PROBLEM SET 6

Suggested reading

As usual, chapter and section indications in Lee refer to the 2003 edition and may differ in the 2013 edition.

- Friedrich and Agricola: §2.6 and §3.8–3.9
- Lee: Chapter 15 (up to "Homotopy Invariance") and Chapter 18 ("Lie derivatives of Tensor Fields")

Problems

1. Let's start with something easy: suppose M is a compact oriented n-manifold with boundary, $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$ with $k + \ell = n - 1$. Prove the n-dimensional *integration by parts* formula:

$$\int_M d\alpha \wedge \beta = \int_{\partial M} \alpha \wedge \beta - (-1)^k \int_M \alpha \wedge d\beta.$$

2. In lecture last Thursday I got the definition of orientations somewhat muddled, so here is a corrected version. Assume M is a smooth n-manifold (possibly with boundary), and denote by $\mathcal{A} = \{(\mathcal{U}_\alpha, x_\alpha)\}_{\alpha \in I}$ its maximal atlas of smoothly compatible charts $x_\alpha : \mathcal{U}_\alpha \to x(\mathcal{U}_\alpha) \subset \mathbb{H}^n$. An *orientation* on M is then a choice of subatlas $\mathcal{A}_+ \subset \mathcal{A}$, i.e. a subcollection $\{(\mathcal{U}_\alpha, x_\alpha)\}_{\alpha \in I_+}$ with $I_+ \subset I$, satisfying the following conditions:

   - $M = \bigcup_{\alpha \in I_+} \mathcal{U}_\alpha$;
   - For every $\alpha, \beta \in I_+$, the transition map $x_\alpha \circ x_\beta^{-1}$ is orientation preserving;
   - $\mathcal{A}_+$ is maximal in the sense that every $(\mathcal{U}, x) \in \mathcal{A}$ for which $x \circ x_\alpha^{-1}$ is orientation preserving for every $(\mathcal{U}_\alpha, x_\alpha) \in \mathcal{A}_+$ also belongs to $\mathcal{A}_+$.

   Given an orientation $\mathcal{A}_+ \subset \mathcal{A}$, we refer to the charts in $\mathcal{A}_+$ as *orientation preserving* (or "positively oriented"), and define the collection $\mathcal{A}_- \subset \mathcal{A}$ of *orientation-reversing* charts by the condition that $(\mathcal{U}, x) \in \mathcal{A}_-$ if and only if $x \circ x_\alpha^{-1}$ is an orientation-reversing map for every $(\mathcal{U}_\alpha, x_\alpha) \in \mathcal{A}_+$. Notice that $\mathcal{A}_+ \cap \mathcal{A}_- = \emptyset$, though it is also possible for a chart $(\mathcal{U}, x) \in \mathcal{A}$ to be in neither $\mathcal{A}_+$ nor $\mathcal{A}_-$, e.g. this may happen if $\mathcal{U}$ has more than one connected component, as x could restrict to an orientation-preserving chart on one connected component of its domain and an orientation-reversing chart on a different component.[1] It remains true however that an orientation on M determines orientations of all the vector spaces T_pM for $p \in M$, namely via the requirement that for any $(\mathcal{U}, x) \in \mathcal{A}_+$, the vector space isomorphism

$$x_*|_{T_pM} : T_pM \to T_{x(p)}\mathbb{R}^n$$

   should be orientation preserving with respect to the canonical orientation on $T_{x(p)}\mathbb{R}^n$ defined via its natural identification with $\mathbb{R}^n$.

   (a) Show that if M is an oriented manifold, then every chart $x : \mathcal{U} \to \mathbb{H}^n$ whose domain is *connected* is either orientation preserving or orientation reversing.

   (b) The *Klein bottle* $\mathbb{K}^2$ is a smooth 2-manifold which can be defined as the following quotient of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (see Figure 1):

$$\mathbb{K}^2 := \mathbb{T}^2/\sim \qquad \text{where } [(\theta, \phi)] \sim [(\theta + 1/2, -\phi)] \text{ for all } [(\theta, \phi)] \in \mathbb{T}^2.$$

   Find a pair of charts $(\mathcal{U}_1, x_1)$ and $(\mathcal{U}_2, x_2)$ on $\mathbb{K}^2$ such that the subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ are both connected but $\mathcal{U}_1 \cap \mathcal{U}_2$ has two connected components, and the transition map $x_1 \circ x_2^{-1}$ is neither orientation preserving nor orientation reversing.

[1] I overlooked this detail in last Thursday's lecture, which is why even the revised definition I gave there was not quite right.

