PROBLEM SET 14
SOLUTIONS

1. (a) Suppose $\gamma : \mathbb{R} \to \Sigma$ is a geodesic with period $T > 0$, whose image $\gamma(\mathbb{R}) \subset \Sigma$ bounds an embedded disk $D \subset \Sigma$. We can then regard D as a polygonal region bounded by a single smooth curve ℓ parametrized by $\gamma|_{[0,T]}$, hence beginning and ending at the same point $p := \gamma(0) = \gamma(T)$. Moreover, since γ is periodic, the angle formed at the vertex p must be π, as $\dot{\gamma}(T) = \dot{\gamma}(0)$. The Gauss-Bonnet formula thus gives
$$\pi = (1-2)\pi + \int_D K_G \, dA,$$
where the geodesic curvature terms do not appear since ℓ is a geodesic. In particular, this implies $\int_D K_G \, dA$ is positive, which is impossible since $K_G \le 0$ everywhere.

   (b) If a family of paths $\{\gamma_s : [0,1] \to \Sigma\}_{s\in[0,1]}$ with the stated properties exists, then the geodesic segments γ_0 and γ_1 are disjoint except at p and q, and the set $\mathrm{P} := \{\gamma_s(t) \in \Sigma \mid s \in [0,1],\ t \in [0,1]\}$ is a polygonal region in Σ bounded by the two edges $\gamma_0([0,1])$ and $\gamma_1([0,1])$, which meet at the two vertices p and q. Since geodesics are uniquely determined up to parametrization by their tangent vectors at any given point, the fact that γ_0 and γ_1 are distinct implies that the vectors $\dot{\gamma}_0(0)$ and $\dot{\gamma}_1(0)$ must be linearly independent, hence the angle α_p between them is nonzero (and therefore positive). By a similar argument, the angle α_q at which they meet at q is also positive. Plugging all this into the Gauss-Bonnet formula and dropping the geodesic curvature terms since γ_0 and γ_1 are geodesics, we have
$$0 < \alpha_p + \alpha_q = \int_P K_G \, dA,$$
once again contradicting the assumption that K_G is everywhere nonpositive.

   (c) The simplest closed surface with nonpositive curvature is the "flat" torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ with its natural Euclidean metric g_E (it descends to the quotient because the translation maps $\mathbb{R}^2 \to \mathbb{R}^2$ defined by elements of $\mathbb{T}^2$ are Euclidean isometries). The geodesics on $(\mathbb{T}^2, g_E)$ are then simply the curves of the form $\pi \circ \gamma$ where $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a line with constant velocity and $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the natural quotient projection. In particular, any curve of the form $t \mapsto [(t, y_0)] \in \mathbb{T}^2$ for a constant $y_0 \in \mathbb{R}$ is then a periodic geodesic.

   (d) Fix on S^2 the natural metric induced by the Euclidean metric on $\mathbb{R}^3$ via the embedding of S^2 into the latter as the unit sphere. Its geodesics are then the "great circles," i.e. intersections of the unit sphere $S^2 \subset \mathbb{R}^3$ with 2-dimensional linear subspaces of $\mathbb{R}^3$. There is an infinite set of such great circles connecting any two antipodal points, and they trace out polygonal regions as in part (b).

2. (a) Suppose $\gamma(t) \in \mathcal{V}_\alpha$ is a smooth path and $v(t) \in (f^*E)_{\gamma(t)} = E_{f\circ\gamma(t)}$ is an arbitrary smooth section along this path. Expressed in the trivialization Ψ_α, we can write $v(t)$ as a function $v_\alpha(t) \in \mathbb{F}^m$, and by construction we get the same function if we instead view v as a section of E along $f \circ \gamma$ and express it in the trivialization Φ_α. To show that f^*A_α is the connection 1-form with respect to Ψ_α, it suffices to show that
$$(\nabla_t v(0))_\alpha = \partial_t v_\alpha(0) + (f^*A_\alpha)(\dot{\gamma}(0))v_\alpha(0).$$
Using the definition of ∇_t via parallel transport, we have
$$\nabla_t v(0) = \frac{d}{dt}\left((P_\gamma^t)^{-1}(v(t))\right)\Big|_{t=0}.$$

