Differential Geometry I | Humboldt-Universität zu Berlin
C. Wendl | Winter Semester 2016–17

TAKE-HOME MIDTERM

Instructions

To receive credit for this assignment, you must hand it in on **Tuesday, January 24** in lecture, or beforehand in my office (slipping it under the door is fine if I'm not there). You are free to use any resources at your disposal and discuss the problems with your comrades, but you must write up your solutions alone. Solutions may be written up in German or English, this is up to you. We will discuss the solutions on January 24 in the Übung.

There are 100 points in total: a score of 50 points or better will boost your final exam grade according to the (approximate) formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is approximately proportional to its conceptual importance/difficulty. A piece of advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

Please feel free to e-mail me if questions arise. (I will not be in Berlin during the week January 13–19, but I should still be reachable by e-mail.)

Suggested reading for the next two weeks

Lecture notes (on the website): Chapter 4, *Natural Constructions on Vector Bundles*

(Note: This suggestion is not relevant to the take-home midterm, but will be important for keeping up with new course material in the mean time.)

Scheduling note

The class will be canceled on **Tuesday, January 17** (both lecture and Übung) and **Thursday, January 26** (lecture). The class will meet as usual on Thursday, January 19, but with a substitute lecturer.

Problems

1. [20 pts total] In class we defined the real projective n-space $\mathbb{RP}^n$ as $S^n/\sim$, where the equivalence relation identifies antipodal points $\mathbf{x} \sim -\mathbf{x}$ on the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Since every line through the origin in $\mathbb{R}^{n+1}$ passes through exactly two points on S^n, which are antipodal, an equivalent definition is

$$\mathbb{RP}^n := (\mathbb{R}^{n+1} \setminus \{0\}) \big/ \mathbb{R}^*,$$

where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ denotes the multiplicative group of nonzero real numbers, acting on elements of $\mathbb{R}^{n+1} \setminus \{0\}$ by scalar multiplication. In other words, two nonzero vectors in $\mathbb{R}^{n+1}$ represent the same element of $\mathbb{RP}^n$ if and only if one is a scalar multiple of the other, and it is thus natural to denote elements of $\mathbb{RP}^n$ in the form

$$[x_0 : \ldots : x_n] \in \mathbb{RP}^n,$$

meaning the equivalence class represented by $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, where we assume always that $x_j \neq 0$ for at least one of the $j = 0, \ldots, n$, and one must keep in mind that $[x_0 : \ldots : x_n] = [\lambda x_0 : \ldots : \lambda x_n]$ for any $\lambda \in \mathbb{R}^*$.

   (a) [2 pts] For each $n \geq 1$ and $j = 0, \ldots, n$, write down a bijective map from $\mathbb{RP}^{n-1}$ to the subset

$$\Sigma_j := \left\{ [x_0 : \ldots : x_n] \in \mathbb{RP}^n \;\middle|\; x_j = 0 \right\} \subset \mathbb{RP}^n.$$

(b) [2 pts] Notice that since every $[x_0 : \ldots : x_n] \in \mathbb{RP}^n$ satisfies $x_j \neq 0$ for some $j \in \{0, \ldots, n\}$, $\mathbb{RP}^n = \bigcup_{j=0}^n \mathcal{U}_j$, where $\mathcal{U}_j := \mathbb{RP}^n \setminus \Sigma_j$. One can now define a smooth structure on $\mathbb{RP}^n$ via the following charts: for each $j = 0, \ldots, n$, define the map

$$\varphi_j : \mathcal{U}_j \to \mathbb{R}^n : [x_0 : \ldots : x_n] \mapsto \left(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \right).$$

Show that φ_j is bijective, and write down its inverse.

(c) [7 pts] Write down a formula for the transition map $\varphi_1 \circ \varphi_0^{-1}$. It should be clear from the formula that this map is smooth. What are its domain and image? Is it orientation preserving, orientation reversing, or neither? (Note that the formulas for all other transition maps $\varphi_j \circ \varphi_k^{-1}$ will be the same just with some of the coordinates switched around; in particular, they are all smooth.)

