Topology II — Humboldt-Universität zu Berlin
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PROBLEM SET 13
To be discussed: 13.02.2019

Instructions

This homework will not be collected or graded, but it is highly advisable to at least think through all of the problems before the next Wednesday lecture after they are distributed, as they will often serve as mental preparation for the material in that lecture. Solutions will be discussed in the Übung.

1. Use the nonsingularity of the intersection form to establish the following isomorphisms of $\mathbb{Z}$-graded rings.

   (a) $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 2$

   (b) $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$

   Hint for both: You need to show in both cases that if $\alpha \in H^k(M;R) \cong R$ and $\beta \in H^\ell(M;R) \cong R$ are generators with $k + \ell \leqslant \dim M$, then $\alpha \cup \beta$ is a generator of $H^{k+\ell}(M;R) \cong R$. Start with the case $k + \ell = \dim M$, and then deduce the general case from this using the fact that for each $m = 0, \ldots, n$, there are natural inclusions $\mathbb{RP}^m \hookrightarrow \mathbb{RP}^n$ and $\mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$ which are also cellular maps. (The induced homomorphisms on cohomology should be easy to compute.)

   Now use the obvious (cellular) inclusions $\mathbb{RP}^n \hookrightarrow \mathbb{RP}^\infty$ and $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$ to compute:

   (c) $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$ with $|\alpha| = 2$

   (d) $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$ with $|\alpha| = 1$

2. A closed and connected 3-manifold M is called a **rational homology sphere** if $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. Prove that this condition holds if and only if M is orientable and $H_1(M;\mathbb{Z})$ is torsion.

3. In lecture we defined the **compactly supported cohomology** $H^*_c(X)$ of a space X via the direct limit
$$H^k_c(X;G) := \varinjlim \left\{H^k(X \,|\, K;G)\right\}_K ,$$
   where $H^k(X \,|\, K;G)$ is an abbreviation for $H^k(X, X\backslash K;G)$, and K ranges over the set of all compact subsets of X. These subsets are ordered by inclusion $K \subset K' \subset X$ and form a direct system via the maps $H^k(X, X\backslash K;G) \to H^k(X, X\backslash K';G)$ induced by inclusions $(X, X\backslash K') \hookrightarrow (X, X\backslash K)$.

   (a) Construct a canonical isomorphism between $H^*_c(X;G)$ and the homology of the subcomplex $C^*_c(X;G) \subset C^*(X;G)$ consisting of every cochain $\varphi : C_k(X) \to G$ for which there exists a compact subset $K \subset X$ with $\varphi|_{C_k(X\backslash K)} = 0$. (Note that K may depend on φ).

   (b) Prove that $H^n_c(\mathbb{R}^n;G) \cong G$ and $H^k_c(\mathbb{R}^n;G) = 0$ for all $k \neq n$.
   Hint: The disks $\mathbb{D}^n_r \subset \mathbb{R}^n$ of all possible radii $r > 0$ form a cofinal family[1] in the directed set of compact subsets of $\mathbb{R}^n$. Use Problem Set 7 #4.

   (c) Recall that a continuous map $f : X \to Y$ is called **proper** if for every compact set $K \subset Y$, $f^{-1}(K) \subset X$ is also compact. Show that proper maps $f : X \to Y$ induce homomorphisms $f^* : H^*_c(Y;G) \to H^*_c(X;G)$, making $H^*_c(\cdot;G)$ into a contravariant functor on the category of topological spaces with morphisms defined as proper maps.

   (d) Deduce from part (c) that $H^*_c(\cdot;G)$ is a topological invariant, i.e. $H^*_c(X;G)$ and $H^*_c(Y;G)$ are isomorphic whenever X and Y are homeomorphic. Give an example showing that this need not be true if X and Y are only homotopy equivalent.

[1] In a directed set $(I, \prec)$, a subset $S \subset I$ is called a **cofinal family** if for every $\alpha \in I$, there exists a $\beta \in S$ such that $\alpha \prec \beta$.