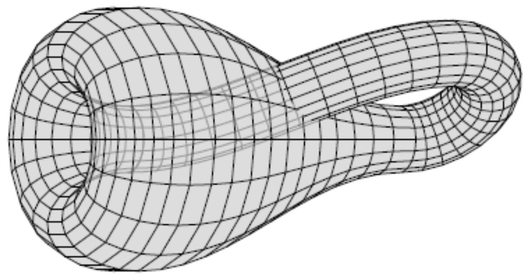

Figure 1: The image of a (non-injective) immersion of the Klein bottle into $\mathbb{R}^3$. (Picture borrowed from *The Manifold Atlas*, `http://www.map.mpim-bonn.mpg.de/2-manifolds`)

(c) Explain why part (b) implies that $\mathbb{K}^2$ is not orientable, i.e. it does not admit an orientation.

(d) Find a continuous path $\gamma : [0,1] \to \mathbb{K}^2$ with $\gamma(1) = \gamma(0) =: p$ and a continuous family of bases $(X_1(t), X_2(t))$ of $T_{\gamma(t)}\mathbb{K}^2$ such that $(X_1(0), X_2(0))$ and $(X_1(1), X_2(1))$ determine distinct orientations of the vector space $T_p\mathbb{K}^2$, i.e. they are not related to each other by any continuous family of bases of $T_p\mathbb{K}^2$.

3. If M and N are oriented manifolds of dimensions m and n respectively, the *product orientation* of $M \times N$ is uniquely determined by the following property: given any point $(p,q) \in M \times N$ and any positively oriented bases $(X_1, \ldots, X_m)$ of T_pM and $(Y_1, \ldots, Y_n)$ of T_qN, the basis $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ of $T_{(p,q)}(M \times N)$ is positively oriented. This definition uses the fact that there is a natural isomorphism $T_{(p,q)}(M \times N) = T_pM \times T_qN$; take a moment to convince yourself that this is true, and that the resulting notion of product orientation is well defined. Then show:

(a) $M \times N = (-1)^{mn} N \times M$, where for any oriented manifold Q, we denote by $-Q$ the same manifold with its orientation reversed.

(b) If M and/or N has boundary, then assuming all boundaries carry the natural boundary orientations and products carry the natural product orientations,

$$\partial(M \times N) = (\partial M \times N) \cup (-1)^m (M \times \partial N).$$

Remark: If both M and N have nonempty boundary then we are cheating slightly with this notation, as $M \times N$ is not technically a manifold with boundary, but a more general object called a "manifold with boundary and corners". (In particular its structure near $\partial M \times \partial N$ does not fit the definition of a manifold with boundary). There is no need to worry about this detail right now—just show that the boundary orientations indicated above are correct at all points on the boundary of $(M \times N) \setminus (\partial M \times \partial N)$.

4. Recall that if $(x^1, \ldots, x^n) : \mathcal{U} \to \mathbb{R}^n$ is a chart defined on an open subset in some n-manifold M, any k-form $\omega \in \Omega^k(M)$ can be written on $\mathcal{U}$ as

$$\omega = \omega_{i_1 \ldots i_k}\, dx^{i_1} \otimes \ldots \otimes dx^{i_k} = \frac{1}{k!} \omega_{i_1 \ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where the first two expressions use the Einstein summation convention and the third one does not. Here the component functions $\omega_{i_1 \ldots i_k} : \mathcal{U} \to \mathbb{R}$ can be written in terms of the coordinate vector fields $\partial_1, \ldots, \partial_n$ as $\omega_{i_1 \ldots i_k} = \omega(\partial_{i_1}, \ldots, \partial_{i_k})$. In order to write down a coordinate formula for the exterior derivative, we introduce the following notation: given any collection of functions $T_{i_1 \ldots i_k}$ on $\mathcal{U}$ labeled by the indices $i_1, \ldots, i_k$, define

$$T_{[i_1 \ldots i_k]} := \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{|\sigma|} T_{i_{\sigma(1)} \ldots i_{\sigma(k)}},$$

so for instance if $T_{i_1\dots i_k}$ are the components of a tensor field T, then $\mathrm{Alt}(T)_{i_1,\dots,i_k} = T_{[i_1\dots i_k]}$, and the wedge product of $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$ can now be written in coordinates as

$$(\alpha \wedge \beta)_{i_1\dots i_k j_1 \dots j_\ell} = \alpha_{[i_1\dots i_k}\beta_{j_1\dots j_\ell]}.$$