(d) [4 pts] Using the charts $\varphi_j : \mathcal{U}_j \to \mathbb{R}^n$, show that Σ_0 is a smooth submanifold of $\mathbb{RP}^n$ and is diffeomorphic to $\mathbb{RP}^{n-1}$. *(Note: You may choose freely between the two equivalent definitions of "submanifold" that we have discussed in this class—see Problem Set 3#1.)*

(e) [5 pts] Show that $\mathbb{RP}^n$ is non-orientable for every even n.
Hint: You might find some helpful inspiration in Problem Set 6#2.

Remark: One can also show that $\mathbb{RP}^n$ is orientable for every odd n, but the charts φ_j do not furnish the most convenient proof of this. It is obvious at least for $n = 1$, since $\mathbb{RP}^1 \cong S^1$.

2. [30 pts total] Since the spaces of closed or exact k-forms on a smooth manifold M are both infinite dimensional, it may seem surprising that their quotient, the de Rham cohomology $H^k_{\mathrm{dR}}(M)$, is finite dimensional for most examples of interest; in particular it turns out that this is always true when M is compact. In this problem, we shall compute that for every $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$,

$$\dim H^k_{\mathrm{dR}}(S^n) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In the big picture, one usually thinks of this as a consequence of *de Rham's theorem*, which gives an isomorphism between de Rham cohomology and the *singular cohomology* of M with real coefficients; the latter is a topological invariant that is easy to compute using methods of algebraic topology. But that would take us somewhat far afield, so instead we'll work out a self-contained differential geometric proof of (1), based mainly on Stokes' theorem.

Recall that we showed $H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$ in Problem Set 6#5, so part (a) below establishes (1) in the case $n = 1$. We will obtain the rest by induction on n.

(a) [3 pts] Show that $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$ for every connected manifold M.
Hint: Recall that since there is no such thing as a "(-1)-form," we set $\Omega^{-1}(M) := \{0\}$ by convention, hence the space of "exact 0-forms" is defined as the trivial subspace of $\Omega^0(M)$.

(b) [6 pts] Show that if M is any closed and oriented n-manifold, then there is a well-defined linear map

$$H^n_{\mathrm{dR}}(M) \to \mathbb{R} : [\omega] \mapsto \int_M \omega, \tag{2}$$

and the following conditions are equivalent:

i. $H^n_{\mathrm{dR}}(M) \cong \mathbb{R}$;
ii. The map (2) is an isomorphism;
iii. Every $\omega \in \Omega^n(M)$ satisfying $\int_M \omega = 0$ is exact.

Hint: Recall that every n-form is closed since $\Omega^{n+1}(M) = \{0\}$. You will need to use the theorem that every orientable manifold admits a volume form.

(c) [8 pts] Suppose M is a closed n-manifold and ω_+, ω_- is a pair of k-forms on $M \times [-1, 1]$ such that $d\omega_+ = d\omega_-$. Show that the following conditions are equivalent:

i. $\omega_+ - \omega_-$ is exact;

ii. $\iota_t^*\omega_+ - \iota_t^*\omega_-$ is an exact k-form on M for every $t \in [-1,1]$, where $\iota_t : M \to M \times [-1,1]$ denotes the inclusion $p \mapsto (p,t)$.

iii. There exists a k-form ω on $M \times [-1,1]$ which matches $\omega_\pm$ near $M \times \{\pm 1\}$ and satisfies $d\omega = d\omega_+ = d\omega_-$.

Hint: First prove the equivalence of (i) and (ii), using the fact that $\iota_t : M \to M \times [-1,1]$ is a smooth homotopy equivalence. (Do not give a detailed proof of this fact, but sketch the idea.)

(d) [5 pts] Under the same assumptions as in part (c), suppose also that M is oriented and $k = n$. Show that the number $\int_{M\times\{t\}} \omega_+ - \int_{M\times\{t\}} \omega_- \in \mathbb{R}$ is the same for any choice of $t \in [-1,1]$.
Hint: Given $-1 \le t_- < t_+ \le 1$, integrate something over $M \times [t_-, t_+]$ and apply Stokes' theorem.