But since $P^t_\gamma : (f^*E)_{\gamma(0)} \to (f^*E)_{\gamma(t)}$ is defined to match the parallel transport map $E_{f(\gamma(0))} \to E_{f(\gamma(t))}$ for the connection on E along the path $f \circ \gamma(t) \in \mathcal{U}_\alpha$, this covariant derivative is exactly the same as what one obtains by viewing v instead as a section of E along $f \circ \gamma$, and computing the latter in the trivialization Φ_α gives

$$(\nabla_t v(0))_\alpha = \partial_t v_\alpha(0) + A_\alpha(\partial_t(f \circ \gamma)(0))v_\alpha(0).$$

Since $A_\alpha(\partial_t(f \circ \gamma)(0)) = A_\alpha(Tf(\dot\gamma(0))) = (f^*A_\alpha)(\dot\gamma(0))$, the desired result follows.

(b) Choose any Hermitian bundle metric $\langle\ ,\ \rangle$ on E, thus reducing the structure group of E to U(1), and fix a U(1)-compatible connection ∇. By definition, $c_1(E)$ is the cohomology class represented by the closed 2-form $-\frac{1}{2\pi i}F \in \Omega^2(M)$, where the "curvature 2-form" $F \in \Omega^2(M, \mathfrak{u}(1))$ matches the exterior derivative of the connection 1-form A_α on $\mathcal{U}_\alpha$ for any choice of U(1)-compatible local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$. Note that while the 2-form F depends on the choices of Hermitian bundle metric and U(1)-compatible connection, the cohomology class $c_1(E)$ does not. Note also that the bundle metric on E induces a bundle metric on f^*E since every fiber of the latter is also a fiber of the former, and the pullback connection on f^*E is then also U(1)-compatible since its parallel transport maps are unitary by construction. Part (a) then implies that the curvature 2-form for the pullback connection is f^*F, hence

$$c_1(f^*E) = \left[-\frac{1}{2\pi i} f^*F\right] = f^*\left[-\frac{1}{2\pi i}F\right] = f^*c_1(E).$$

(c) If there is a bundle isomorphism $\Phi : E \to F$, it suffices to observe that for any choice of bundle metric and compatible connection on E, one can use the isomorphism to define a corresponding bundle metric and connection on F; moreover, this isomorphism associates to every local trivialization of E a corresponding local trivialization of F such that the connection 1-forms for E and F with respect to corresponding trivializations become identical. This implies that they have identical curvature 2-forms, hence $c_1(E) = c_1(F)$.

3. (a) We first need to digress for a moment and consider how local trivializations are actually defined for tensor product bundles. To that end, suppose more generally that $E_1, E_2 \to M$ are bundles of rank m and n respectively over the field $\mathbb{F}$, and $\Phi^1_\alpha : E_1|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$ and $\Phi^2_\alpha : E_2|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^n$ are trivializations over the same open subset $\mathcal{U}_\alpha \subset M$. The most natural way to define a trivialization of $E_1 \otimes E_2$ over this same subset is to identify the vector spaces $\mathbb{F}^{mn}$ and $\mathbb{F}^m \otimes \mathbb{F}^n$ and define

$$\Phi^{12}_\alpha : (E_1 \otimes E_2)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times (\mathbb{F}^m \otimes \mathbb{F}^n)$$

as the unique linear bundle map such that for all $x \in \mathcal{U}_\alpha$, $v \in (E_1)_x$ and $w \in (E_2)_x$, if $\Phi^1_\alpha(v) = (x, v_\alpha)$ and $\Phi^2_\alpha(w) = (x, w_\alpha)$, then

$$\Phi^{12}_\alpha(v \otimes w) = (x, v_\alpha \otimes w_\alpha). \tag{1}$$

The fact that this defines a *bilinear* map with respect to v and w implies that it extends uniquely to a well-defined linear map $(E_1)_x \otimes (E_2)_x \to \mathbb{F}^m \otimes \mathbb{F}^n$ for each $x \in \mathcal{U}_\alpha$. Now it is easy to check that if Φ^1_β and Φ^2_β are trivializations on an overlapping neighborhood $\mathcal{U}_\beta$ with smooth transition maps $g^1_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{F}^{m\times m}$ and $g^2_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathbb{F}^{n\times n}$, then the transition map relating Φ^{12}_α to Φ^{12}_β is a smooth map

$$g^{12}_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{End}(\mathbb{F}^m \otimes \mathbb{F}^n)$$

uniquely determined by the property that $g^{12}_{\beta\alpha}(x)(v \otimes w) = \left(g^1_{\beta\alpha}(x)v\right) \otimes \left(g^2_{\beta\alpha}(x)w\right)$ for all $v \in \mathbb{F}^m$ and $w \in \mathbb{F}^n$.

