Seminar announcement

General information

Instructor: Prof. Chris Wendl
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Office hour: Tuesdays 15:00–16:00

Seminar webpage: http://www.mathematik.hu-berlin.de/~wendl/Winter2016/hPrinzip/

Time and place: Wednesdays 13:00–15:00 in 3.007 (Rudower Chaussee 25)

Language: The seminar can be run in German or in English (or a mixture), depending on the preferences of the participants. If you would like to participate and have a strong preference about the language, feel free to discuss it with me ahead of time.

Prerequisites: Contents of the HU's courses *Differentialgeometrie I* and *Topologie I*; taking *Topologie II* concurrently is also recommended if you have not seen much algebraic topology before.
All participants will be assumed to be comfortable with smooth n-dimensional manifolds, tensors, differential forms and Riemannian metrics, as well as basic notions from topology such as homotopy equivalence and the fundamental group. Some familiarity with more advanced topics such as fiber bundles, CW-complexes, higher homotopy groups, cohomology and characteristic classes would be helpful but will not be assumed—we will spend some extra time on these topics in the first few weeks to make sure everyone's knowledge is sufficiently solid.

Overview

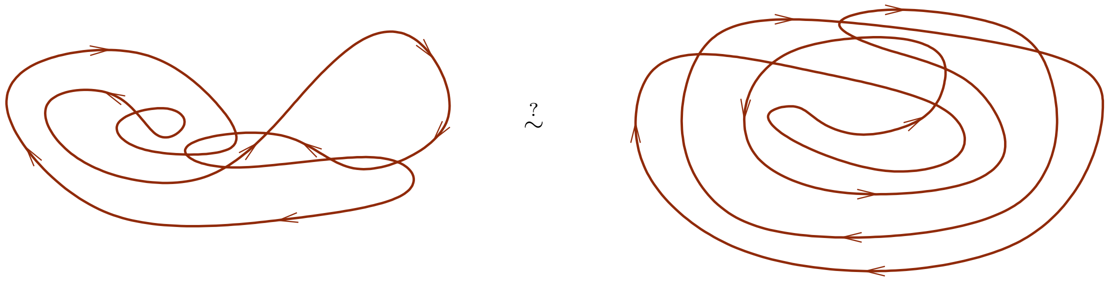
Many natural problems in differential geometry and topology—for example the existence of immersions, symplectic forms, or isometric maps—can be formulated in terms of partial differential relations (PDRs), i.e. equations or inequalities that constrain the partial derivatives of a map. In solving such problems, there is typically a much easier problem that must be solved first: the classification of *formal solutions*, maps which satisfy a corresponding algebraic relation without any constraints on derivatives. To illustrate the idea, consider the following simple problem:

Question*: What conditions are necessary and sufficient for two smooth immersions $\gamma : S^1 \looparrowright \mathbb{R}^2$ to be "regularly" homotopic, i.e. related to each other via a smooth 1-parameter family of immersions?*

The PDR in this example is the condition that all loops $\gamma : S^1 \to \mathbb{R}^2$ under consideration should be immersions, meaning $\dot{\gamma}(t) \neq 0$ for all $t \in S^1$. If two such loops $\gamma_1, \gamma_2 : S^1 \looparrowright \mathbb{R}^2$ are regularly homotopic, then the corresponding loops of tangent vectors $\dot{\gamma}_1, \dot{\gamma}_2 : S^1 \to \mathbb{R}^2 \setminus \{0\}$ must obviously be homotopic in the punctured plane, which (using the fact that $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \pi_1(S^1) = \mathbb{Z}$) is true if and only if they have the same winding number,

$$\mathrm{wind}(\dot{\gamma}_1) = \mathrm{wind}(\dot{\gamma}_2) \in \mathbb{Z}.$$

Of course, most loops in an arbitrary homotopy from $\dot{\gamma}_1$ to $\dot{\gamma}_2$ will not be actual loops of tangent vectors to immersions $S^1 \looparrowright \mathbb{R}^2$, thus a formal solution seems at first quite far from being an answer to the question that was actually asked. But it is a remarkable fact, known as the *Whitney-Graustein theorem*, that formal solutions to this problem imply genuine solutions: two immersed loops in $\mathbb{R}^2$ are regularly homotopic if and only if the winding numbers of their first derivatives are the same. The condition on winding numbers is generally quite easy to check, even though for a given pair of immersed loops, it may seem impossible in practice to find an explicit regular homotopy between them!



In general, we say that a differential-geometric problem based on a PDR **satisfies the h-principle** if it can be reduced to the problem of finding formal solutions. The latter is typically a matter of standard algebraic topology, with no need for deeper analytical or geometric techniques—as a topological problem it may be easy or hard, but on the surface it is at least much simpler than solving a PDE or PDR. The fact that some geometric problems are "flexible" in this sense was first recognized by Gromov around 1970, though various seemingly unrelated examples of it had been known since the 1950's or earlier. The h-principle comes in a variety of flavors and has applications in a wide range of subjects. Famous examples include the following:

- The Smale-Hirsch immersion theory in differential topology: this is a generalization of the Whitney-Graustein theorem and also includes the notoriously counterintuitive fact that the 2-sphere can be "turned inside out" by a smooth family of immersions $S^2 \looparrowright \mathbb{R}^3$! This is known as the *Smale eversion.* (For a very entertaining and informative video about it, see `http://www.youtube.com/watch?v=wO61D9x6lNY`.)
- The Nash-Kuiper C^1-isometric embedding theorem: among other things, this implies that any Riemannian n-manifold admits C^1-smooth isometric embeddings into arbitrarily small balls in $\mathbb{R}^{n+1}$. This cannot be done with C^2-smooth embeddings; curvature prevents it.
- Existence and uniqueness (up to homotopy) of symplectic and contact structures on open manifolds.