(e) [8 pts] Now given an integer $n \ge 2$, assume (1) is true for S^{n-1}, and fix $k \in \{1, \dots, n\}$. Regarding S^n as the unit sphere in $\mathbb{R}^{n+1}$ with standard coordinates $(x^1, \dots, x^{n+1})$, we can decompose it into two overlapping n-dimensional disks $S^n = D_+ \cup D_-$ whose intersection looks like $S^{n-1} \times [-1,1]$; specifically, define

$$D_+ := \{x^1 \ge -1/2\} \cap S^n, \qquad D_- := \{x^1 \le 1/2\} \cap S^n.$$

Take a moment to convince yourself that there is a diffeomorphism $D_+ \cap D_- \cong S^{n-1} \times [-1,1]$. Recall next that by the Poincaré lemma, any closed k-form ω on S^n will then by exact over each of D_+ and D_-, giving $\alpha_\pm \in \Omega^{k-1}(D_\pm)$ such that $d\alpha_\pm = \omega$ on $D_\pm$. The difficulty is that α_+ and α_- need not match on $D_+ \cap D_-$. Use the inductive hypothesis and the previous steps in this problem to show that if either $1 \le k \le n-1$ or $k = n$ with $\int_{S^n} \omega = 0$, then there exists $\alpha \in \Omega^{k-1}(S^n)$ satisfying $d\alpha = \omega$; show in fact that α can be chosen to match $\alpha_\pm$ on the portions of $D_\pm$ where D_+ and D_- do not overlap. This completes the inductive proof of (1).
Hint: The case $k = n$ is trickiest, as you need to use the hypothesis $\int_{S^n} \omega = 0$ to deduce something about α_+ and α_-. What can you say about the integrals of $\alpha_\pm$ over the "equator" $S^{n-1} \cong \{x^1 = 0\} \subset S^n$? Try Stokes' theorem, but be careful with orientations!

3. [25 pts total] On a $2n$-dimensional manifold M, we've previously defined the term "symplectic form" to mean a 2-form $\omega \in \Omega^2(M)$ that is closed and nondegenerate (cf. Problem Set 7 #4(e)). By a fundamental result in symplectic geometry called Darboux's theorem, the following definition is equivalent: $\omega \in \Omega^2(M)$ is a *symplectic form* if every point $x \in M$ is contained in the domain $\mathcal{U} \subset M$ of a chart of the form $(q^1, p^1, \dots, q^n, p^n) : \mathcal{U} \to \mathbb{R}^{2n}$ in which

$$\omega = \sum_{j=1}^{n} dp^j \wedge dq^j.$$

Any local coordinate chart in which ω takes this form is called a *Darboux chart.*

(a) [2 pts] Show that if ω is symplectic, then $\omega^n := \omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M)$ is a volume form.

(b) [5 pts] Show that every closed manifold M admitting a symplectic form satisfies $H^2_{\mathrm{dR}}(M) \neq \{0\}$. Be sure to point out where you are using the assumption that M is compact and without boundary. Show also (by example) that the result is not true for noncompact manifolds in general.
Remark: By the result of Problem 2, it follows that the spheres S^{2n} do not admit symplectic forms except in the case $n = 1$ (on an oriented surface, every volume form is closed and nondegenerate and is thus a symplectic form). By contrast, a theorem of Gromov from 1969 based on the h-principle implies that a noncompact manifold always admits a symplectic form if it admits an almost complex structure; no cohomological condition is necessary. Popular examples of closed manifolds that do admit symplectic forms include the tori $\mathbb{T}^{2n} := \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, and the complex projective spaces $\mathbb{CP}^n$ (i.e. the complex version of $\mathbb{RP}^n$, cf. Problem 1).

(c) [3 pts] A smooth function $H : M \to \mathbb{R}$ on a symplectic manifold (M, ω) gives rise to a so-called *Hamiltonian vector field* $X_H \in \mathrm{Vec}(M)$, defined as the unique vector field satisfying[1]

$$\omega(X_H, \cdot) = -dH.$$

[1] Some authors define X_H instead by $\omega(X_H, \cdot) = dH$. There are good motivations to include the minus sign as I did here, but it is to some extent a matter of taste.

