PROBLEM SET 2

Suggested reading

This week I'm suggesting reading some sections in Helga Baum's Vorlesungsskript for Differentialgeometrie I (see `https://www.mathematik.hu-berlin.de/~baum/Skript/diffgeo1.pdf`*) instead of Friedrich and Agricola, as the former fits better with our lecture. As usual, chapter and section indications in Lee refer to the 2003 edition and may differ in the 2013 edition.*
(Actually the original version of this problem sheet did erroneously refer to the 2013 edition, but that mistake has been corrected in this updated version!)

- Baum: §2.3–2.5
- Lee: Chapter 3, Chapter 4 (excluding "The Lie Algebra of a Lie Group"), Chapter 17 (up to "The Fundamental Theorem on Flows") and Chapter 18 (up to "Commuting Vector Fields").

By now I'm sure you've noticed that Lee contains quite a lot more material than we can cover in lecture—it is probably worth reading all of it someday, but you shouldn't feel you need to learn all of it right now.

Problems

1. Denote by $\mathbb{R}^{n\times n}$ the vector space of all real n-by-n matrices; this is isomorphic to $\mathbb{R}^{n^2}$. The n-dimensional *orthogonal group* $\mathrm{O}(n) \subset \mathbb{R}^{n\times n}$ is the set of all real n-by-n matrices $\mathbf{A}$ with the property
$$\mathbf{A}^T\mathbf{A} = \mathbb{1},$$
where $\mathbb{1}$ is the n-by-n identity matrix and $\mathbf{A}^T$ denotes the *transpose* of $\mathbf{A}$, i.e. if $\mathbf{A}$ has entries A_{ij}, then the corresponding entries of $\mathbf{A}^T$ are A_{ji}. This is precisely the set of all linear transformations $\mathbb{R}^n \to \mathbb{R}^n$ which preserve dot products $\mathbf{v}\cdot\mathbf{w}$, which means geometrically that they preserve *lengths* of vectors and *angles* between them. We will show in this problem that $\mathrm{O}(n)$ is a smooth submanifold of $\mathbb{R}^{n\times n}$.

   (a) Let $\Sigma(n) \subset \mathbb{R}^{n\times n}$ denote the set of all real *symmetric* n-by-n matrices, i.e. those which satisfy $\mathbf{A} = \mathbf{A}^T$. Show that $\Sigma(n)$ is a linear subspace of $\mathbb{R}^{n\times n}$ (i.e. it is closed under addition and scalar multiplication). What is its dimension?

   (b) Consider the map
$$f : \mathbb{R}^{n\times n} \to \Sigma(n) : \mathbf{A} \mapsto \mathbf{A}^T\mathbf{A}.$$
The orthogonal group is then precisely $\mathrm{O}(n) = f^{-1}(\mathbb{1})$. The entries of $f(\mathbf{A})$ are quadratic functions of the entries of $\mathbf{A}$, thus f is clearly a smooth map. Show that its derivative at any $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the linear map
$$df(\mathbf{A}) : \mathbb{R}^{n\times n} \to \Sigma(n) : \mathbf{H} \mapsto \mathbf{A}^T\mathbf{H} + \mathbf{H}^T\mathbf{A}.$$
Hint: in theory you can do this by computing all the partial derivatives of f with respect to the entries of $\mathbf{A}$, but it's much, *much* easier to use the definition of the derivative, i.e. regarding $\mathbb{R}^{n\times n}$ and $\Sigma(n)$ simply as Euclidean spaces, use the definition of *df* stated in Problem Set 1. One useful thing you may assume: defining the "length" $|\mathbf{A}|$ of a matrix via identification with vectors in $\mathbb{R}^{n^2}$, this length satisfies $|\mathbf{AB}| \le |\mathbf{A}||\mathbf{B}|$.

   (c) Show that $df(\mathbf{A})$ is surjective if $\mathbf{A} \in \mathrm{O}(n)$. In fact, you won't even need to assume $\mathbf{A} \in \mathrm{O}(n)$, but it *is* useful to assume that $\mathbf{A}$ is invertible (which is automatically true for orthogonal matrices). It is also *crucial* that the target space is $\Sigma(n)$ rather than the entirety of $\mathbb{R}^{n\times n}$—$df(\mathbf{A})$ is certainly *not* surjective onto $\mathbb{R}^{n\times n}$.

   (d) It follows now from the implicit function theorem that $\mathrm{O}(n)$ is a smooth submanifold of $\mathbb{R}^{n\times n}$. What is its dimension? (For a sanity check I will tell you: $\dim \mathrm{O}(2) = 1$ and $\dim \mathrm{O}(3) = 3$.)

