A NOTE ON OBSTRUCTION BUNDLES

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1. Finite dimensions

In order to understand the geometric idea behind obstruction bundles, it's useful to start with a finite-dimensional analogue. All objects considered in the following discussion have natural counterparts in the infinite-dimensional setting of Gromov-Witten theory described in §2.

1.1. Transversality.

Let's suppose
$$E \to B$$
is a smooth oriented vector bundle of real rank n over a closed oriented n-dimensional manifold, and we would like to compute its Euler number
$$\langle e(E), [B] \rangle \in \mathbb{Z}.$$
The Euler number is the algebraic count of zeroes of any smooth section $\eta : B \to E$ that is generic enough for its zeroes $p \in \eta^{-1}(0) \subset B$ to be isolated, in which case each zero has a well-defined **order** $\operatorname{ord}(\eta; p) \in \mathbb{Z}$ as defined in [Mil97], and the algebraic count is
$$Z(\eta) := \sum_{p \in \eta^{-1}(0)} \operatorname{ord}(\eta; p).$$
In particular, if $\eta \in \Gamma(E)$ is transverse to the zero-section, then the linearization
$$D\eta(p) : T_pB \to E_p \tag{1.1}$$
defined by $D\eta(p)X = \nabla_X \eta$ for any choice[1] of connection ∇ is an isomorphism[2] for every $p \in \eta^{-1}(0)$, and one can then deduce from the inverse function theorem that every zero is isolated and satisfies $\operatorname{ord}(\eta; p) = \pm 1$.

The above situation is especially simple because $\operatorname{rank} E = \dim B$, but more generally, if $E \to B$ has arbitrary finite rank $m \le n$, the condition of $\eta \in \Gamma(E)$ being transverse to the zero-section is equivalent to the linearization (1.1) being surjective for all $p \in \eta^{-1}(0)$, and the implicit function theorem then gives $\mathcal{M}(\eta) := \eta^{-1}(0) \subset M$ the structure of a smooth submanifold with dimension $n - m$, along with a canonical identification between $T_p\mathcal{M}(\eta)$ and $\ker D\eta(p) \subset T_pB$ for every $p \in \eta^{-1}(0)$. All of this is also true if $m > n$, but η being transverse to the zero-section in this case just means that it never intersects it, so $\mathcal{M}(\eta) = \emptyset$.

1.2. Clean intersections and the obstruction bundle.

Obstruction bundles arise naturally in various situations where one wants to allow sections $\eta \in \Gamma(E)$ that are not generic. A common example is when the bundle $E \to B$ is equipped with a group action: if one prefers to consider only sections that are equivariant with respect to this group action, then it may be impossible to make η transverse to the zero-section, but it may still be possible to achieve a weaker condition known as "clean intersection". In general, one says that two submanifolds $P, Q \subset M$ in a smooth manifold M have a **clean intersection** if their intersection $P \cap Q \subset M$ is also a smooth submanifold such that for every $p \in P \cap Q$, the obvious inclusion
$$T_p(P \cap Q) \hookrightarrow T_pP \cap T_pQ \tag{1.2}$$

[1] The linearization $D\eta(p)$ at $p \in \eta^{-1}(0)$ does not depend on the choice of connection since for any other choice ∇', $\nabla_X\eta - \nabla'_X\eta$ depends linearly on $\eta(p)$ and thus vanishes.

[2] also known as a "surjective Fredholm operator of index zero"

is an isomorphism. The implicit function theorem implies that this is true whenever P and Q intersect transversely, but in this more general definition we are not requiring $T_pP+T_pQ = T_pM$, thus there is no general theorem dictating what the dimension of $P \cap Q$ must be; it may even be a manifold with different dimensions on different connected components. All that can be said about this in general is the following easy exercise in linear algebra: the dimension of each component of $P \cap Q$ will always be *at least* as large as what it would be if P and Q were transverse. Without the clean intersection condition, it may also happen that $P \cap Q$ is a submanifold, in which case the inclusion (1.2) still makes sense, but it may fail to be surjective. Of course, it can also happen in the non-transverse case that $P \cap Q$ is not a submanifold at all, but just a subset with no especially nice structure. We will have nothing useful to say about this level of generality, but we claim that the case of a clean intersection between $\eta \in \Gamma(E)$ and the zero-section gives enough information to compute the Euler number of the bundle $E \to B$.

