The Index Problem

Seminar "Atiyah-Singer Index Theorem"

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1 Gradings and Clifford Modules

Definition 1.1. Recall that a module W over a Clifford algebra $\mathrm{Cl}(V)$ is said to be graded if it is provided with a decomposition $W = W_+ \oplus W_-$ such that multiplication with any $v \in V$ exchanges W_+ and W_-. Let S be a Clifford bundle on a Riemannian manifold. Then S is *graded* if it has a decomposition $S = S_+ \oplus S_-$, such that

- the metric and connection on S respect this decomposition,
- and each fiber S_x is a graded Clifford module over $\mathrm{Cl}(T_xM)$.

We define the *grading operator* $\varepsilon \in \mathrm{End}(S)$ associated to the grading by requiring that the $S_\pm$ are the ± 1 eigenspaces of ε. Conversely, any self-adjoint involution ε of S which

- commutes with covariant derivation and
- with $\varepsilon c(v) + c(v)\varepsilon = 0$ for all $v \in TM$,

gives rise to a grading on S.

For even-dimensional oriented manifolds with $\dim M = 2m$, there is a canonical grading on any clifford bundle, which is obtained as follows: The volume element $\omega \in \mathrm{Cl}(T_xM)$ is given as $\omega = e_1 \dots e_{2m} \in \mathrm{Cl}(T_xM)$ for a positively oriented orthonormal basis $e_1, \dots, e_{2m}$. It satisfies $\omega^2 = (-1)^m$ and $v\omega = -\omega v$ for $v \in T_xM$. Thus multiplication with $i^m\omega$ defines a grading operator ε_0 on S. Other gradings exist however. Given a second grading ε, then $\varepsilon\varepsilon_0$ is has almost all the properties of a grading operator, except it commutes with the Clifford algebra i.e. $c(v)\varepsilon\varepsilon_0 = \varepsilon\varepsilon_0 c(v)$. Considering its eigenvalues yields the following:

Lemma 1.2. *Any graded Clifford bundle S is split in the sum of two graded Clifford sub-bundles, on one of which $\varepsilon = \varepsilon_0$ (canonically graded part of S) and on the other $\varepsilon = -\varepsilon_0$ (anticanonically graded).*

Thus it often suffices to consider only canonically graded Clifford bundles.

2 The Supertrace

Definition 2.1. Let A be a trace-class operator on $L^2(S)$, where S is a graded Clifford bundle with grading operator ε, then the *supertrace* of A is defined by

$$\mathrm{Tr}_s(A) = \mathrm{Tr}(\varepsilon A)$$

Recall that the normal trace vanishes on commutators, it is easy to check that the supertrace vanishes on supercommutators $[A, B]_s$ if one of A or B is trace-class.

Theorem 2.2. *Let A be a smoothing operator on $L^2(S)$ with kernel $k \in C^\infty(S \boxtimes S^*)$, i.e.*

$$Au(m_1) = \int_M k(m_1, m_2) u(m_2) \,\mathrm{vol}(m_2)$$

then its supertrace is given by

$$\mathrm{Tr}_s(A) = \int_M \mathrm{tr}_s(k(x, x)) \,\mathrm{vol}(x)$$

where the 'local supertrace' $\mathrm{tr}_s(a), a \in \mathrm{End}(S_x)$ is defined to be $\mathrm{tr}(\varepsilon a)$.

Recall from the representation theory of the Clifford Algebra (chapter 4) that we can write $S_x = \Delta \otimes V$ where Δ is the spin representation and V some auxiliary vectory space. We also know

$$\mathrm{End}_{\mathbb{C}}(S_x) = \mathrm{Cl}(T_x M) \otimes \mathrm{End}_{\mathbb{C}}(V), \qquad \mathrm{End}_{\mathbb{C}}(V) = \mathrm{End}_{Cl}(S_x)$$

If S_x is canonically graded, then the eigenspaces $\Delta_\pm \subseteq \Delta$ of multiplication with $i^m \omega$ (wrt ± 1) give the grading of $S_x = (\Delta_+ \otimes V) \oplus (\Delta_- \otimes V)$. Since the trace is multiplicative on tensor products, we obtain the following proposition:

Proposition 2.3. *If $a = c \otimes F \in \mathrm{End}_{\mathbb{C}}(S_x)$ with $c \in \mathrm{Cl}(T_x M)$ and $F \in \mathrm{End}_{\mathrm{Cl}}(S_x)$ and S canonically graded, we have*

$$\mathrm{tr}_s(a) = \tau_s(c) \,\mathrm{tr}^{S/\Delta}(F)$$

where $\tau_s(c)$ is the supertrace of the action of $\mathrm{Cl}(T_x M)$ on Δ and $\mathrm{tr}^{S/\Delta}$ is the relative trace, i.e. the trace of the endomorphism of V corresponding to F via $\mathrm{End}_{\mathrm{Cl}}(S_x) = \mathrm{End}_{\mathbb{C}}(V)$.

