Topics covered in the lectures

1. We. 18.10.2017: definitions and examples of categories, functors and natural transformations, definition of singular homology $H_*(X, A; G)$ for a pair of spaces (X, A) with coefficients in an abelian group G
2. Fr. 20.10.2017: LECTURE CANCELED
3. We. 25.10.2017: properties of (relative) singular homology: computation of $H_0(X; G)$ and $H_*(\{\mathrm{pt}\}; G)$, Hurewicz homomorphism $\pi_1(X, p) \to H_1(X)$ and its kernel, functoriality and homotopy invariance via subdivision, excision and barycentric subdivision
4. Fr. 27.10.2017: exact sequences, long exact sequence of the pair (X, A), from short exact sequences of complexes to long exact sequences of homology, application to suspension: isomorphism $H_{k+1}(\Sigma X) \to H_k(X)$ for every $k \geq 1$, computation of $H_*(S^n; \mathbb{Z})$, definition of the fundamental class $[S^n] \in H_n(S^n; \mathbb{Z})$
5. We. 1.11.2017: the Mayer-Vietoris sequence in singular homology, new proof that $H_{k+1}(\Sigma X) \cong H_k(X)$, split exact sequences, reduced homology $\widetilde{H}_*(X)$ and its basic properties (functoriality, $H_0(X; G) \cong \widetilde{H}_0(X; G) \oplus G$, Mayer-Vietoris sequence), proof that $\widetilde{H}_n(S^n; G) = G$ and $\widetilde{H}_k(S^n; G) = 0$ for $k \neq n$
6. Fr. 3.11.2017: LECTURE CANCELED
7. We. 8.11.2017: review of properties of reduced homology, long exact sequence of a triple (X, A, B), $H_*(X, A) \cong \widetilde{H}_*(X/A)$ for good pairs, the long exact sequence of a mapping torus, application to homology of the Klein bottle
8. Fr. 10.11.2017: degree of a map $S^n \to S^n$ and its properties (homotopy invariance, $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, computation for constant maps and orthogonal transformations, suspensions, winding numbers for $S^1 \to S^1$), if $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$, the "hairy sphere" theorem, topological n-manifolds, local homology $H_n(M, M \setminus \{x\}) \cong \mathbb{Z}$ and local orientations, local degree of maps, proof that $\deg(f) = \sum_{x \in f^{-1}(y)} \deg(f; x)$
9. We. 15.11.2017: dualization and cohomology of chain complexes, definition of singular cohomology $H^*(X, A; G)$, $H^0(X; G)$ as a product over path-components, computation of $H^*(\{\mathrm{pt}\}; G)$, functoriality, reduced cohomology $\widetilde{H}^*(X; G)$, statement of the excision theorem, long exact sequences, computation of $H^*(S^n; G)$, the natural homomorphism $h : H^k(X; G) \to \mathrm{Hom}(H_k(X), G)$
10. Fr. 17.11.2017: $H^1(X; G) \cong \mathrm{Hom}(\pi_1(X), G) = \mathrm{Hom}(H_1(X), G)$ for path-connected spaces X, proof of exactness for long exact sequences of pairs in cohomology, relation between connecting homomorphisms $\delta^* : H^{k-1}(A; G) \to H^k(X, A; G)$ and $\partial_* : H_k(X, A) \to H_{k-1}(A)$
11. We. 22.11.2017: constructing chain homotopy equivalences via subdivision, proof of Mayer-Vietoris and excision for cohomology, the Eilenberg-Steenrod axioms for (co)homology theories on admissible categories
12. Fr. 24.11.2017: sketch of Čech homology and cohomology, computation of $\check{H}_1(S^1; \mathbb{Z})$ and $\check{H}^1(S^1; G)$, $\check{H}^0(X; G)$ as a direct product over connected (not path-connected!) components of X, direct and inverse limits of systems of abelian groups
13. We. 29.11.2017: relative Čech (co)homology and its functoriality, the direct limit functor commutes with the homology functor and why this implies $\check{H}^*(\,\cdot\,; G)$ satisfies the exactness axiom, inverse limits of exact sequences need not be exact unless all groups are finite, why this implies $\check{H}_*(\,\cdot\,; G)$ only sometimes satisfies the exactness axiom

14. Fr. 1.12.2017: CW-complexes and cell decompositions, examples (two cell decompositions of S^n, the infinite-dimensional complex S^∞, the surface of genus g, any simplicial complex), characterizing the topology of a CW-complex via characteristic maps, any compact subset is contained in a finite subcomplex, the cellular chain complex and cellular (co)homology, statement of the natural isomorphism between $H_*^{\mathrm{CW}}(X, A; G)$ and $h_*(X, A)$ or $H^*_{\mathrm{CW}}(X, A; G)$ and $h^*(X, A)$ for any axiomatic (co)homology theory h with coefficient group G