This determines X_H uniquely due to the nondegeneracy of ω, i.e. for any $x \in M$, the linear map $T_x M \to T_x^* M : Y \mapsto \omega(Y, \cdot)$ is injective and is therefore an isomorphism, whose inverse determines $X_H(x)$ from $-dH|_{T_x M}$. Given another function $F : M \to \mathbb{R}$, we say that F is *conserved* under the Hamiltonian flow of H if for every solution $x : \mathbb{R} \to M$ of the differential equation $\dot{x}(t) = X_H(x(t))$, the function $F(x(t))$ is constant in t. Show that H itself is conserved. *(In applications to mechanical systems, H is typically interpreted as the "energy" of the system.)*

(d) [4 pts] In 1915, Emmy Noether established a beautiful correspondence between the conserved quantities of a mechanical system and its symmetries. A simple version of this theorem in the Hamiltonian context takes the following form: fixing $H : M \to \mathbb{R}$ as above, suppose $F : M \to \mathbb{R}$ is another function which is conserved under the Hamiltonian flow of H. Then the flow of the vector field X_F determined by $\omega(X_F, \cdot) = -dF$ gives a smooth 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ which are *symplectic* and preserve H, meaning

$$\varphi_t^* \omega = \omega \quad \text{and} \quad H \circ \varphi_t = H \tag{3}$$

for all t. Prove this, assuming that the flow of X_F exists for all $t \in \mathbb{R}$.[2]

(e) [5 pts] In some settings, there is a converse to the result proved in part (d). Suppose (M, ω) is diffeomorphic to $\mathbb{R}^{2n}$, and $Y \in \mathrm{Vec}(M)$ is a vector field with a well-defined flow $\varphi_t : M \to M$ satisfying (3) for all t. Show that there exists a function $F : M \to \mathbb{R}$, uniquely defined up to addition of a constant, which satisfies $\omega(Y, \cdot) = -dF$ and is conserved under the Hamiltonian flow of H.

Let's work out a concrete example. Let $M = \mathbb{R}^4$ with coordinates (x, p_x, y, p_y) and the standard symplectic form

$$\omega_{\mathrm{std}} = dp_x \wedge dx + dp_y \wedge dy.$$

We can think of $\mathbb{R}^4$ as the "position-momentum space" (also called *phase space*) representing the motion of a single particle of mass $m > 0$ in a plane: its position is given by $\mathbf{q} := (x, y) \in \mathbb{R}^2$, and $\mathbf{p} := (p_x, p_y) \in \mathbb{R}^2$ are the corresponding "momentum variables". Given a "potential" function $V : \mathbb{R}^2 \to \mathbb{R}$, the total energy of the system is given by the function

$$H(\mathbf{q}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{q}).$$

Suppose now that the potential V is chosen to be *rotationally symmetric*, e.g. this is the case if $\mathbf{q}$ represents the position of the Earth moving around the sun (with the latter positioned at the origin). To express this condition succinctly, one can transform to polar coordinates (r, θ) on $\mathbb{R}^2$, related to the (x, y)-coordinates as usual by $x = r\cos\theta$ and $y = r\sin\theta$. The condition imposed on V is then $\partial_\theta V \equiv 0$.

(f) [3 pts] Regarding r and θ as real-valued functions on (a suitable subdomain of) $\mathbb{R}^4$ that depend on the coordinates x and y but not on p_x and p_y, define two additional functions on the same domain by

$$p_r := \frac{x}{r} p_x + \frac{y}{r} p_y, \qquad p_\theta := -y p_x + x p_y.$$

Show that $(r, p_r, \theta, p_\theta)$ is then a Darboux chart with respect to the symplectic form ω_{std}. *Hint: It suffices to compute ω_{std} in the new coordinates and show that it satisfies the right formula, but this computation is a bit long. You could make your life easier by observing that $\omega_{\mathrm{std}} = d\lambda_{\mathrm{std}}$ for $\lambda_{\mathrm{std}} := p_x\, dx + p_y\, dy$, and then computing λ_{std} in the new coordinates.*

(g) [3 pts] Write down H as a function of $(r, p_r, \theta, p_\theta)$ and show that the family of diffeomorphisms defined in these coordinates by $\varphi_t(r, p_r, \theta, p_\theta) := (r, p_r, \theta + t, p_\theta)$ satisfy (3). Derive a formula for the corresponding conserved quantity F as promised by part (e). *(It is called the "angular momentum," for reasons that should now appear somewhat natural.)*

[2] If M is not closed, then the flow of X_F might not exist for all time, but let's not worry about this detail right now.

