Fourier transform: $f : \mathbb{R}^n \to V$

$$(\mathcal{F} f)(p) = \hat{f}(p) := \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx$$

$$(\mathcal{F}^* f)(x) = \check{f}(x) := \int_{\mathbb{R}^n} e^{2\pi i p \cdot x} f(p)\, dp$$

$\mathcal{F}, \mathcal{F}^* : L^1(\mathbb{R}^n) \longrightarrow C_b^0(\mathbb{R}^n)$ bdd lin. ops.

rk: The fns. $\{x \longmapsto e^{2\pi i p \cdot x}\}_{p \in \mathbb{R}^n}$ are not an O-N basis of $L^2(\mathbb{R}^n)$; they are not in $L^2(\mathbb{R}^n)$!

defn: $\mathcal{S}(\mathbb{R}^n) := \{ f : \mathbb{R}^n \to V \mid f$ smooth s.t. $\forall$ multi-indices α, β, the fn. $\mathbb{R}^n \to V : x \longmapsto x^\alpha \partial^\beta f$ is bdd $\}$

"smooth & rapidly decreasing fns" = "Schwartz space"

EXS: (1) $f \in \mathcal{S}(\mathbb{R}^n) \iff \forall\ \alpha,\ \forall\ k \in \mathbb{N},\ \exists\ C > 0$ (dep. on α, k)

s.t. $|\partial^\alpha f(x)| \leq \dfrac{C}{1 + |x|^k}$.

(2) $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n) \quad \forall\ p \in [1, \infty]$

(3) $\forall\ f \in \mathcal{S}(\mathbb{R}^n),\ \forall\ \alpha,\quad \partial^\alpha f \in \mathcal{S}(\mathbb{R}^n)$ & $x^\alpha f \in \mathcal{S}(\mathbb{R}^n)$.

(4) $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ is dense if $p < \infty$.

(follows since $C_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$.)

ex: $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$.

main thm: $\mathcal{S}(\mathbb{R}^n) \underset{\mathcal{F}^* = \mathcal{F}^{-1}}{\overset{\mathcal{F}}{\underset{1:1}{\rightleftarrows}}} \mathcal{S}(\mathbb{R}^n)$ & $\underbrace{\forall f, g \in \mathcal{S}(\mathbb{R}^n),\ \langle \hat{f}, \hat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}}_{\text{Plancherel's thm}}$

Density of $\mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ + Plancherel:

cor: $\exists!$ contin. extension of $\mathcal{F}, \mathcal{F}^*$ to bdd linear unitary isomorphism

$$L^2(\mathbb{R}^n) \underset{\mathcal{F}^* = \mathcal{F}^{-1}}{\overset{\mathcal{F}}{\underset{1:1}{\rightleftarrows}}} L^2(\mathbb{R}^n).$$

$\boxed{}$

rk: For $f \in L^2(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx$ might not converge!

Instead, $\hat{f} = L^2\text{-}\lim_{j \to \infty} \hat{f}_j$ for any seq. $f_j \in \mathcal{S}(\mathbb{R}^n)$ s.t. $f_j \xrightarrow{L^2} f$.

$\hat{f} \in L^2(\mathbb{R}^n)$ is only def'd up to equality a.e.!

rk: For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have 2 defns of $\hat{f}$:

(1) $\hat{f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx$ $\quad (\leadsto \hat{f} \in C^0_b(\mathbb{R}^n))$

(2) $\hat{f} = L^2\text{-}\lim \hat{f}_j$ for $f_j \in \mathcal{S}(\mathbb{R}^n)$ s.t. $f_j \xrightarrow{L^2} f$. $\quad (\leadsto \hat{f} \in L^2(\mathbb{R}^n))$.

prop: For $f \in L^1 \cap L^2$, (1) = (2) a.e.

pf: Choose $f_j \in C^\infty_0(\mathbb{R}^n)$ s.t. $f_j \to f$ in L^1 & L^2.

$f_j \xrightarrow{L^1} f \implies \hat{f}_j \xrightarrow{C^0} (1)$.

