Topology II Humboldt-Universität zu Berlin
C. Wendl Winter Semester 2017–18

TAKE-HOME MIDTERM

Instructions

To receive credit for this assignment, you must hand it in by **Wednesday, January 24** before the Übung. The solutions will be discussed in the Übung on that day.

You are free to use any resources at your disposal and to discuss the problems with your comrades, but **you must write up your solutions alone**. Solutions may be written up in German or English, this is up to you.

There are 100 points in total; a score of 75 points or better will boost your final exam grade according to the formula that was indicated in the course syllabus. Note that the number of points assigned to each part of each problem is usually proportional to its conceptual importance/difficulty.

If a problem asks you to prove something, then unless it says otherwise, a **complete argument** is typically expected, not just a sketch of the idea. Partial credit may sometimes be given for incomplete arguments if you can demonstrate that you have the right idea, but for this it is important to write as clearly as possible. Less complete arguments can sometimes be sufficient, e.g. in cases where you want to show that two spaces are homotopy equivalent and can justify it with a very convincing picture (use your own judgement). You are free to make use of all results we've proved in lectures or problem sets, without reproving them. (When using a result from a problem set, say explicitly which one.)

One more piece of general advice: if you get stuck on one part of a problem, it may often still be possible to move on and do the next part.

You are free to ask for clarification or hints via e-mail or in office hours; of course I reserve the right not to answer such questions.

Problems

1. [70 pts total] Consider a knot $K \subset \mathbb{R}^3$, i.e. the image of a topological embedding[1] $S^1 \hookrightarrow \mathbb{R}^3$. For technical reasons, it is conventional in knot theory to assume that K is not too "wild," for instance it is good enough to assume that the embedding $S^1 \hookrightarrow \mathbb{R}^3$ is smooth (meaning C^∞).

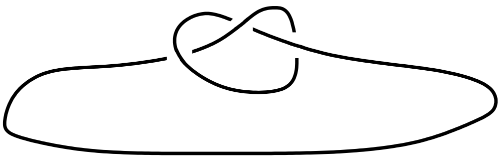

Figure 1: A smooth knot.

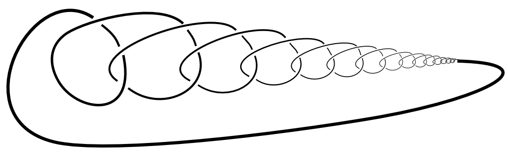

Figure 2: A "wild" knot, which is continuous, but not smooth. We will not consider these.

The smoothness condition has the following advantage: if K is the image of $f : S^1 \hookrightarrow \mathbb{R}^3$, we can always assume there exists an extension of f to a topological embedding $S^1 \times \mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ that matches f along $S^1 \times \{0\}$. (Take a moment to convince yourself that no such extension exists for the knot in Figure 2.) We shall denote the image of this extension by $N \subset \mathbb{R}^3$, so

$$K \subset \mathring{N} \subset N \subset \mathbb{R}^3 \quad \text{where} \quad N \cong S^1 \times \mathbb{D}^2.$$

[1]Recall that a map $f : X \to Y$ between two topological spaces is called a **topological embedding** if it is continuous and injective and the inverse $f^{-1} : f(X) \to X$ is also continuous with respect to the subspace topology on $f(X) \subset Y$.

One way to distinguish topologically between two knots is via their **knot groups**, meaning the group $\pi_1(\mathbb{R}^3 \setminus K)$. As you might recall from a homework problem in *Topology I*, one can equivalently extend $\mathbb{R}^3$ to its one-point compactification S^3 and replace $\pi_1(\mathbb{R}^3 \setminus K)$ with $\pi_1(S^3 \setminus K)$, as it is easy to show via the Seifert-van Kampen theorem that these two groups are isomorphic. With this in mind, we shall regard all knots as subsets of S^3.

(a) [20 pts] Show that $H_1(S^3 \setminus K) \cong \mathbb{Z}$.
Hint: Consider the Mayer-Vietoris sequence for $S^3 = N \cup (S^3 \setminus K)$.[2]

A union of multiple disjoint knots $K_1 \cup \ldots \cup K_n$ in S^3 is called a **link**, and we say it is **isotopic** to another link $K_1' \cup \ldots \cup K_n' \subset S^3$ if there exists a continuous family of homeomorphisms $\varphi_t : S^3 \to S^3$ for $t \in [0, 1]$ such that $\varphi_0 = \mathrm{Id}$ and $\varphi_1(K_i) = K_i'$ for $i = 1, \ldots, n$. We say moreover that $K_1 \cup \ldots \cup K_n$ is **unlinked** if it is isotopic to a link whose connected components are each contained in disjoint balls.

