PROBLEM SET 5

Suggested reading

As usual, chapter and section indications in Lee refer to the 2003 edition and may differ in the 2013 edition.

- Baum: §2.8–2.10
- Friedrich and Agricola: §2.1–2.5 and §3.2–3.6
- Lee: Chapter 12 ("Exterior Derivatives"), Chapter 13 (skip "The Orientation Covering", then up to "Boundary Orientations"), Chapter 14 (up to "Stokes's Theorem")

Problems

1. Given an n-dimensional vector space V, we denote by $\Lambda^k V^* \subset V_k^0$ the vector space of alternating k-multilinear maps $V \times \ldots \times V \to \mathbb{R}$. For a definition of the wedge product $\wedge : \Lambda^k V^* \times \Lambda^\ell V^* \to \Lambda^{k+\ell} V^*$, we write

$$\alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!} \operatorname{Alt}(\alpha \otimes \beta),$$

where the skew-symmetric projection $\operatorname{Alt} : V_k^0 \to \Lambda^k V^*$ is defined by

$$\operatorname{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}),$$

with S_k denoting the group of permutations of $\{1, \ldots, k\}$ and the even/odd parity of each permutation $\sigma \in S_k$ labeled by $|\sigma| = 0$ or 1 respectively.

   (a) Show that a set of dual vectors $\alpha^1, \ldots, \alpha^k \in \Lambda^1 V^* = V^*$ is linearly independent in V^* if and only if $\alpha^1 \wedge \ldots \wedge \alpha^k \neq 0$.
   Hint: Given $c_1, \ldots, c_k \in \mathbb{R}$, how can you simplify the expression $\sum_{i=1}^k c_i \alpha^i \wedge \alpha^2 \wedge \ldots \wedge \alpha^k$?

   (b) Prove by induction on k that if $\alpha^1, \ldots, \alpha^k \in V^*$ are linearly independent, then

$$\alpha^1 \wedge \ldots \wedge \alpha^k = \sum_{\sigma \in S_k} (-1)^{|\sigma|} \alpha^{\sigma(1)} \otimes \ldots \otimes \alpha^{\sigma(k)}.$$

   Then explain why the linear independence assumption is unnecessary.
   Hint: Extend $\alpha^1, \ldots, \alpha^k$ to a basis of V^* and evaluate forms using its dual basis.

2. The following topological fact lies in the background of the notion of *orientations* on vector spaces or manifolds: *For all $n \in \mathbb{N}$, the group $\mathrm{GL}(n, \mathbb{R})$ of real invertible n-by-n matrices has exactly two connected components.* Let's prove it.

   Recall that in general, a topological space X is said to be *path-connected* if for every pair of points $x, y \in X$, there exists a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A subset $A \subset X$ is then called a *connected component* if it is path-connected and is not contained in any strictly larger path-connected subset.[1] It should be clear that if the statement about $\mathrm{GL}(n, \mathbb{R})$ having two components is true, then the two components are

$$\mathrm{GL}_+(n, \mathbb{R}) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \det \mathbf{A} > 0\} \quad \text{and} \quad \mathrm{GL}_-(n, \mathbb{R}) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \det \mathbf{A} < 0\}.$$

   (a) Show that $\mathrm{GL}_+(n, \mathbb{R})$ is path-connected if and only if $\mathrm{GL}_-(n, \mathbb{R})$ is path-connected.

[1] Technically, the notions of "connectedness" and "path-connectedness" of topological spaces are not generally equivalent, but one has to consider fairly weird spaces before this distinction becomes relevant. For manifolds they are equivalent notions, and since $\mathrm{GL}_\pm(n, \mathbb{R})$ can each be identified with open subsets of an n^2-dimensional Euclidean space, they are both manifolds and we will therefore not worry about this distinction any further.