(a) Prove that the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies

$$(d\omega)_{i_1\dots i_{k+1}} = (k+1)\partial_{[i_1}\omega_{i_2\dots i_{k+1}]}.$$

(b) Show that for $\lambda \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$, the above formula reduces to

$$(d\lambda)_{ij} = \partial_i\lambda_j - \partial_j\lambda_i, \qquad \text{and} \qquad (d\omega)_{ijk} = \partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij}.$$

(c) It now follows from Problem Set 4 #1(a) that the exterior derivative of a 1-form λ can also be written as

$$d\lambda(X,Y) = L_X(\lambda(Y)) - L_Y(\lambda(X)) - \lambda([X,Y]).$$

Indeed, the right hand side is C^∞-linear with respect to vector fields $X, Y \in \mathrm{Vec}(M)$ and thus defines a tensor field, whose component functions we've seen match the formula from part (b). Prove the corresponding formula for the exterior derivative of a 2-form,

$$d\omega(X,Y,Z) = L_X\left(\omega(Y,Z)\right) + L_Y\left(\omega(Z,X)\right) + L_Z\left(\omega(X,Y)\right) - \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y).$$

Remark: Similar formulas exist for the exterior derivatives of k-forms for all $k > 2$, though I cannot recall ever having needed to use them.

5. Recall that the *kth de Rham cohomology group* $H^k_{\mathrm{dR}}(M)$ of a smooth manifold M is a real vector space defined as the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ modulo the image of $d : \Omega^{k-1}(M) \to \Omega^k(M)$. Here we adopt the convention $\Omega^{-1}(M) = \{0\}$ so that this is also well defined for $k = 0$. Show that the map

$$H^1_{\mathrm{dR}}(S^1) \to \mathbb{R} : [\lambda] \mapsto \int_{S^1} \lambda$$

is a well-defined vector space isomorphism.
Hint: You might find some inspiration on Problem Set 4, #3.

6. Given a volume form $\mu \in \Omega^n(M)$ on an n-manifold M, one can define volumes of compact regions $\mathcal{U} \subset M$ by

$$\mathrm{Vol}(\mathcal{U}) := \int_{\mathcal{U}} \mu.$$

The *divergence* of a vector field $X \in \mathrm{Vec}(M)$ can then be defined in terms of the Lie derivative of μ with respect to X: let $\mathrm{div}(X) : M \to \mathbb{R}$ be the unique real-valued function such that

$$L_X\mu = \mathrm{div}(X)\,\mu.$$

Note that this is well defined since $L_X\mu$ is an n-form and the space $\Lambda^n T_p^*M$ of n-forms at each point $p \in M$ is 1-dimensional. Observe also that $d\mu = 0$ since $\Omega^{n+1}(M) = \{0\}$, so Cartan's formula implies $\mathrm{div}(X)\,\mu = d\iota_X\mu$, which matches the formula we saw in lecture for the case $M = \mathbb{R}^n$.

(a) Show that if $\varphi^t_X : M \to M$ denotes the flow of X, then for any compact region $\mathcal{U} \subset M$,

$$\frac{d}{dt}\mathrm{Vol}(\varphi^t_X(\mathcal{U})) = \int_{\varphi^t_X(\mathcal{U})} \mathrm{div}(X)\,\mu.$$

(b) Show that in the case $M = \mathbb{R}^3$ with $\mu = dx \wedge dy \wedge dz$ and $\mathbf{X} = X^x\partial_x + X^y\partial_y + X^z\partial_z \in \mathrm{Vec}(\mathbb{R}^3)$ in standard Cartesian coordinates (x,y,z),

$$\mathrm{div}(\mathbf{X}) = \partial_x X^x + \partial_y X^y + \partial_z X^z.$$

The latter expression is sometimes also denoted by $\nabla \cdot \mathbf{X}$.
Note: One can show more generally that if $\mu = dx^1 \wedge \ldots \wedge dx^n$ on $\mathbb{R}^n$ then $\mathrm{div}(X) = \partial_i X^i$.