It will be convenient in the following to assume that our knots $K \subset S^3$ are endowed with *orientations*, meaning we have a distinguished class of embeddings $f : S^1 \hookrightarrow S^3$ for $K = f(S^1)$ that are all related to each other by orientation-preserving reparametrizations $S^1 \to S^1$. It follows that all the distinguished parametrizations of K are homotopic, hence for any subset $U \subset S^3$ containing K, there is a uniquely determined homology class $[K] := f_*[S^1] \in H_1(U)$, where $[S^1]$ denotes a fixed generator of $H_1(S^1) \cong \mathbb{Z}$. Changing the orientation of K changes $[K] \in H_1(U)$ by a sign.

(b) [15 pts] For a given oriented knot $K_0 \subset S^3$, fix[3] a generator $[S^3 \setminus K_0] \in H_1(S^3 \setminus K_0) \cong \mathbb{Z}$. Then if $K_1 \subset S^3$ is another oriented knot which is disjoint from K_0, it represents a homology class $[K_1] \in H_1(S^3 \setminus K_0)$. We define the **linking number** of the link $K_0 \cup K_1$ by

$$\mathrm{link}(K_0, K_1) := m \in \mathbb{Z} \quad \text{where} \quad [K_1] = m[S^3 \setminus K_0] \in H_1(S^3 \setminus K_0).$$

Show that if $K_0 \cup K_1$ is unlinked, then $\mathrm{link}(K_0, K_1) = 0$.

For any oriented knot $K \subset S^3$ with its neighborhood $S^1 \times \mathbb{D}^2 \cong N \subset S^3$, an oriented knot $\mu \subset S^3$ contained in $\partial N \cong \mathbb{T}^2$ is called a **meridian** for K if it is nullhomotopic in N and satisfies $\mathrm{link}(K, \mu) = 1$. For example, the circles $\{\text{const}\} \times \partial \mathbb{D}^2 \subset N$ with a suitable choice of orientation are meridians. In contrast, an oriented knot $\lambda \subset S^3$ contained in ∂N is called a **longitude** if it is of the form $S^1 \times \{\text{const}\}$ for some identification of $S^1 \times \mathbb{D}^2$ with N that maps $S^1 \times \{0\}$ to K and preserves the orientation.

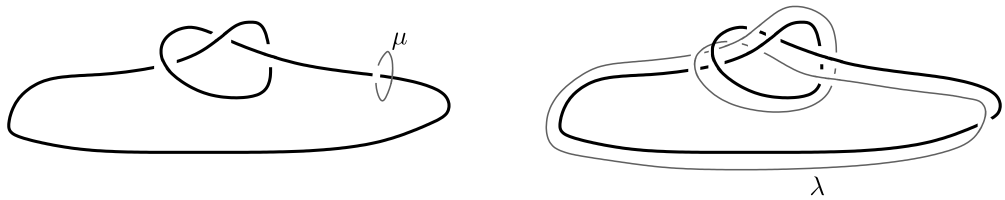

Figure 3: The knot from Figure 1 with a meridian μ and a longitude λ.

(c) [20 pts] Show that meridians for $K \subset S^3$ are unique up to homotopy of loops $S^1 \to \partial N \cong \mathbb{T}^2$, but there are infinitely many homotopy classes of longitudes; in fact, for every $m \in \mathbb{Z}$, there exists a longitude λ with $\mathrm{link}(K, \lambda) = m$ and it is unique up to homotopy through loops $S^1 \to \partial N$.
Hint 1: The homotopy classes of maps $S^1 \to \mathbb{T}^2$ *are easy to classify by considering lifts of loops in* $\mathbb{T}^2$ *to paths in its universal cover* $\mathbb{R}^2$. *They are in bijective correspondence with* $\mathbb{Z}^2$. *(Why?)*

[2]Note that since the result of Problem 1(a) does not depend on the knot K, it is bad news if your goal is to distinguish inequivalent knots: you cannot do so by distinguishing the abelianizations of their knot groups, as these are all isomorphic to $\mathbb{Z}$. One has to find cleverer algebraic tricks for distinguishing two non-isomorphic knot groups, e.g. one such trick involving the *center* of $\pi_1(S^3 \setminus K)$ is used for torus knots in Example 1.24 of Hatcher.

[3]Actually there is a canonical way to choose the generator $[S^3 \setminus K_0] \in H_1(S^3 \setminus K_0)$ that depends on the orientation of K_0, but let's not worry about this.