(b) Show that for every $\mathbf{A} \in \mathrm{GL}_+(n, \mathbb{R})$, there exists a continuous path from $\mathbf{A}$ to a matrix in the special orthogonal group

$$\mathrm{SO}(n) := \{\mathbf{A} \in \mathrm{GL}(n, \mathbb{R}) \mid \mathbf{A}^T\mathbf{A} = \mathbb{1} \text{ and } \det \mathbf{A} = 1\}.$$

(c) It will now suffice to show that $\mathrm{SO}(n)$ is path-connected for all $n \in \mathbb{N}$. We shall prove this by induction. For $n = 1$ it is obvious, so assume now that $\mathrm{SO}(n-1)$ is path-connected. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ denote the standard basis of unit vectors on $\mathbb{R}^n$, which we can also regard as points in the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Since matrices $\mathbf{A} \in \mathrm{SO}(n)$ act on $\mathbb{R}^n$ by orthogonal transformations, we can define a continuous map

$$\pi : \mathrm{SO}(n) \to S^{n-1} : \mathbf{A} \mapsto \mathbf{A}\mathbf{e}_1.$$

Show that π is surjective.
Hint: You may take for granted the existence of the Gram-Schmidt orthogonalization algorithm.

(d) Show that for each $\mathbf{v} \in S^{n-1}$, $\pi^{-1}(\mathbf{v})$ is homeomorphic to $\mathrm{SO}(n-1)$.
Hint: Start with the case $\mathbf{v} = \mathbf{e}_1$, and then for more general $\mathbf{v} \in S^{n-1}$, write $\mathbf{v} = \mathbf{A}\mathbf{e}_1$ for some $\mathbf{A} \in \mathrm{SO}(n)$, using part (c).

(e) Show that S^{n-1} is path-connected. (This is easy.)

(f) Suppose $\mathbf{v} : [0, 1] \to S^{n-1}$ is a continuous path. Show that for every $t_0 \in [0, 1]$ and every $\mathbf{A}_0 \in \pi^{-1}(\mathbf{v}(t_0))$, there exists a neighborhood $\mathcal{U} \subset [0, 1]$ of t_0 and a continuous path $\mathcal{U} \to \mathrm{SO}(n) : t \mapsto \mathbf{A}(t)$ such that $\mathbf{A}(t_0) = \mathbf{A}_0$ and $\pi(\mathbf{A}(t)) = \mathbf{v}(t)$ for all $t \in \mathcal{U}$.
Hint: Gram-Schmidt is useful again here.

(g) Observe that since $[0, 1]$ is compact, it can be covered by finitely many of the neighborhoods from part (f). Use this, and the assumption that $\mathrm{SO}(n-1)$ is path-connected, to show that $\mathrm{SO}(n)$ is path-connected.

If you like algebraic topology, you may be interested to know that you have just carried out the first step in proving that $\pi : \mathrm{SO}(n) \to S^{n-1}$ is a Serre fibration, and you then secretly made use of the "π_0 part" of its homotopy exact sequence to prove $\pi_0(\mathrm{SO}(n)) \cong \pi_0(\mathrm{SO}(n-1))$. Whatever that means.

3. Recall that if M is a smooth oriented n-manifold (possibly with boundary), $\omega \in \Omega^n(M)$ is a differential n-form with compact support and $A \subset M$ is an open or closed subset,[2] one can define the integral of ω over A as

$$\int_A \omega = \sum_{\alpha \in I} \int_{x_\alpha(\mathcal{U}_\alpha \cap A)} \varphi_\alpha^*(\psi_\alpha \omega),$$

where $\{\mathcal{U}_\alpha \xrightarrow{x_\alpha} x_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n\}_{\alpha \in I}$ is a finite collection of orientation-preserving charts such that each $\mathcal{U}_\alpha \subset M$ has compact closure,

$$\mathrm{supp}(\omega) \subset N := \bigcup_{\alpha \in I} \mathcal{U}_\alpha,$$

φ_α denotes the diffeomorphisms $x_\alpha^{-1} : x_\alpha(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha$ for each α, and $\{\psi_\alpha : N \to [0, 1]\}_{\alpha \in I}$ is a partition of unity on N subordinate to the open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$. We've seen in lecture that the result does not depend on our choice of charts and partition of unity. We would now like to establish a few more properties of this integral, some of which were stated in lecture but not proved.