We show how the super trace τ_s is computed. Let $\{e_1, \ldots, e_2 m\}$ be an orthonormal basis of $\mathbb{R}^{2m}$. If $E \subseteq \{1, \ldots, 2m\}$ then let $\tilde{E} = \prod_{i \in E} e_i \in \mathrm{Cl}(\mathbb{R}^{2m})$. The $\tilde{E}$ generate the clifford algebra as a $\mathbb{C}$-vector space.

Lemma 2.4. *Let $c = \sum_E c_E \tilde{E} \in \mathrm{Cl}(\mathbb{R}^{2m})$ with $c_E \in \mathbb{C}$, then $\tau_s(c) = (-2i)^m c_{12\ldots 2m}$ i.e. the supertrace is (up to scalar) the projection to the 'top degree part' of c.*

Proof. Since the grading operator is given by multiplication with $i^m\omega$, we have $\tau_s(c) = \tau(i^m\omega c)$, where τ is the ordinary trace. Thus it is equivalent to show $\tau(c) = 2^m c_\varnothing$, i.e.

$$\tau(\tilde{E}) = \begin{cases} 2^m & \text{if } E = \varnothing \\ \tau(\tilde{E}) = 0 & \text{otherwise} \end{cases}$$

Note $\tilde{\varnothing} = 1$ acts trivially on Δ and hence its trace is 2^m. If $E \neq \varnothing$, consider $\tilde{E}$ acting on Δ. Then $\tilde{E}$ permutes the basis elements without a fixed point, so $\operatorname{tr}(\tilde{E}) = 0$. Recall the Clifford algebra as a representation is equal to $\Delta \otimes \Delta^*$ with left action on the first factor Δ. So $\tau(\tilde{E}) = 2^m \operatorname{tr}(E) = 0$. □

3 Graded Dirac Operators

The Dirac operator of a graded Clifford bundle anticommutes with the grading operator and hence exchanges S_+ and S_-. We thus have maps

$$C^\infty(S_+) \xrightarrow{D_+} C^\infty(S_-) \xrightarrow{D_-} C^\infty(S_+)$$

where D_+ and D_- are restrictions of D, and D_- is the adjoint of D_+.

Definition 3.1. The *index* of a graded Dirac operator D is defined as

$$\operatorname{Ind}(D) = \dim\ker D_+ - \dim\ker D_-$$

Example 3.2. Consider $D = d + d^*$ with the grading operator $\varepsilon = (-1)^q$ on $\Omega^q(M)$. We know by Hodge theory that the index of D is just the Euler characteristic of M. Notice that this grading, which is called the *Euler grading* of the de Rham operator, is not the canonical grading. For example, the element $1 \in \Omega^0(M)$ is positively graded with respect to ϵ, but it is not fixed under multiplication with $i^m\omega$.

Let P be the orthogonal projection of $L^2(S)$ to $\ker(D)$. Then $\operatorname{Ind}(D) = \operatorname{Tr}_s(P)$. More generally,

Proposition 3.3. *Let F be a rapidly decreasing smooth function on $\mathbb{R}^+$ with $f(0) = 1$. Then* $\operatorname{Ind}(D) = \operatorname{Tr}_s(f(D^2))$.

Proof. For an eigenvalue λ of D^2 let $n_+(\lambda)$ denote the dimension of the λ-eigenspace of D^2 restricted to S_+ and similarly for $n_-(\lambda)$. Then

$$\operatorname{Tr}_s f(D^2) = \sum_\lambda f(\lambda)(n_+(\lambda) - n_-(\lambda)) = \operatorname{Ind}(D) + \sum_{\lambda>0} f(\lambda)(n_+(\lambda) - n_-(\lambda))$$

But for $\lambda \neq 0$, the operator D is an isomorphism between λ-eigenspaces of D^2 on S_+ and S_-, so $n_+(\lambda) = n_-(\lambda)$.