Hint 2: Given a homeomorphism $f : S^1 \times \mathbb{D}^2 \to N$, one can write down another one in the form $(t, z) \mapsto f(t, e^{2\pi ikt} z)$ for any $k \in \mathbb{Z}$, where S^1 is identified with $\mathbb{R}/\mathbb{Z}$ and $\mathbb{D}^2$ with the closed unit disk in $\mathbb{C}$.

(d) [15 pts] The following standard example of a 2-component link is often called the *Hopf link*: regarding S^3 as the unit sphere in $\mathbb{R}^4$, let $K_0 = \partial\mathbb{D}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^2$ and $K_1 = \{0\} \times \partial\mathbb{D}^2 \subset \mathbb{R}^2 \times \mathbb{R}^2$. Show that $\text{link}(K_0, K_1)$ is either 1 or -1. (It is conventional to choose orientations so that the answer is $+1$.)

Remark: Part (c) in this problem is one of the basic observations underlying the notion of Dehn surgery *along knots in S^3, a fundamental technique for constructing interesting examples in low-dimensional topology. The idea is to form a new 3-manifold M by cutting the interior of N out of S^3 and replacing it with another copy of $S^1 \times \mathbb{D}^2$, attached along its boundary $\partial(S^1 \times \mathbb{D}^2) = \mathbb{T}^2$ to $\partial(S^3 \setminus \mathring{N}) \cong \mathbb{T}^2$ via some homeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$. The topological type of M then depends on the homotopy class of this "attaching" homeomorphism, and the unique longitude λ with $\text{link}(K, \lambda) = 0$ serves as a kind of "normalization" for describing these homotopy classes. If you're curious, a good place to read about surgery along knots is the book "Lectures on the Topology of 3-Manifolds" by Nikolai Saveliev (de Gruyter 1999).*

2. [30 pts total] Recall that on any path-connected space X with a fixed base point, the *Hurewicz map* $\Phi : \pi_1(X) \to H_1(X)$ is defined by identifying closed paths $\gamma : [0, 1] \to X$ with singular 1-cycles $\langle\gamma\rangle \in C_1(X)$ after identifying $[0, 1]$ with the standard 1-simplex Δ^1. We've seen in our study of singular homology that this map is a homomorphism, and that it descends to the abelianization of $\pi_1(X)$ as an isomorphism. In this problem, we consider to what extent this discussion can be extended to an arbitrary axiomatic homology theory h_* with coefficients $h_0(\{\text{pt}\}) \cong \mathbb{Z}$.

There is an obvious definition for a map $\Phi : \pi_1(X) \to h_1(X)$. Indeed, we know from the usual exact sequence arguments that $h_1(S^1) \cong \tilde{h}_0(S^0) \cong \mathbb{Z}$, so we can fix a generator $[S^1] \in h_1(S^1)$, regard elements of $\pi_1(X)$ as pointed homotopy classes of loops $\gamma : S^1 \to X$, and set

$$\Phi([\gamma]) := \gamma_*[S^1] \in h_1(X).$$

This map is well defined due to the homotopy axiom for h_*, but it will not generally descend to an isomorphism of the abelianization of $\pi_1(X)$ to $h_1(X)$; indeed, we've seen in lecture that the cohomological analogue of this statement fails to hold in general for Čech or Alexander-Spanier cohomologies on suspensions of spaces that are connected but not path-connected. But right now we have an even more basic problem: it is not obvious whether Φ is a homomorphism. Let us prove that it is.

(a) [20 pts] Given two distinct points $x, y \in S^1$, fix an identification of $S^1 \vee S^1$ with the quotient $S^1/\{x, y\}$ and consider the resulting quotient projection $p : S^1 \to S^1 \vee S^1$. Show that there is a natural isomorphism of $\tilde{h}_1(S^1 \vee S^1)$ to $h_1(S^1) \oplus h_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that $p_* : \tilde{h}_1(S^1) \to \tilde{h}_1(S^1 \vee S^1)$ is determined by the formula

$$p_*[S^1] = ([S^1], [S^1]).$$

Hint: You could probably do this using only the Eilenberg-Steenrod axioms, but it might be easier to make use of cellular homology for a well-chosen cell decomposition of S^1.

(b) [10 pts] Prove that for any two base-point preserving maps $f, g : S^1 \to X$ and their concatenation $f \cdot g : S^1 \to X$, $(f \cdot g)_*[S^1] = f_*[S^1] + g_*[S^1] \in h_1(X)$, thus implying that $\Phi : \pi_1(X) \to h_1(X)$ is a homomorphism.
Hint: Can you factor $f \cdot g : S^1 \to X$ through $S^1 \vee S^1$?