(a) If M has boundary, show $\displaystyle\int_M \omega = \int_{M \setminus \partial M} \omega$. Similarly, show that if $X \subset M$ is a finite union of submanifolds with dimensions at most $n-1$, then

$$\int_M \omega = \int_{M \setminus X} \omega.$$

[2] More generally, one can make sense of this discussion whenever $A \subset M$ is a *Borel* subset of M, but I don't really want to get into measure theory here.

(b) Show that if $M = A \cup B$ where $A, B \subset M$ are each closed or open subsets and $A \cap B = \emptyset$, then

$$\int_M \omega = \int_A \omega + \int_B \omega.$$

Observe that by part (a), this formula remains true if $M = A \cup_\Sigma B$, meaning $\Sigma \subset M$ is an $(n-1)$-dimensional submanifold with $\Sigma = A \cap B$ and $M = A \cup B$.

(c) If $\overline{M}$ denotes the same smooth manifold as M but endowed with the opposite orientation, prove

$$\int_{\overline{M}} \omega = -\int_M \omega.$$

Hint: Given an orientation-preserving chart $x_\alpha : \mathcal{U}_\alpha \to x_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$ on M, what is the easiest way to produce from this an orientation-reversing chart?

(d) Prove the general *change of variables* formula: if M and N are smooth n-manifolds, $f : M \to N$ is an orientation-preserving diffeomorphism, $\omega \in \Omega^n(N)$ is an n-form with compact support and $A \subset M$ is an open or closed subset,

$$\int_A f^*\omega = \int_{f(A)} \omega.$$

Caution: Always remember that this is only true when f is both a diffeomorphism and orientation preserving! How does the formula change if f is still a diffeomorphism but reverses orientation?

(e) The definition of $\int_M \omega$ via partitions of unity is cumbersome, and in practice we almost always use something simpler. One commonly occurring situation is as follows: suppose M is a compact n-manifold with a chart

$$\mathcal{U} := M \setminus X \xrightarrow{x} x(\mathcal{U}) \subset \mathbb{R}^n$$

where the subset $X \subset M$ is a finite union of compact submanifolds that each have dimension at most $n-1$. In this case part (a) shows

$$\int_M \omega = \int_{M \setminus X} \omega,$$

and it now seems intuitive that one could use the change of variables formula and compute

$$\int_{M \setminus X} \omega = \int_{x(\mathcal{U})} \varphi^*\omega,$$

where $\varphi := x^{-1}$. However, the validity of this formula is not immediately obvious since ω need not have compact support on $M \setminus X$. One possible remedy is to find a sequence of compact subsets $A_k \subset M \setminus X$ for $k \in \mathbb{N}$ such that

$$A_k \subset A_{k+1} \text{ for all } k \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} A_k = M \setminus X.$$

Show that under this assumption, $\displaystyle\int_{A_k} \omega = \int_{x(A_k)} \varphi^*\omega$ and $\displaystyle\lim_{k\to\infty} \int_{A_k} \omega = \int_{M \setminus X} \omega$, hence

$$\int_M \omega = \int_{x(\mathcal{U})} \varphi^*\omega,$$

where the latter is understood as an improper integral over the noncompact open set $x(\mathcal{U}) \subset \mathbb{R}^n$.

(f) Use part (e) to show the following: if M is a closed (i.e. compact without boundary) and oriented 1-manifold, and $\gamma : [a, b] \to M$ is a smooth map such that $\gamma(a) = \gamma(b) = p \in M$ and $\gamma|_{(a,b)}$ is an orientation-preserving diffeomorphism $(a, b) \to M \setminus \{p\}$, then for any $\lambda \in \Omega^1(M)$,

$$\int_M \lambda = \int_a^b \lambda(\dot{\gamma}(t))\, dt.$$

Now go back to Problem Set 4 and think again about #3(b).