This story simplifies considerably when E_1 and E_2 are both complex line bundles, due to the fact that

$$\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} : v \otimes w \mapsto vw$$

is a complex vector space isomorphism. Using this to rewrite Φ^{12}_α as a bundle map $(E_1 \otimes E_2)|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}$, we replace (1) with

$$\Phi^{12}_\alpha(v \otimes w) = (x, v_\alpha w_\alpha).$$

With this understood, suppose $s_1 \in \Gamma(E_1)$ and $s_2 \in \Gamma(E_2)$ are both sections with only finitely many zeroes, where without loss of generality (e.g. after small adjustments to s_2 in local coordinates) their zero sets are disjoint, and consider the section $s_1 \otimes s_2 \in \Gamma(E_1 \otimes E_2)$. Choosing trivializations of both bundles over a region $\mathcal{U}_\alpha$ and writing the sections accordingly as $s_1^\alpha, s_2^\alpha : \mathcal{U}_\alpha \to \mathbb{C}$, their tensor product is now expressed in the corresponding local trivialization of $E_1 \otimes E_2$ as

$$(s_1 \otimes s_2)^\alpha(z) = s_1^\alpha(z) s_2^\alpha(z).$$

From this, one deduces the following: if $s_1(z_0) = 0$ but $s_2(z_0) \neq 0$ for some point $z_0 \in \mathcal{U}_\alpha$, then $\mathrm{ind}(s_1 \otimes s_2; z_0) = \mathrm{ind}(s_1; z_0)$ since on a sufficiently small disk containing z_0, s_2 can be continuously deformed without touching zero until $s_2^\alpha \equiv 1$, implying that the winding of $s_1^\alpha s_2^\alpha$ about the boundary of this disk matches that of s_1^α. The same argument proves $\mathrm{ind}(s_1 \otimes s_2; z_0) = \mathrm{ind}(s_2; z_0)$ if $s_2(z_0) = 0$ but $s_1(z_0) \neq 0$. Note finally that $s_1(z) \otimes s_2(z)$ can only be zero when either $s_1(z)$ or $s_2(z)$ is zero. In summary, this proves

$$c_1(E_1 \otimes E_2) = \sum_{z \in (s_1 \otimes s_2)^{-1}(0)} \mathrm{ind}(s_1 \otimes s_2; z) = \sum_{z \in s_1^{-1}(0)} \mathrm{ind}(s_1; z) + \sum_{z \in s_2^{-1}(0)} \mathrm{ind}(s_2; z) = c_1(E_1) + c_1(E_2).$$

(b) Observe first that if $E \to \Sigma$ is any trivial complex line bundle then $c_1(E) = 0$: indeed, triviality implies the existence of a nowhere zero section $s \in \Gamma(E)$, hence $\sum_{z \in s^{-1}(0)} \mathrm{ind}(s; z) = 0$. Next recall from Problem Set 9 #3(c) that for any complex line bundle $E \to \Sigma$, the product $E \otimes E^*$ is a trivial bundle. The reason is very simple: for every complex vector bundle there exists a natural linear bundle map of $E^* \otimes E$ to the trivial line bundle,

$$E^* \otimes E \to \Sigma \otimes \mathbb{C} : \lambda \otimes v \mapsto \lambda(v)$$

which is manifestly surjective. For dimensional reasons, it is also injective if E has rank 1. Now part (a) gives us

$$0 = c_1(E^* \otimes E) = c_1(E^*) + c_1(E),$$

implying $c_1(E^*) = -c_1(E)$.

(c) By part (b) and the result of Problem 2(c), E cannot be isomorphic to E^* unless $c_1(E) = 0$. From the Gauss-Bonnet formula, we know in the case at hand that $c_1(E) = \chi(\Sigma_g) = 2 - 2g$, thus vanishing if and only if $g = 1$. We also know in the genus 1 case that E is a trivial bundle, as $T\mathbb{T}^2$ admits a vector field that is nowhere zero, giving a global frame for E. Since a global frame for E naturally gives rise to a global frame for E^*, defined by taking the dual basis at every point, it follows that E^* is also trivial and thus $E \cong E^*$.