(c) Recall that on $\mathbb{R}^3$, the *gradient* of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is the vector field

$$\operatorname{grad}(f) = \nabla f := (\partial_x f)\partial_x + (\partial_y f)\partial_y + (\partial_z f)\partial_z,$$

and the *curl* of a vector field $\mathbf{X} = X^x\partial_x + X^y\partial_y + X^z\partial_z$ is the vector field

$$\operatorname{curl}(\mathbf{X}) = \nabla \times \mathbf{X} := (\partial_y X^z - \partial_z X^y)\partial_x + (\partial_z X^x - \partial_x X^z)\partial_y + (\partial_x X^y - \partial_y X^x)\partial_z.$$

Using the relations of these operations to differential forms and the exterior derivative, deduce from $d^2 = 0$ the formulas

$$\nabla \times (\nabla f) = 0 \qquad \text{and} \qquad \nabla \cdot (\nabla \times \mathbf{X}) = 0$$

for all $f \in C^\infty(\mathbb{R}^3)$ and $\mathbf{X} \in \operatorname{Vec}(\mathbb{R}^3)$.

(d) Use the Poincaré lemma to deduce that on $\mathbb{R}^3$, any vector field with zero curl is the gradient of a function, and any vector field with zero divergence is the curl of another vector field.

(e) Find an example of a vector field $\mathbf{X}$ on $\mathbb{R}^3 \setminus \{x = y = 0\}$ that has zero curl but is not the gradient of a function.

7. In this problem we shall work through a proof of the fact that for a smooth map $f : M \to N$, the induced homomorphism on de Rham cohomology

$$f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$

depends on f only up to smooth homotopy. Recall that two maps $f, g : M \to N$ are *smoothly homotopic* if there exists a smooth homotopy between them, meaning a smooth map $h : [0,1] \times M \to N$ such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$.

Assume throughout the following that $h : \mathbb{R} \times M \to N$ is a smooth map, let $f_t := h(t,\cdot) : M \to N$ and $j_t : M \hookrightarrow \mathbb{R} \times M : p \mapsto (t,p)$ for each $t \in \mathbb{R}$, and define

$$\Phi : \Omega^k(N) \to \Omega^{k-1}(\mathbb{R} \times M) : \omega \mapsto \iota_{\partial_t}(h^*\omega) := h^*\omega(\partial_t, \ldots),$$

where $t : \mathbb{R} \times M \to \mathbb{R}$ denotes the standard coordinate function on the first factor (i.e. the natural projection $\mathbb{R} \times M \to \mathbb{R}$) and $\partial_t \in \operatorname{Vec}(\mathbb{R} \times M)$ is the corresponding coordinate vector field. Notice that the flow $\varphi_s : \mathbb{R} \times M \to \mathbb{R} \times M$ is well defined for all times $s \in \mathbb{R}$ and is very simple, namely $\varphi_s(t,p) = (t+s,p)$. We also define

$$\Phi_t := j_t^*\Phi : \Omega^k(N) \to \Omega^{k-1}(M)$$

for every $t \in \mathbb{R}$, and let ω denote an arbitrary k-form on N.

(a) Use Cartan's formula to derive the expression

$$L_{\partial_t}(h^*\omega) = d\Phi\omega + \Phi d\omega. \tag{1}$$

(b) For any $(t,p) \in \mathbb{R} \times M$, the tangent space $T_{(t,p)}(\mathbb{R} \times M)$ has a subspace of codimension 1 that is naturally identified with T_pM, namely the space of all vectors tangent at (t,p) to the submanifold $\{t\} \times M$. With this understood, show that for all tuples $(X_1, \ldots, X_k) \in T_pM \subset T_{(t,p)}(\mathbb{R} \times M)$ and all $s \in \mathbb{R}$,

$$(h \circ \varphi_s)^*\omega(X_1, \ldots, X_k) = f^*_{t+s}\omega(X_1, \ldots, X_k). \tag{2}$$

(c) Use (2) and the definition of the Lie derivative of k-forms to show that for all $(t,p) \in \mathbb{R} \times M$ and tuples $(X_1, \ldots, X_k) \in T_pM \subset T_{(t,p)}(\mathbb{R} \times M)$,

$$L_{\partial_t}(h^*\omega)(X_1, \ldots, X_k) = \frac{d}{dt} f_t^*\omega(X_1, \ldots, X_k). \tag{3}$$

Combining this (1) and applying the operator j_t^*, this implies the formula

$$\frac{d}{dt} f_t^*\omega = d\Phi_t\omega + \Phi_t d\omega \quad \text{ for all } \quad t \in \mathbb{R},\ \omega \in \Omega^k(N). \tag{4}$$

(d) Integrate (4) with respect to t in order to show that there exists a homomorphism $H : \Omega^k(N) \to \Omega^{k-1}(M)$ satisfying $f_1^*\omega - f_0^*\omega = (d \circ H + H \circ d)\omega$ for all $\omega \in \Omega^k(N)$. Explain why this implies that f_0^* and f_1^* descend to the same map $H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$.