Some more advanced applications include results in foliation theory and the existence of metrics with negative curvature, as well as several groundbreaking advances in symplectic and contact topology involving flexibility of certain classes of contact structures, Legendrian submanifolds and Stein manifolds. A common feature of many h-principles is that they are very hard to visualize, and for this reason it tends to seem surprising whenever an h-principle holds. On the other hand, when the h-principle fails, it often indicates the existence of interesting geometric invariants that contain more than purely topological information.

The goal of this seminar will be to learn the basics of the subject, including at least the three classic applications listed above, and two powerful methods for proving fairly general h-principles: holonomic approximation and convex integration theory. We will focus somewhat (but not exclusively) on applications to symplectic geometry, where the h-principle has been especially instrumental in shaping the development of the subject.

Literature

The main portion of the seminar will follow:

- Yakov Eliashberg and Nikolay Mishachev, *Introduction to the h-principle*, AMS 2002
  This is a very good book for learning the essentials of the subject, with particularly good coverage of

the applications to symplectic geometry. In addition to a detailed discussion of convex integration, it introduces the method of holonomic approximation, which simplifies the proofs of results that previously required the covering homotopy method.
(online access to this book is available for free from within the HU network at
`http://www.ams.org/books/gsm/048/`)

Here are some additional sources that are worth looking at:

- Hansjörg Geiges, *h-principles and flexibility in geometry*, Memoirs of the AMS, 2003
  A very short book that provides an introduction to the basic ideas, focusing on the covering homotopy method and its applications, and a sketch of convex integration.
- Mikhael Gromov, *Partial differential relations*, Springer 1986
  This is the bible of the subject and includes a large amount of interesting material, but is a difficult read for beginners.
  (online access available via the HU library)
- Vincent Borelli, lecture notes on convex integration (from a 2012 workshop in Les Diablerets), available at `http://math.univ-lyon1.fr/~borrelli/Diablerets/`

For topics in algebraic topology that may be unfamiliar to some participants, I suggest either of the following:

- Glen E. Bredon, *Topology and Geometry*, Springer GTM 1993
  (online access available via the HU library)
- Allen Hatcher, *Algebraic Topology*, Cambridge University Press 2002
  (electronic version freely downloadable from the author's website at
  `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`)

We do not plan to go very deeply into *obstruction theory* in this seminar, but you may often notice it lurking in the background and thus feel a desire to read more about it. The classic book on this topic is:

- Norman Steenrod, *The topology of fibre bundles*, Princeton University Press 1957

As for differential geometry, if you have not seen smooth fiber bundles and structure groups before, then one possible place to learn about them is in Chapter 2 of:

- Chris Wendl, *Lecture Notes on Bundles and Connections*
  (available at `http://www.mathematik.hu-berlin.de/~wendl/connections.html`)

Tentative plan of topics

The schedule of talks and assignment of topics will be decided at the end of the introductory meeting on October 19. The plan for the first few weeks will be to cover some standard topics from algebraic topology and differential geometry which some participants may not have seen before. After that we will dive into the book by Eliashberg and Mishachev.

1. General introduction
2. Higher homotopy groups, weak homotopy equivalences, the long exact sequence for fibrations (Hatcher, portions of chapter 4)
3. CW complexes and basic examples of obstruction theory (excerpts from Steenrod)

4. Jet bundles and PDRs, general formulations of the h-principle (Eliashberg-Mishachev, portions of chapters 1, 5 and 6)
5. Holonomic approximation theorem, part 1 (Eliashberg-Mishachev, chapter 3)
6. Holonomic approximation theorem, part 2 (Eliashberg-Mishachev, chapter 3)
7. The h-principle for open Diff-invariant relations (Eliashberg-Mishachev, chapter 7)
8. The Smale-Hirsch immersion theorem and other applications to closed manifolds (Eliashberg-Mishachev, chapter 8)
9. Symplectic and contact structures on open manifolds (Eliashberg-Mishachev, portions of chapters 9 and 10)
10. Microflexibility and isotropic immersions (Eliashberg-Mishachev, portions of chapters 13 and 14)
11. Lagrangian and Legendrian immersions (Eliashberg-Mishachev, portions chapters 15 and 16)
12. One-dimensional convex integration (Eliashberg-Mishachev, chapter 17)
13. The h-principle for ample differential relations (Eliashberg-Mishachev, chapter 18)
14. Directed immersions and embeddings (Eliashberg-Mishachev, chapter 19)
15. The Nash-Kuiper theorem on isometric immersions (Eliashberg-Mishachev, chapter 21)
16. (to be decided)