4. [25 pts total] Recall that the *special unitary group* is defined for each $n \in \mathbb{N}$ by

$$\mathrm{SU}(n) = \left\{ \mathbf{A} \in \mathrm{GL}(n, \mathbb{C}) \;\middle|\; \mathbf{A}^\dagger \mathbf{A} = \mathbb{1} \text{ and } \det \mathbf{A} = 1 \right\},$$

where $\mathbf{A}^\dagger$ denotes the complex congugate of the transpose of $\mathbf{A}$. Its Lie algebra is the vector space of traceless anti-Hermitian matrices

$$\mathfrak{su}(n) = \left\{ \mathbf{A} \in \mathbb{C}^{n \times n} \;\middle|\; \mathbf{A} + \mathbf{A}^\dagger = 0 \text{ and } \operatorname{tr} \mathbf{A} = 0 \right\}.$$

Note that $\mathfrak{su}(n)$ is only a *real* vector space, not complex, as the anti-Hermitian condition is not invariant under multiplication by i.

In this problem, we will investigate an interesting relationship between the Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$.

(a) [2 pts] Write down a diffeomorphism between $\mathrm{SU}(2)$ and S^3, and explain briefly why it is bijective. *Hint: Think of S^3 as the unit sphere in $\mathbb{R}^4$ with the latter identified with $\mathbb{C}^2$. Then each column of a matrix in $\mathrm{SU}(2)$ is also a vector in S^3.*

(b) [2 pts] Show that for every $n \in \mathbb{N}$, the pairing

$$\langle \mathbf{A}, \mathbf{B} \rangle := \operatorname{tr}(\mathbf{A}^\dagger \mathbf{B}) \tag{4}$$

defines a Hermitian inner product on $\mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}^{n \times n}$ which restricts to a Euclidean (i.e. real) inner product on the real subspace $\mathfrak{su}(n) \subset \mathfrak{gl}(n, \mathbb{C})$.

(c) [2 pts] Recall from Problem Set 8 #7 that $\mathfrak{so}(3)$ admits a Lie algebra isomorphism to $\mathbb{R}^3$, with the Lie algebra structure of the latter defined via the cross product. This means in particular that $\mathfrak{so}(3)$ admits a basis $J_1, J_2, J_3 \in \mathfrak{so}(3)$ satisfying the relations

$$[J_1, J_2] = J_3, \qquad [J_2, J_3] = J_1, \qquad [J_3, J_1] = J_2.$$

Find a basis of $\mathfrak{su}(2)$ that satisfies these same relations, and deduce that there exists a Lie algebra isomorphism

$$\Phi : \mathfrak{su}(2) \to \mathfrak{so}(3). \tag{5}$$

(d) [2 pts] Despite having isomorphic Lie algebras and both being connected, it is not true that $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ are isomorphic, nor diffeomorphic. The following observation gives a hint of this: find an element $\mathbf{A} \in \mathfrak{su}(2)$ such that $[0, 2\pi] \to \mathrm{SU}(2) : t \mapsto e^{t\mathbf{A}}$ is an embedded path from $\mathbb{1}$ to $-\mathbb{1}$, while $[0, 2\pi] \to \mathrm{SO}(3) : t \mapsto e^{t\Phi(\mathbf{A})}$ is a loop from $\mathbb{1}$ to itself.