4. The standard spherical coordinates on $\mathbb{R}^3$ are defined in terms of the Cartesian coordinates (x, y, z) via the relations

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta.$$

One can make various choices in order to define (r, θ, ϕ) as a chart on some open subset $\mathcal{U} \subset \mathbb{R}^3$, e.g. one option is to set

$$\mathcal{U}_0 = \mathbb{R}^3 \setminus \{x \geq 0 \text{ and } y = 0\},$$

so that we obtain a diffeomorphism $(r, \theta, \phi) : \mathcal{U}_0 \to (0, \infty) \times (0, \pi) \times (0, 2\pi) \subset \mathbb{R}^3$. Alternatively, setting

$$\mathcal{U}_1 = \mathbb{R}^3 \setminus \{x \leq 0 \text{ and } y = 0\},$$

one can regard (r, θ, ϕ) instead as a diffeomorphism $\mathcal{U}_1 \to (0, \infty) \times (0, \pi) \times (-\pi, \pi) \subset \mathbb{R}^3$. Notice that r and θ are both well defined on $\mathcal{U}_0 \cup \mathcal{U}_1$; for ϕ this is not true, and the two charts described above define two versions of the coordinate $\phi : \mathcal{U}_0 \cap \mathcal{U}_1 \to \mathbb{R}$ which differ on certain regions, though only by addition of the constant 2π. As a consequence, the 1-forms dr, $d\theta$ and $d\phi$ are well defined everywhere on $\mathbb{R}^3$ except at the z-axis, where all conceivable definitions of both θ and ϕ become singular.

Note: This situation is analogous to what we saw in #3 on Problem Set 4, where the 1-form λ could be interpreted as $d\phi$ with respect to the polar coordinates (r, ϕ) on $\mathbb{R}^2$ and is well defined on $\mathbb{R}^2 \setminus \{0\}$, even though ϕ itself requires a smaller domain in order define a smooth real-valued function.

(a) Compute formulas for dx, dy and dz in terms of dr, $d\theta$ and $d\phi$. On what subsets of $\mathbb{R}^3$ are these formulas valid?

(b) Show that on the complement of the z-axis,

$$dx \wedge dy \wedge dz = r^2 \sin\theta \, dr \wedge d\theta \wedge d\phi.$$

(c) Restricting to the unit sphere $S^2 = \{r = 1\} \subset \mathbb{R}^3$, (θ, ϕ) and (ϕ, θ) can now be regarded as charts on the intersection of S^2 with either of $\mathcal{U}_0$ or $\mathcal{U}_1$. The 1-forms $d\theta$ and $d\phi$ are thus well defined on the entire sphere except at the north and south poles $\{z = \pm 1\} \subset S^2$. Assuming $\mathbb{R}^3$ carries its canonical orientation and S^2 inherits the natural orientation as the boundary of the unit ball, which chart is orientation preserving: (θ, ϕ) or (ϕ, θ)?

(d) For $p \in S^2$, use the canonical identification of $T_p\mathbb{R}^3$ with $\mathbb{R}^3$ to define $\nu(p) \in T_p\mathbb{R}^3$ as the outward-pointing unit vector in $\mathbb{R}^3$ orthogonal to the subspace $T_pS^2 \subset \mathbb{R}^3$. Show that ν is the same as the coordinate vector field ∂_r at every point along S^2 where the coordinates (r, θ, ϕ) can be defined.

(e) We can now define a (positively oriented) area form ω on S^2 in terms of the standard volume form on $\mathbb{R}^3$ by

$$\omega := \iota_\nu \left(dx \wedge dy \wedge dz|_{TS^2} \right),$$

in other words, for $p \in S^2$ and $X, Y \in T_pS^2$, $\omega(X, Y) := dx \wedge dy \wedge dz(\nu(p), X, Y)$. Show that

$$\omega = \sin\theta \, d\theta \wedge d\phi$$

away from the north and south poles.

(f) Use the results of Problem 3 to compute

$$\int_{S^2} \omega = 4\pi$$

by integrating over a suitable open subset on which a chart (θ, ϕ) can be defined.

(g) Find a 1-form λ on $S^2 \setminus \{z = \pm 1\}$ such that $\omega = d\lambda$ on this region. In this case we say λ is a *primitive* of ω.

(h) Given $\epsilon > 0$ small, consider the compact subset

$$A_\epsilon = \{\epsilon \leq \theta \leq \pi - \epsilon\} \subset S^2.$$

Use Stokes' theorem and the primitive λ from part (g) to compute $\int_{A_\epsilon} \omega$, then take the limit as $\epsilon \to 0$ and make sure you get the right answer. (If you don't get the right answer, it probably means you're not paying careful enough attention to orientations.)