15. We. 6.12.2017: proof that $H_*^{\mathrm{CW}}(X; G) \cong h_*(X)$ when either X is finite dimensional or h_* is singular homology $H_*(\,\cdot\,; G)$, maps $f : S^n \to S^n$ of degree $k \in \mathbb{Z}$ act on $\tilde{h}_n(S^n) = G$ as multiplication by k

16. Fr. 8.12.2017: clarification on the definition of $\partial : C_1^{\mathrm{CW}}(X; G) \to C_0^{\mathrm{CW}}(X; G)$, cellular chain maps, consequences of cellular homology: $H_*(X, A)$ is finitely generated for any compact CW-pair (X, A), $H_k(X, A; G) = 0$ for all $k > n$ if (X, A) is an n-dimensional CW-pair, definition of Betti numbers $b_k(X) \ge 0$ and Euler characteristic $\chi(X) \in \mathbb{Z}$, $\chi(X) = \sum_k (-1)^k$(number of k-cells), three computations of $\chi(S^n)$, #vertices − #edges = 1 for any graph that is a tree, $\chi(\Sigma_g) = 2 - 2g$, $\chi(Y) = d\chi(X)$ for a d-fold covering map of a compact CW-complex (with applications: d-fold covers of Σ_g are $\Sigma_{d(g-1)+1}$, and covers $\pi : S^n \to X$ have degree 1 or 2 if n is even)

17. We. 13.12.2017: tensor products of R-modules, submodules of free modules over a PID are free (statement without proof), short exact sequences need not remain exact under $\otimes G$ or $\mathrm{Hom}(\,\cdot\,, G)$ unless they split, right- and left-exactness of $\otimes G$ and $\mathrm{Hom}(\,\cdot\,, G)$, statement of the universal coefficient theorems for homology and cohomology of a chain complex of free modules over a PID, $H_*(X, A; \mathbb{Z})$ determines $H_*(X, A; G)$ and $H^*(X, A; G)$, $H^n(X, A; \mathbb{K})$ is the vector space dual of $H_n(X, A; \mathbb{K})$ for $\mathbb{K}$ a field, proof of the UCT modulo definitions of Tor_R and Ext_R

18. Fr. 15.12.2017: projective resolutions of R-modules (existence and uniqueness up to chain homotopy equivalence), the functors Tor_R and Ext_R, proof of the universal coefficient theorem (both homology and cohomology versions), A a free R-module $\Rightarrow \mathrm{Tor}_R(A, G) = \mathrm{Ext}_R(A, G) = 0$ for all G, application to the equivalence of singular and cellular cohomology for all (not just finite-dimensional) CW-complexes

19. We. 10.1.2018: computing Tor and Ext, review of $H^n(X; \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(H_n(X; \mathbb{K}), \mathbb{K})$ for fields $\mathbb{K}$, $H_n(X; \mathbb{K}) \cong H_n(X) \otimes \mathbb{K}$ for fields $\mathbb{K}$ of characteristic zero, shift of torsion from $H_{k-1}(X)$ to $H^k(X)$, fixed point problems, the Hopf trace formula, definition of Lefschetz numbers $L_{\mathbb{K}}(f) \in \mathbb{K}$ and statement of the Lefschetz fixed point theorem

20. Fr. 12.1.2018: LECTURE CANCELLED DUE TO ILLNESS

21. We. 17.1.2018: Lefschetz fixed point theorem and proof, why $L(f) := L_{\mathbb{Q}}(f)$ is an integer, extension to Euclidean neighborhood retracts (statement without proof), counterexamples for X not compact or not a simplicial complex, simplicial approximation theorem and sketch of the proof, applications of the fixed point theorem: no nowhere zero vector field on a closed smooth manifold with $\chi(M) \neq 0$, $f : S^n \to S^n$ has a fixed point whenever $\deg(f) \neq (-1)^{n+1}$, acyclic spaces and the fixed point property, $\mathbb{RP}^n$ and $\mathbb{CP}^n$ have the fixed point property for n even (in the latter case using the cup product without proof)

22. Fr. 19.1.2018: big theorems on products and duality (stated so far without proof): (1) existence of the cup product on $H^*(X; R)$ for R a commutative ring with unit, (2) Poincaré duality $H^k(M; R) \xrightarrow{\cong} H_{n-k}(M; R)$ for M a closed connected topological n-manifold and either $R = \mathbb{Z}_2$ or $R = \mathbb{Z}$ with M oriented, homology classes represented by submanifolds, transversality of smooth submanifolds and the implicit function theorem, definition of the homological intersection product in terms of the cup product and Poincaré duality, (3) meaning of the intersection product for transversely intersecting submanifolds. Application to computing the cohomology rings of $\mathbb{RP}^n$ and $\mathbb{CP}^n$. Product CW-complexes and the cellular cross product on homology and cohomology, cellular definition of the cup product

23. We. 24.1.2018: graded commutativity, tensor product of chain complexes, statement and proof of the algebraic Künneth formula, computation of $H_*(\Sigma_g \times S^1)$ and representing homology classes via submanifolds