We must first reformulate the definition of clean intersections in terms of an equivalent condition on the linearization (1.1).

Definition 1.1. We say that $\eta \in \Gamma(E)$ intersects the zero-section **cleanly** if $\mathcal{M}(\eta) := \eta^{-1}(0) \subset B$ is a smooth orientable submanifold such that for every $p \in \mathcal{M}(\eta)$, the natural inclusion

$$T_p\mathcal{M}(\eta) \hookrightarrow \ker D\eta(p)$$

is an isomorphism.

In the situation described by this definition, $\mathcal{M}(\eta) \subset B$ may again have different dimensions on different connected components, which will in general satisfy the lower bound

$$\dim \mathcal{M}(\eta) \geq \text{vir-dim}\, \mathcal{M}(\eta) := \dim B - \operatorname{rank} E.$$

Here we have defined the **virtual dimension** vir-dim $\mathcal{M}(\eta)$ as the dimension that $\mathcal{M}(\eta)$ *would* have if η were transverse to the zero-section; equivalently, this is the Fredholm index of $D\eta(p) : T_pB \to E_p$. The inequality is clear since for a clean intersection the dimension of $\mathcal{M}(\eta)$ near $p \in \mathcal{M}(\eta)$ is

$$\dim T_p\mathcal{M}(\eta) = \dim \ker D\eta(p) = \operatorname{ind} D\eta(p) + \operatorname{codim} \operatorname{im} D\eta(p) \geq \operatorname{ind} D\eta(p).$$

An additional consequence is that since $\dim T_p\mathcal{M}(\eta)$ depends only on the connected component of $\mathcal{M}(\eta)$ that p lies in, the dimension of

$$\operatorname{coker} D\eta(p) := E_p \big/ \operatorname{im} D\eta(p)$$

is also fixed on connected components; specifically, it equals the local dimension of $\mathcal{M}(\eta)$ minus $\operatorname{ind} D\eta(p)$. One can use the implicit function theorem to show that these cokernels fit together to form a smooth vector bundle over $\mathcal{M}(\eta)$, which we denote by

$$\mathcal{O}b \to \mathcal{M}(\eta), \qquad \mathcal{O}b_p := \operatorname{coker} D\eta(p)$$

and call the **obstruction bundle** for η. In the situation of most interest to us, we have $\operatorname{ind} D\eta(p) = \dim B - \operatorname{rank} E = 0$, thus the rank of the obstruction bundle over each component of $\mathcal{M}(\eta)$ matches the dimension of that component. Note that since E and B were both assumed to be oriented and $D\eta(p)$ descends to an isomorphism $T_pB \big/ \ker D\eta(p) \to \operatorname{im} D\eta(p)$, the resulting isomorphism

$$\frac{T_pB}{\ker D\eta(p)} \oplus \mathcal{O}b_p \cong E_p$$

associates to any choice of orientation for $\mathcal{M}(\eta)$ an orientation for the bundle $\mathcal{O}b \to \mathcal{M}(\eta)$. Since reversing either of these orientations forces the other to reverse as well, the Euler number

$$\langle e(\mathcal{O}b), [\mathcal{M}(\eta)] \rangle \in \mathbb{Z}$$

of $\mathcal{O}b \to \mathcal{M}(\eta)$ is defined independently of this choice.

Remark 1.2. One can show that for any smooth group action by bundle isomorphisms on $E \to B$ such that the isotropy group of the underlying action on B at each point has order at most 3, generic equivariant sections intersect the zero-section cleanly. But it is also easy to find examples with isotropy of order 2 in which no equivariant section can ever be *transverse* to the zero-section; see [Wena] for details.

1.3. **Local Euler numbers.** We now consider an arbitrary smooth section $\eta \in \Gamma(E)$ without any transversality or clean intersection condition, but suppose

$$C \subset \mathcal{M}(\eta) := \eta^{-1}(0) \subset B$$

is an open and closed subset of its zero-set. If η is transverse to the zero-section, this just means C is a finite set of zeroes; if η only intersects the zero-section cleanly, then a natural choice for C would be any connected component of the zero-set, which is then a closed connected and orientable submanifold of B.