(e) [6 pts] In order to see the global relationship between $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ more clearly, we shall use the following general construction. Given a Lie group G with Lie algebra $\mathfrak{g}$, let $\mathrm{Aut}(\mathfrak{g})$ denote the group of vector space isomorphisms $\mathfrak{g} \to \mathfrak{g}$. For each $g \in G$ and $X \in \mathfrak{g}$, define

$$\mathrm{Ad}(g)X := \frac{d}{dt} \left(g \exp(tX) g^{-1} \right) \Big|_{t=0} \in \mathfrak{g}.$$

Show that $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is linear and invertible for every $g \in G$, and that the resulting map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a group homomorphism. It is called the *adjoint representation* of G.
Hint: Relate $\mathrm{Ad}(g)$ to the smooth map $C_g : G \to G$ defined by $C_g(h) := ghg^{-1}$ for every $g \in G$.

(f) [2 pts] For matrix groups $G \subset \mathrm{GL}(n, \mathbb{F})$, the definition of the adjoint representation reduces to the straightforward formula

$$\mathrm{Ad}(\mathbf{A})\mathbf{B} = \mathbf{A}\mathbf{B}\mathbf{A}^{-1}.$$

Choosing any basis of $\mathfrak{g}$ then identifies $\mathrm{Aut}(\mathfrak{g})$ with $\mathrm{GL}(n, \mathbb{R})$ for $n := \dim G$, thus we obtain a group homomorphism $G \to \mathrm{GL}(n, \mathbb{R})$ which is also manifestly a smooth map. Applying this to $\mathrm{SU}(2)$ gives

$$\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{Aut}(\mathfrak{su}(2)) \cong \mathrm{GL}(3, \mathbb{R}).$$

Show that the image of this map is actually in $\mathrm{O}(\mathfrak{su}(2))$, the group of isomorphisms on $\mathfrak{su}(2)$ that preserve the inner product defined in part (b).

(g) [4 pts] Compute the derivative of $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{O}(\mathfrak{su}(2))$ at $\mathbb{1}$ and show that it is a Lie algebra isomorphism from $\mathfrak{su}(2)$ to the Lie algebra $\mathfrak{o}(\mathfrak{su}(2))$ of $\mathrm{O}(\mathfrak{su}(2))$.
Hint: You might find the Jacobi identity useful.

(h) [5 pts] It follows that any choice of orthogonal isomorphism between $\mathfrak{su}(2)$ and $\mathbb{R}^3$ transforms $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{Aut}(\mathfrak{su}(2))$ into a Lie group homomorphism

$$\Psi : \mathrm{SU}(2) \to \mathrm{O}(3),$$

whose image is actually in $\mathrm{SO}(3)$ since $\mathrm{SU}(2)$ is connected. The derivative of Ψ at $\mathbb{1}$ is in turn a Lie algebra isomorphism

$$\Psi_* : \mathfrak{su}(2) \to \mathfrak{so}(3).$$

Show that $\Psi(\exp(t\mathbf{A})) = \exp(t\Psi_*\mathbf{A})$ for all $\mathbf{A} \in \mathfrak{su}(2)$ and $t \in \mathbb{R}$, and deduce that Ψ is surjective onto $\mathrm{SO}(3)$. (Recall from Problem Set 8 #7(d) that $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective.)
Hint: Use the characterization of $\exp : \mathfrak{g} \to G$ *in terms of smooth group homomorphisms* $\mathbb{R} \to G$.

(i) [0 pts, but included for the sake of completeness] Observe however that $\Psi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is not injective, as $\Psi(-\mathbf{A}) = \Psi(\mathbf{A})$ for every $\mathbf{A} \in \mathrm{SU}(2)$. Show that $\ker \Psi = \{\pm\mathbb{1}\}$ and deduce that Ψ is globally two-to-one.

Combined with part (a), the last step implies that $\mathrm{SO}(3)$ *is diffeomorphic to* $\mathbb{RP}^3$. *There are also more explicitly geometric ways to see this if you think of* $\mathrm{SO}(3)$ *as the group of rotations on* $\mathbb{R}^3$, *e.g. you might find the explanation at* `https://en.wikipedia.org/wiki/Rotation_group_SO(3)#Topology` *interesting. (But I do not recommend reading the section on that page about the "Connection between* $\mathrm{SO}(3)$ *and* $\mathrm{SU}(2)$*"; it is dreadfully ugly.)*