24. Fr. 26.1.2018: computing $H_*(\mathbb{RP}^2 \times \mathbb{RP}^2)$ via Künneth, the cross product on relative cellular homology and cohomology, acyclic models and the unique (up to chain homotopy) chain map $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ for singular chains

25. We. 31.1.2018: definition of cross product $H_k(X,A) \otimes H_\ell(Y,B) \xrightarrow{\times} H_{k+\ell}((X,A) \times (Y,B))$ for singular homology, acyclic models for chain maps $C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$, the Eilenberg-Zilber theorem, topological Künneth formula, associativity and graded commutativity of the cross product, definition of $H^k(X,A;R) \otimes H^\ell(Y,B;R) \xrightarrow{\times} H^{k+\ell}((X,A) \times (Y,B);R)$, the formula $\langle \alpha \times \beta, A \times B \rangle = (-1)^{|\beta||A|}\langle \alpha, A \rangle \langle \beta, B \rangle$, definition of $H^k(X,A;R) \otimes H^\ell(X,B;R) \xrightarrow{\cup} H^{k+\ell}(X, A \cup B; R)$ in terms of the cross product and its main properties: naturality, graded commutativity, associativity, relation to the connecting homomorphism for a pair, the formula $\pi_X^* \alpha \cup \pi_Y^* \beta = \alpha \times \beta$

26. Fr. 2.2.2018: topological n-manifolds have finitely generated homology, local orientations, the orientation bundle $p : \Theta^G \to M$ and its sections, the orientation double cover $\widetilde{M} \to M$, R-orientations along subsets $A \subset M$, $H_k(M, M \setminus A; G) = 0$ for $k > n$ and the isomorphism $H_n(M, M \setminus A; G) \to \Gamma_c(\Theta^G|_A)$ for closed sets $A \subset M$, corollaries: $H_k(M;G) = 0$ for $k > n$ and also for $k = n$ if M noncompact, computation of $H_n(M;G)$ when M is compact, existence of fundamental classes $[M] \in H_n(M;R)$

27. We. 7.2.2018: intuition on Poincaré duality via triangulations, definition and properties of the cap product $\cap : H^k(X,A;R) \otimes H_\ell(X, A \cup B; R) \to H_{\ell-k}(X,B;R)$, the duality map $\mathrm{PD} : H^k(M;R) \to H_{n-k}(M;R) : \alpha \mapsto \alpha \cap [M]$ for a closed manifold M, simple corollaries of the isomorphism ($b_k(M) = b_{n-k}(M)$ and $\chi(M) = 0$ when $\dim M$ is odd), singular cohomology with compact support (defined as a direct limit), the noncompact duality map $\mathrm{PD}_M : H_c^k(M;R) \to H_{n-k}(M;R)$, Mayer-Vietoris sequences for $H_c^*(A \cup B)$ and the inductive lemma for PD, proof that $\mathrm{PD}_{\mathbb{R}^n}$ is an isomorphism

28. Fr. 9.2.2018: LECTURE CANCELED

29. We. 14.2.2018: the intersection form on $H^*(M;R)$ and intersection product on $H_*(M;R)$, nonsingularity, Poincaré duality for compact manifolds with boundary, normal bundles of smooth submanifolds and the Thom class, expressing $\mathrm{PD}^{-1}[A] \in H^*(M)$ for a submanifold $A \subset M$ in terms of the Thom class, sketch of the proof that $[A] \cdot [B] = [A \cap B]$ for transverse smooth submanifolds $A, B \subset M$, applications: closed hypersurfaces are nullhomologous if and only if they separate, all closed hypersurfaces in spheres are orientable, intersections of the graph of a map $f : M \to M$ with the diagonal and the geometric meaning of the Lefschetz number

30. Fr. 16.2.2018: the homotopy groups $\pi_n(X)$ and $\pi_n(X,A)$, $\pi_n(X)$ is abelian for $n \geq 2$, independence of the base point, examples of calculations ($\pi_k(\mathbb{T}^n) = 0$ for $k \geq 2$, $\pi_k(S^n) = 0$ for $k < n$, $\pi_n(S^n) \cong \mathbb{Z}$ generated by the identity, $\pi_3(S^2) \cong \mathbb{Z}$ generated by the Hopf fibration, $\pi_{n+k}(S^n)$ unknown for $k > 64$), n-connectedness, all maps $X \to Y$ are homotopic if X is an n-dimensional CW-complex and Y is n-connected (induction on skeleta), statement of Whitehead's theorem on weak homotopy equivalences, the Hurewicz map and statement of the Hurewicz theorem, applications: $\pi_2(X) \cong H_2(\widetilde{X})$ for the universal cover $\widetilde{X} \to X$, simply connected CW-complexes with vanishing (reduced) homology are contractible, a map between simply connected CW-complexes is a homotopy equivalence if it induces the identity on homology, all simply connected closed 3-manifolds are homotopy equivalent to S^3, the Poincaré conjecture, Freedman's and Donaldson's results on (topological or smooth) 4-manifolds and their intersection forms