Proposition 1.3. *Given $\eta \in \Gamma(E)$ and an open and closed subset $C \subset \mathcal{M}(\eta)$, there exists a unique integer*

$$e(E|_C) \in \mathbb{Z}$$

with the following significance. For any open set $\mathcal{U} \subset B$ satisfying $\mathcal{U} \cap \mathcal{M}(\eta) = C$, there exists a neighborhood $\mathcal{V} \subset \Gamma(E)$ of η such that for every $\eta_\epsilon \in \mathcal{V}$ with only finitely many zeroes in $\mathcal{U}$,

$$\sum_{p \in \eta_\epsilon^{-1}(0) \cap \mathcal{U}} \operatorname{ord}(\eta_\epsilon; p) = e(E|_C).$$

Proof. Given $\mathcal{U}$, choose a smaller neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of C and choose the neighborhood $\mathcal{V} \subset \Gamma(E)$ to be small enough so that any two sections $\xi_0, \xi_1 \in \mathcal{V}$ are homotopic through a family of sections $\{\xi_s \in \mathcal{V}\}_{s \in [0,1]}$ with

$$\xi_s(p) \neq 0 \quad \text{for all} \quad p \in \mathcal{U} \setminus \mathcal{U}_0,\ s \in [0,1].$$

Now if ξ_0 and ξ_1 both have at most finitely many zeroes in $\mathcal{U}$, we can first perturb both without changing the algebraic count of zeroes so as to assume that both are transverse to the zero-section over $\mathcal{U}$, and then perturb the above homotopy between them so that a smooth, compact, oriented 1-dimensional cobordism between $\xi_0^{-1}(0) \cap \mathcal{U}$ and $\xi_1^{-1}(0) \cap \mathcal{U}$ is defined by

$$\mathcal{M}(\{\xi_s\}\,; \mathcal{U}) := \left\{ (s,p) \in [0,1] \times \mathcal{U} \;\middle|\; \xi_s(p) = 0 \right\}.$$

The key point here is that $\mathcal{M}(\{\xi_s\}\,; \mathcal{U})$ is compact because zeroes of $\xi_s|_{\mathcal{U}}$ cannot escape from the smaller region $\mathcal{U}_0$. It follows since the signed count of boundary points in a compact oriented 1-manifold with boundary is always zero that the counts of zeroes of ξ_0 and ξ_1 over $\mathcal{U}$ match. □

Remark 1.4. If you found any detail in the proof above non-obvious, then now might be a good time to read [Mil97].

One could sensibly call $e(E|_C) \in \mathbb{Z}$ in the above proposition the **local Euler number** of the bundle $E \to B$ along $C \subset \mathcal{M}(\eta)$. It is now easy to see that if $\mathcal{M}(\eta)$ can be decomposed into a finite disjoint union of open and closed subsets $C_1 \amalg \ldots \amalg C_N$, then

$$\langle e(E), [B] \rangle = \sum_{i=1}^{N} e(E|_{C_i}). \tag{1.3}$$

Indeed, choosing suitable disjoint open neighborhoods $\mathcal{U}_i \subset B$ of the components $C_i \subset \mathcal{M}(\eta)$, a generic small perturbation $\eta_\epsilon \in \Gamma(E)$ of η will remain nonzero outside these neighborhoods, and its count of zeroes in $\mathcal{U}_i$ is exactly $e(E|_{C_i})$.

1.4. **An obstruction bundle computation.** We can now prove the most important result about obstruction bundles.

Theorem 1.5. *Suppose $\eta \in \Gamma(E)$ intersects the zero-section cleanly and $C \subset \mathcal{M}(\eta) := \eta^{-1}(0)$ is a connected component of its zero-set. Then*

$$e(E|_C) = \langle e(\mathcal{O}b|_C), [C] \rangle.$$

Sketch of the proof. Choose a small tubular neighborhood $\mathcal{U} \subset B$ of C with projection $\pi : \mathcal{U} \to B$ and use a deformation retraction to identify $E|_{\mathcal{U}}$ with $\pi^*(E|_B)$. Choose also a smooth family of subspaces complementary to $\operatorname{im} D\eta(p) \subset E_p$ for $p \in C$; these subspaces have a canonical identification with fibers of the obstruction bundle, thus giving a splitting

$$E_p \cong \operatorname{im} D\eta(\pi(p)) \oplus \mathcal{O}b_{\pi(p)} \quad \text{for all} \quad p \in \mathcal{U}.$$