□

In particular, this shows the *McKean-Singer* formula, for any $t > 0$:

$$\mathrm{Ind}(D) = \mathrm{Tr}_s\, e^{-tD^2}$$

We now consider the variation of the index as the operator D varies. Let $D_t, t \in [0,1]$ be a continuous family of graded Dirac operators on (M, S). This means that the Riemannian metric, the Clifford action and the metric and connection on S all vary continuously with t. Then $t \mapsto D_t$ is a continuous map $[0,1] \to B(W^{k+1}, W^k)$ for any k.

Proposition 3.4. *Let D_t be a continuous family of graded Dirac operators, as above. Then* $\mathrm{Ind}(D_0) = \mathrm{Ind}(D_1)$.

Proof (Sketch). Since the assignment $t \mapsto D_t$ is continuous, it can be shown that the index $\mathrm{Ind}(D_t) = \mathrm{Tr}_s\, e^{-t' D_t^2}$ depends continuously on t. But the index is an integer, so it is constant. □

This shows that the index of D is a topological invariant, that is it depends only upon homotopy-theoretic data of the manifold M and the bundle S.

At this point, Roe makes a number of remarks on the relevance and the proof of the index theorem. The index theorem allowed a vigorous exchange between analysis and topology. On the one hand, information about the index derived from PDE theory (even the information that $\mathrm{Ind}(D)$ is an integer) could be used to constrain the characteristic classes and hence the topology of M. On the other hand topological conditions could force the existence of solutions to differential equations. Milnor's construction of the exotic sqheres and the Kodaira embedding theorem are examples of these two phenomena which predate the general form of the index theorem itself.

The original proof of the index theorem relied on algebraic topology (either cobordism theory or K-theory) to organize the possible pairs (M, S) into some kind of group and then check the index theorem only on specific generators. Thus the proofs were essentially global and topological in nature. This book proves the index theorem by an alternative method: As McKean and Singer pointed out, the asymptotical expression of the heat kernel is in principle locally computable and the index can be computed with the local super-trace of certain coefficients in that expansion. The computation of the coefficients is almost impossible by brute force, but Getzler showed that it can be made more tractable by paying careful attention to the role of the Clifford algebra. Using the fact that the local supertrace corresponds to the top degree part of the Clifford algebra reduces the computation to a model that is essentially the harmonic oscillator.

4 The heat equation and the index theorem

Recall from 7.15 the asymptotic expansion near 0 for the heat kernel k_t associated to the smoothing operator e^{-tD^2},

$$k_t(p,q) \sim h_t(p,q) \left(\Theta_0(p,q) + t\Theta_1(p,q) + t^2\Theta(p,q) + \dots \right)$$

for certain smooth sections Θ_i of $S \boxtimes S^*$. Thus we obtain:

$$\begin{aligned}\operatorname{Ind}(D) &= \operatorname{Tr}_s(e^{-tD^2}) = \int_M \operatorname{tr}_s k_t(p,p) \operatorname{vol}(p) \\ &\sim \frac{1}{(4\pi t)^{n/2}} \Big(\int \operatorname{tr}_s \Theta_0 \operatorname{vol} + t \int \operatorname{tr}_s \Theta_1 \operatorname{vol} + \dots \Big)\end{aligned}$$

(recall $h_t(p,p) = 1/(4\pi t)^{n/2}$). But $\operatorname{Tr}_s(e^{-tD^2})$ is constant, so letting t go to 0 we obtain the following theorem, which is the main result of this chapter:

Proposition 4.1. *The index of the graded Dirac operator D is zero if $n = \dim M$ is odd, and equal to*

$$\operatorname{Ind}(D) = \frac{1}{(4\pi)^{n/2}} \int \operatorname{tr}_s \Theta_{n/2} \operatorname{vol}$$

if n is even, where $\Theta_{n/2}$ is a certain algebraic expression in the metrics and connection coefficients and their derivatives.

Corollary 4.2. *The index is multiplicative under coverings, i.e. if $\tilde{M}$ is a k-fold covering of M and $\tilde{S}, \tilde{D}$ are the natural lifts of S and D to $\tilde{M}$, then $\operatorname{Ind}(\tilde{D}) = k \operatorname{Ind}(D)$.*

This is obvious from the preceding theorem, since $\Theta_{n/2}$ is a local expression involving the metric and connection, hence the same on M and $\tilde{M}$. But it is not at all obvious from the definition of the index.