(e) Since $\mathrm{O}(n)$ is embedded into $\mathbb{R}^{n\times n}$ as a smooth submanifold, we can regard the tangent space $T_{\mathbb{1}}\,\mathrm{O}(n)$ to the identity as a linear subspace of $\mathbb{R}^{n\times n}$, i.e. "tangent vectors" to $\mathrm{O}(n)$ are literally n-by-n matrices. Show that every matrix $\mathbf{H} \in T_{\mathbb{1}}\,\mathrm{O}(n)$ is *antisymmetric*, i.e.

$$\mathbf{H}^T = -\mathbf{H}.$$

Can you now say precisely which space of matrices $T_{\mathbb{1}}\,\mathrm{O}(n)$ is?

2. For this problem define the circle S^1 to be $\mathbb{R}/\mathbb{Z}$, i.e. the set of equivalence classes $[t]$ of real numbers $t \in \mathbb{R}$, where $s \sim t$ if and only if $s - t \in \mathbb{Z}$. There is a natural projection map $\pi : \mathbb{R} \to S^1 : t \mapsto [t]$. The tangent bundle TS^1 can then be identified with $S^1 \times \mathbb{R}$ as follows: any tangent vector $X \in TS^1$ is a velocity vector $\dot{\gamma}(0)$ for some smooth path $\gamma : (-\epsilon, \epsilon) \to S^1$, and this path can be *lifted* (in multiple ways) to a smooth path $\tilde{\gamma} : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $\pi \circ \tilde{\gamma} = \gamma$. Now identify the tangent vector $X = \dot{\gamma}(0) \in TS^1$ with the pair

$$\left(\gamma(0), \left.\frac{d\tilde{\gamma}}{dt}\right|_{t=0} \right) \in S^1 \times \mathbb{R}.$$

This gives a bijection $TS^1 \cong S^1 \times \mathbb{R}$. (Take a moment to convince yourself of this.)

Next we define the 2-torus as $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and, using the same ideas as above, identify its tangent bundle with $(S^1 \times S^1) \times (\mathbb{R} \times \mathbb{R}) = \mathbb{T}^2 \times \mathbb{R}^2$. Consider now the smooth map

$$F : \mathbb{T}^2 \to S^1 : ([s], [t]) \mapsto [3s + \sin(2\pi t)].$$

It's well defined since equivalent pairs $(s, t) \sim (s', t')$ give rise to equivalent images $3s + \sin(2\pi t) \sim 3s' + \sin(2\pi t')$.

(a) Using the identifications described above, write down an explicit expression for $TF : T\mathbb{T}^2 \to TS^1$ as a map $\mathbb{T}^2 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$.

(b) Show that F is a submersion. A slight generalization of the implicit function theorem then implies that for any $p \in S^1$, $F^{-1}(p)$ is a smooth submanifold of $\mathbb{T}^2$. Verify this for $F^{-1}([0])$ in particular, i.e. what precisely is this set? To which well known manifold is it diffeomorphic?

3. (a) Denote by x the standard coordinate on $\mathbb{R}$ and consider the smooth vector field

$$X(x) = x^2 \frac{\partial}{\partial x}.$$

Find an expression for the flow φ_X^t as a function of x. Given $x \in \mathbb{R}$, what is the largest interval $t \in (-t_0, t_0)$ for which $\varphi_X^t(x)$ is defined? Is there any value of t for which φ_X^t is well defined on all of $\mathbb{R}$?

This illustrates one of the dangerous things about flows: in general φ_X^t is only locally defined. The trouble here is that our manifold $\mathbb{R}$ is not compact*; on compact manifolds, φ_X^t is a globally defined diffeomorphism for all $t \in \mathbb{R}$.*

(b) Consider now a continuous but *nonsmooth* vector field on $\mathbb{R}$:

$$X(x) = \sqrt{|x|}\frac{\partial}{\partial x}.$$

Find two distinct solutions to the initial value problem

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)), \\ \gamma(0) = 0. \end{cases}$$

This shows that flows are not necessarily well defined when X is not smooth. That's one of a few reasons why we always assume vector fields are smooth.