Using this identification, we then choose a section $\xi \in \Gamma(\mathcal{O}b|_C)$ transverse to the zero-section and define a perturbation of η in $\mathcal{U}$ by

$$\eta_\epsilon(p) := \eta(p) + \epsilon \xi(\pi(p))$$

for $\epsilon \in \mathbb{R}$ close to 0. Since $\eta(p)$ can be assumed to lie arbitrarily close to the subspace $\operatorname{im} D\eta(\pi(p)) \subset E_p$ and $\mathcal{O}b_{\pi(p)}$ is complementary to it, the zero-set of η_ϵ in $\mathcal{U}$ is precisely the zero-set of ξ. One can now check that each such zero is nondegenerate and gets counted with the same sign for η_ϵ as it does for ξ. □

In conjunction with (1.3), one now obtains the formula

$$\langle e(E), [B] \rangle = \sum_{i=1}^{N} \langle e(\mathcal{O}b|_{C_i}), [C_i] \rangle,$$

where the sum is over all connected components of the zero-set of an arbitrary smooth section $\eta \in \Gamma(E)$ that intersects the zero-section cleanly.

2. Gromov-Witten theory and super-rigid curves

In a loose sense, each Gromov-Witten invariant can be interpreted as a computation of the "Euler number" of a certain infinite-dimensional Banach space bundle $\mathcal{E} \to \mathcal{B}$, namely by counting zeroes (up to equivalence) of the nonlinear Cauchy-Riemann operator $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$. In general this perspective only makes sense locally, so one should not take it overly seriously, but nonetheless, every object in the finite-dimensional case can be understood to have an analogue in this infinite-dimensional setting.

In particular, suppose (M, ω) is a symplectic Calabi-Yau 3-fold (meaning its first Chern class vanishes and its real dimension is 6), with compatible almost complex structure J, and let $\mathcal{M}_g(A, J)$ denote the moduli space of closed J-holomorphic curves up to parametrization with genus g and homology class $A \in H_2(M)$. If we ignore the automorphism groups of their domains or consider only cases in which these are finite (as is true e.g. for every $g \geq 2$), the linearized operator $D\bar{\partial}_J(j, u)$ that describes the local structure of this moduli space will always have index 0, hence $\text{vir-dim}\, \mathcal{M}_g(A, J) = 0$ and the Gromov-Witten invariant

$$N_A^g(M, \omega) \in \mathbb{Q}$$

is meant to be defined as an algebraic count of the elements in $\mathcal{M}_g(A, J)$.[3] This intuition unfortunately cannot be used as a precise definition, because whenever $A \in H_2(M)$ is not a primitive class, $\mathcal{M}_g(A, J)$ may contain multiple covers for which transversality is not achieved and the set $\mathcal{M}_g(A, J)$ is not even discrete. However, there are various ways of perturbing the usual nonlinear Cauchy-Riemann equation

$$du(z) + J(u(z)) \circ du(z) \circ j(z) = 0$$

[3]Everything being said here also either is true or admits a reasonable generalization to cases where domains have non-discrete automorphism groups.

to a new equation whose solutions form a discrete set that can be counted. The prescription for this in [MS04] is roughly that one should choose a generic smooth family of almost complex structures $\{J_z \in \mathcal{J}(M,\omega)\}_{z\in\Sigma_g}$ and count solutions to the equation

$$du(z) + J_z(u(z)) \circ du(z) \circ j(z) = 0, \tag{2.1}$$

i.e. one makes J into a "domain-dependent" almost complex structure. I am oversimplifying this discussion a bit since e.g. it is not obvious what the notion of equivalence "up to parametrization" should mean for solutions of the equation (2.1), but such issues can be dealt with; I will avoid discussing this here and instead refer to [MS04] or [RT95] for details in the $g = 0$ case and [RT97] for higher genus. The point is that one can make sense of $N_A^g(M,\omega)$ as a signed count of finitely many Fredholm regular solutions to a generalized nonlinear Cauchy-Riemann equation such as (2.1) depending on generic auxiliary choices. In the case of domain-dependent almost complex structures, the reason it works is that allowing $J(z,p)$ to depend explicitly on z eliminates the usual difficulty in the proof that the universal moduli space is a smooth Banach manifold—one need no longer restrict attention to simple curves, and in fact, the notion of a *multiply covered* curve no longer makes sense, as composition of solutions to (2.1) with holomorphic branched covers will not generally satisfy (2.1). In other words, this type of perturbation kills the symmetry and thus makes transversality possible. There is still a small worry involving multiple covers: if a sequence of solutions to (2.1) degenerates to a nodal curve that includes genus zero "bubbles," then some of these bubbles may in general be multiply covered, but this can be dealt with by dimensional arguments due to the fact that all symplectic 6-manifolds are semipositive.