(e) In contrast to part (c), show that $H_c^*(\cdot; G)$ does *not* define a functor on the usual category of topological spaces with morphisms defined to be continuous (but not necessarily proper) maps. *Hint: Think about maps between $\mathbb{R}^n$ and the one-point space.*

(f) We say that two proper maps $f, g : X \to Y$ are **properly homotopic** if there exists a homotopy $h : I \times X \to Y$ between them that is also a proper map. Show that under this assumption, the induced maps $f^*, g^* : H_c^*(Y; G) \to H_c^*(X; G)$ in part (c) are identical if X is Hausdorff and locally compact. In other words, $H_c^*(\cdot; G)$ defines a contravariant functor on the category whose objects are locally compact Hausdorff spaces and whose morphisms are proper homotopy classes of proper maps.
Hint: You might first want to remind yourself how one proves the homotopy axiom for $H^*(\cdot; G)$.[2] *It will then help to show that every compact subset $K \subset I \times X$ is contained in some set of the form $I \times K'$ for a compact subset $K' \subset X$. You will find a helpful lemma for this in Problem Set 4 #2 of last semester's Topologie I class.*

4. Assume M is a compact R-oriented n-manifold with boundary and $[M] \in H_n(M, \partial M; R)$ is the resulting relative fundamental class. The relative cap product with $[M]$ then gives rise to two natural maps

$$\mathrm{PD} : H^k(M, \partial M; R) \to H_{n-k}(M; R), \tag{1}$$

$$\mathrm{PD} : H^k(M; R) \to H_{n-k}(M, \partial M; R), \tag{2}$$

both defined by $\mathrm{PD}(\varphi) = \varphi \cap [M]$. The theorem that both are isomorphisms is sometimes called *Lefschetz duality*.

(a) Find a cofinal family of compact subsets $A \subset \mathring{M}$ such that the natural maps in the diagram

$$H^*(\mathring{M} \mid A; R) \longleftarrow H^*(M \mid A; R) \longrightarrow H^*(M, \partial M; R)$$

are isomorphisms. Use this to find a natural isomorphism

$$H_c^*(M; R) \cong H^*(M, \partial M, R),$$

and deduce via the duality map $H_c^k(\mathring{M}; R) \to H_{n-k}(\mathring{M}; R)$ that (1) is an isomorphism.

(b) Show that the long exact sequenes of the pair $(M, \partial M)$ in homology and cohomology fit together into a commutative diagram of the form

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & H^k(M, \partial M; R) & \xrightarrow{j^*} & H^k(M; R) & \xrightarrow{i^*} & H^k(\partial M; R) & \xrightarrow{\delta^*} & H^{k+1}(M, \partial M; R) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle \cdot\cap[M]} & & \downarrow{\scriptstyle \cdot\cap[M]} & & \downarrow{\scriptstyle \cdot\cap[\partial M]} & & \downarrow{\scriptstyle \cdot\cap[M]} & \\
\ldots \longrightarrow & H_{n-k}(M; R) & \xrightarrow{j_*} & H_{n-k}(M, \partial M; R) & \xrightarrow{\partial_*} & H_{n-k-1}(\partial M; R) & \xrightarrow{i_*} & H_{n-k-1}(M; R) & \longrightarrow \ldots,
\end{array}$$

where $i : \partial M \hookrightarrow M$ and $j : (M, \varnothing) \hookrightarrow (M, \partial M)$ denote the usual inclusions.
Hint: Work directly with chains and cochains. Problem Set 12 #3 implies the (intuitively unsurprising) fact that if $c \in C_n(M; R)$ is a relative n-cycle representing $[M] \in H_n(M, \partial M; R)$, then the $(n-1)$-cycle $\partial c \in C_{n-1}(\partial M; R)$ represents $[\partial M] \in H_{n-1}(\partial M; R)$.

(c) Deduce from the diagram in part (b) that the map in (2) is also an isomorphism.

(d) If M has a triangulation, interpret the isomorphisms (1) and (2) in terms of the dual cell decomposition.

[2] . . . which should in any case be good preparation for the final exam!