4. Suppose M is a manifold with two charts $x = (x^1, \dots, x^n)$ and $(\tilde{x}^1, \dots, \tilde{x}^n)$ defined over the same open set $\mathcal{U} \subset M$. We can then think of $\tilde{x}^1, \dots, \tilde{x}^n$ as a set of n real-valued smooth functions of the n variables $(x^1, \dots, x^n)$, or vice versa; in particular the derivative of $\tilde{x}$ with respect to x at any point in $\mathcal{U}$ is the n-by-n matrix with entries $\frac{\partial \tilde{x}^i}{\partial x^j}$. Regarding the coordinate vector fields $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial \tilde{x}^j}$ as derivations, the chain rule then implies

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial}{\partial \tilde{x}^i}. \tag{1}$$

(a) The *components* of a vector $X \in T_pM$ for $p \in \mathcal{U}$ with respect to the coordinates $x^1, \dots, x^n$ are defined to be the unique real numbers $X^1, \dots, X^n$ such that $X = \sum_j X^j \frac{\partial}{\partial x^j}$. Show that these are related to the components $\widetilde{X}^j$ with respect to $\tilde{x}^1, \dots, \tilde{x}^n$ by

$$\widetilde{X}^i = \sum_j \frac{\partial \tilde{x}^i}{\partial x^j} X^j.$$

(b) If X is a smooth vector field, its components with respect to the coordinates $x^1, \dots, x^n$ are the n smooth functions $X^j : \mathcal{U} \to \mathbb{R}$ such that $X = \sum_j X^j \frac{\partial}{\partial x^j}$ on $\mathcal{U}$. The Lie derivative L_X on functions $f \in C^\infty(M)$ can then be written in coordinates as

$$L_X f = \sum_j X^j \frac{\partial f}{\partial x^j}.$$

Use this to derive the coordinate expression for the Lie bracket of two vector fields:

$$[X, Y]^i = \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right).$$

At this point one could (and many classical differential geometry books do) use the above expression to define *the Lie bracket, but one then has to use the formula of part (a) to verify that the resulting definition of $[X, Y]$ doesn't depend on the choice of coordinates. That's rather a headache and I won't ask you to do it, though I considered it.*

(c) The polar coordinates (r, θ) defined by

$$x = r\cos\theta, \quad y = r\sin\theta$$

define a pair of smooth vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ on $\mathbb{R}^2 \setminus \{0\}$. Show that

$$\frac{\partial}{\partial r} = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),$$

and write down $\frac{\partial}{\partial \theta}$ similarly in terms of the (x, y)-coordinates.

(d) Use the expressions for $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in (x, y)-coordinates together with part (b) to show that

$$\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right] = 0.$$

Then explain why this fact was already practically obvious.

5. This problem deals again with the torus $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ as defined in Problem 2. To simplify notation, we shall drop the usual brackets that indicate equivalence classes and denote points on $\mathbb{T}^2$ by (x, y) for $x, y \in \mathbb{R}$: it should be understood that we really mean $([x], [y]) \in S^1 \times S^1$. In this notation, there are well defined vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ which span every tangent space of $\mathbb{T}^2$. Consider now the vector fields

$$X(x, y) = \frac{\partial}{\partial x}, \qquad Y(x, y) = \sin(2\pi x) \frac{\partial}{\partial y}.$$

(a) Compute $[X, Y]$. (It is not zero.)

(b) Find the diffeomorphisms $\varphi_X^t : \mathbb{T}^2 \to \mathbb{T}^2$ and $\varphi_Y^t : \mathbb{T}^2 \to \mathbb{T}^2$ for arbitrary $t \in \mathbb{R}$.

(c) Fix a point $(x, y) \in \mathbb{T}^2$ and consider the "parallelogram map"

$$\alpha(s,t) := \varphi_Y^{-t} \circ \varphi_X^{-s} \circ \varphi_Y^{t} \circ \varphi_X^{s}(x, y)$$

for real numbers s and t close to 0. Show that $\partial_s \alpha(0, 0)$ and $\partial_t \alpha(0, 0)$ are both 0. Now write $\partial_s \alpha(0, t) = f_1(t)\ \partial_x + f_2(t)\ \partial_y$ and show that

$$\partial_t f_1(0)\ \partial_x + \partial_t f_2(0)\ \partial_y = [X, Y](x, y).$$

Informally, what this says is that the bracket $[X, Y]$ at a given point (x, y) is essentially the "second derivative" of the composition of flows:

$$\partial_t \partial_s \left(\varphi_Y^{-t} \circ \varphi_X^{-s} \circ \varphi_Y^{t} \circ \varphi_X^{s}(x, y)\right)\big|_{s=t=0}$$

This expression doesn't quite make sense as written, but one can make sense of it, and the equality here illustrates a more general theorem, a proof of which can be found e.g. in Spivak pp. 159–163. Observe in any case that it's clearly true if the flows commute, for then $\alpha(s, t)$ is constant in s and t.