The "clean intersection" condition arises if one prefers to avoid the use of domain-dependent almost complex structures and just perturb J generically within $\mathcal{J}(M,\omega)$. In this case multiple covers cannot generally be avoided, but one can show that they contribute something to the Gromov-Witten invariants, and this contribution is always expressible as the Euler number of an obstruction bundle. Given a curve $u \in \mathcal{M}_g(A,J)$ and integers $d \geq 1$ and $h \geq 0$, let

$$\mathcal{M}_h(d;u) \subset \mathcal{M}_h(dA,J)$$

denote the subset consisting of compositions of u with d-fold holomorphic branched covers $\Sigma_h \to \Sigma_g$.

Proposition 2.1. *If $u \in \mathcal{M}_g(A,J)$ is a super-rigid curve, then for every $d \geq 1$ and $h \geq 0$, the space $\mathcal{M}_h(d;u)$ is an open and closed subset of $\mathcal{M}_h(dA,J)$ and corresponds to a clean intersection of the nonlinear Cauchy-Riemann operator $\bar\partial_J$ with the zero-section.*

Proof. For simplicity we shall consider only branched covers of u whose domains have discrete automorphism groups; the argument requires small modifications without this assumption.

The fact that $\mathcal{M}_h(d;u)$ is an open and closed subset of $\mathcal{M}_h(dA,J)$ is [Wenb, Prop. B.1]. The clean intersection condition means the following. For a given parametrization $u : (\Sigma_g, j) \to (M,J)$ of $u \in \mathcal{M}_g(A,J)$, there is a natural identification between $\mathcal{M}_h(d;u)$ and $\mathcal{M}_h(d[\Sigma_g], j)$, the moduli space of holomorphic branched covers of genus h and degree d over (Σ_g, j). Each such cover $\varphi : \Sigma_h \to \Sigma_g$ satisfies

$$Z(d\varphi) = -\chi(\Sigma_h) + d\chi(\Sigma_g)$$

according to the Riemann-Hurwitz formula, where $Z(d\varphi)$ is its algebraic count of branch points; one can see this by interpreting $d\varphi$ as a holomorphic section of the line bundle $\mathrm{Hom}_{\mathbb{C}}(T\Sigma_h, \varphi^*T\Sigma_g)$ over Σ_h and then computing the first Chern number of the latter. Now, it is a classical fact that $\mathcal{M}_h(d[\Sigma_g], j)$ is naturally a smooth orbifold with real dimension

$$\dim_{\mathbb{R}} \mathcal{M}_h(d[\Sigma_g], j) = 2\left[-\chi(\Sigma_h) + d\chi(\Sigma_g)\right].$$

One can deduce this from the standard analysis of J-holomorphic curves after using the similarity principle to prove that elements of $\mathcal{M}_h(d[\Sigma_g], j)$ are always *automatically* Fredholm regular (i.e. no generic perturbation is required for proving this). The dimension formula also has an easy geometric interpretation: two branched covers $\varphi, \varphi' \in \mathcal{M}_h(d[\Sigma_g], j)$ are obviously not equivalent if the images of their branch points are different, and in fact the positions of the critical values of these branched covers can be used to parametrize small neighborhoods in the moduli space $\mathcal{M}_h(d[\Sigma_g], j)$. Since generically there are $Z(d\varphi)$ branch points, this yields $2Z(d\varphi)$ real parameters.

It follows from this discussion that $\mathcal{M}_h(d; u)$ is an orbifold of dimension $2Z(d\varphi)$ for any curve $\widetilde{u} := u \circ \varphi : (\Sigma_h, \widetilde{\jmath}) \to (M, J)$ representing an element of $\mathcal{M}_h(d; u)$, so the condition we now need to check is that the kernel of the linearized operator

$$D\bar{\partial}_J(\widetilde{\jmath}, \widetilde{u}) : T_{\widetilde{\jmath}}\mathcal{T} \oplus W^{1,p}(\widetilde{u}^*TM) \to L^p\big(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma_h, \widetilde{u}^*TM)\big)$$

