that the data defining a smooth manifold include not just the space itself, but also a collection of smoothly compatible charts: this constitutes its *smooth structure*, also known as an *atlas*. Below is a slightly weird example.

Let $M = \mathbb{R}$, which we make into a smooth manifold in the most natural way, choosing the obvious chart $x : \mathbb{R} \to \mathbb{R} : t \mapsto t$, and defining the smooth structure to consist of all charts that are smoothly compatible with this one.

Now define $M' = \mathbb{R}$ as well, but with a different smooth structure, including the chart $y : \mathbb{R} \to \mathbb{R} : t \mapsto t^3$, and all others that are smoothly compatible with y. (Note that y *is* indeed a homeomorphism.)

(a) Show that the two charts x and y are *not* smoothly compatible.

(b) Let $\mathcal{U} = (-1, 1) \subset M'$ and show that the map

$$\varphi : \mathcal{U} \to \mathbb{R} : t \mapsto \tan\left(\frac{\pi}{2} t^3\right)$$

is a smoothly compatible chart on M'. In other words, show that the coordinate transformations $\varphi \circ y^{-1}$ and $y \circ \varphi^{-1}$ are both smooth wherever they are defined. (What are their domains?)

(c) The identity map $M \to M' : t \mapsto t$ is a homeomorphism, clearly. Show that it is also a smooth map, but it is *not* a diffeomorphism. Remember that this notion depends on the particular smooth structures we've chosen.

(d) Show that the map $M' \to M : t \mapsto t^2$ is not smooth.

(e) All is not lost: there *are* diffeomorphisms from M to M'. Find one!