has dimension $2Z(d\varphi)$. Since we are assuming $\dim \mathrm{Aut}(\Sigma_h, \widetilde{\jmath}) = 0$, [Wen10, Theorem 3] gives the relation

$$\dim \ker D\bar{\partial}_J(\widetilde{\jmath}, \widetilde{u}) = 2Z(d\widetilde{u}) + \dim \ker \mathbf{D}^N_{\widetilde{u}},$$

where $Z(d\widetilde{u})$ is the algebraic count of critical points of $\widetilde{u}$ and $\mathbf{D}^N_{\widetilde{u}}$ is the normal Cauchy-Riemann operator. But the super-rigidity condition is that $\mathbf{D}^N_{\widetilde{u}}$ is injective, and since u is immersed by assumption, critical points of $\widetilde{u}$ are in bijective correspondence to branch points of φ; this yields the desired result. □

As in the finite-dimensional case, the (finite-dimensional) cokernels of the operators $D\bar{\partial}(\widetilde{\jmath}, \widetilde{u})$ now fit together to form a smooth finite-rank vector bundle

$$\mathcal{O}b^u \to \mathcal{M}_h(d; u),$$

whose rank matches the dimension of its base. Strictly speaking, the fact that the (always finite) automorphism groups of the covers $\widetilde{u} = u \circ \varphi \in \mathcal{M}_h(d; u)$ can vary as φ moves around in $\mathcal{M}_h(d[\Sigma_g], j)$ means that $\mathcal{O}b^u$ is an *orbibundle* rather than a vector bundle, just as $\mathcal{M}_h(d, u)$ is an orbifold rather than a manifold. One must also extend this construction to the closure $\overline{\mathcal{M}}_h(d; u)$ of $\mathcal{M}_h(d; u)$ in the Gromov compactification $\overline{\mathcal{M}}_h(dA, J)$, since $\mathcal{M}_h(d; u)$ on its own is not compact; full details of how this can be done are explained in [Zin11, LP12]. Since $\overline{\mathcal{M}}_h(d, u)$ is an open and closed subset of $\overline{\mathcal{M}}_h(dA, J)$, one can define the **local Gromov-Witten invariant**

$$N^h_d(u) \in \mathbb{Q}$$

of the curve u in analogy with Proposition 1.3: it is the algebraic count of solutions of the perturbed equation (2.1) that will exist in a small neighborhood of the moduli space $\overline{\mathcal{M}}_h(d; u)$ after making a sufficiently small domain-dependent perturbation. The same type of argument as in Theorem 1.5 then identifies this invariant with the Euler number of the orbibundle $\mathcal{O}b^u \to \overline{\mathcal{M}}_h(d; u)$, which generally lies in $\mathbb{Q}$ rather than $\mathbb{Z}$ since $\mathcal{O}b^u$ is an orbibundle instead of a vector bundle.

Remark 2.2. It can be a bit tricky to define precisely what "Euler number of $\mathcal{O}b^u \to \overline{\mathcal{M}}_h(d; u)$" means since $\overline{\mathcal{M}}_h(d; u)$ is not generally a closed manifold, nor even an orbifold—its "top stratum" $\mathcal{M}_h(d; u)$ is quite well behaved, but a few subtle issues arise when nodal curves are allowed. For instance, there are cases where $\mathcal{M}_h(d; u)$ is *empty* but $\overline{\mathcal{M}}_h(d; u)$ is not. This happens whenever $d = 1$ and $h > g$: holomorphic branched covers of degree 1 are just biholomorphic maps, so there exist none from Σ_h to Σ_g, but $\overline{\mathcal{M}}_h(d[\Sigma_g], j)$ does contain a nodal curve consisting of the identity map $\Sigma_g \to \Sigma_g$ attached by nodes to another component on which the map is constant. It is known that such objects actually contribute to Gromov-Witten invariants: they can give rise to nontrivial solutions of the perturbed equation (2.1) which must be counted in the computation of $N^h_1(u)$. Since holomorphic branched covers are also algebraic curves, one can deal with this issue using methods from algebraic geometry: one thus defines the Euler number of $\mathcal{O}b^u \to \overline{\mathcal{M}}_h(d; u)$ by evaluating its Euler class on a *virtual fundamental cycle* $[\overline{\mathcal{M}}_h(d[\Sigma_g], j)]^{\mathrm{vir}}$, which is defined in [LT98] for computing Gromov-Witten invariants of algebraic varieties.

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