$f_j \xrightarrow{L^2} f \implies \hat{f}_j \xrightarrow{L^2} (2) \implies$ a subseq conv. a.e. to (2)

$\implies (1) = (2)$ a.e. $\boxed{}$

EX (PSET 7): For $f, g \in L^2(\mathbb{R}^n)$,

$\hat{f} = g$ a.e. $\iff \exists$ seq. $R_j \to \infty$ s.t. for almost every $p \in \mathbb{R}^n$,

$$g(p) = \lim_{R_j \to \infty} \int_{B_{R_j}(0)} e^{-2\pi i p \cdot x} f(x)\, dx.$$

(This can hold even if $\int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx$ not convergent!)

__derivatives:__

$$\widehat{\partial_j f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x}\, \partial_j f(x)\, dx \overset{\text{(int. by parts)}}{=} 2\pi i p_j \hat{f}(p)$$

for $f \in C^1 \cap L^1$ & $\partial_j f \in L^1$ & $\lim\limits_{R\to\infty} \sup\limits_{x\in\mathbb{R}^n\setminus B_R(0)} |f(x)| = 0$.

True in particular for $f \in \mathcal{S}(\mathbb{R}^n)$; in fact

$\boxed{\widehat{\partial^\alpha f}(p) = (2\pi i p)^\alpha \hat{f}(p)}$ $\forall$ multi-indices α if $f \in \mathcal{S}(\mathbb{R}^n)$.

Similarly $\boxed{\widecheck{\partial^\alpha f}(x) = (-2\pi i x)^\alpha \check{f}(x)}$

$$\partial_j \hat{f}(p) := \frac{\partial}{\partial p_j}\int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} f(x)\, dx = \int_{\mathbb{R}^n} \frac{\partial}{\partial p_j}\left(e^{-2\pi i p\cdot x}\right) f(x)\, dx$$

$$= -2\pi i \int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} x_j f(x)\, dx = -2\pi i\, \widehat{x_j f}(p)$$

for $f \in L^1$ if also $x \longmapsto x^j f(x)$ is also in L^1.

$\Rightarrow$ For $f \in \mathcal{S}(\mathbb{R}^n)$, $\boxed{\partial^\alpha \hat{f}(p) = (-2\pi i)^{|\alpha|}\, \widehat{x^\alpha f}(p)}$

$\boxed{\partial^\alpha \check{f}(x) = (2\pi i)^{|\alpha|}\, \widecheck{p^\alpha f}(x)}$

__pf that $\mathcal{F}f, \mathcal{F}^* f \in \mathcal{S}(\mathbb{R}^n)$ if $f \in \mathcal{S}(\mathbb{R}^n)$:__

__for $\mathcal{F}$:__ α, β any multi-indices, $p^\beta \partial^\alpha \hat{f}(p) = p^\beta (-2\pi i)^{|\alpha|}\, \widehat{x^\alpha f}(p)$

$$= \frac{(-2\pi i)^{|\alpha|}}{(2\pi i)^{|\beta|}} (2\pi i p)^\beta\, \widehat{x^\alpha f}(p) = \frac{(-2\pi i)^{|\alpha|}}{(2\pi i)^{|\beta|}}\, \widehat{\partial^\beta (x^\alpha f)}(p)$$

is bdd since $\partial^\beta (x^\alpha f) \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. $\Rightarrow \hat{f} \in \mathcal{S}(\mathbb{R}^n)$. ☒

__canonical example__ (Gaussian): Let $f(x) = e^{-a^2|x|^2}$ for a const. $a > 0$.

__prop:__ $\hat{f}(x) = \check{f}(x) = \frac{\pi^{n/2}}{a^n} e^{-(\pi/a)^2 |x|^2}$.

__pf:__ $$\hat{f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i x_1 p_1} \cdots e^{-2\pi i x_n p_n}\, e^{-a^2 x_1^2} \cdots e^{-a^2 x_n^2}\, dx_1 \dots dx_n$$

$$= \int_{\mathbb{R}^n} \left(e^{-2\pi i x_1 p_1} e^{-a^2 x_1^2}\right) \cdot \dots \cdot \left(e^{-2\pi i x_n p_n} e^{-a^2 x_n^2}\right) dx_1 \dots dx_n$$

$$\overset{\text{(Fubini)}}{=} \left(\int_{\mathbb{R}} e^{-2\pi i p_1 x} e^{-a^2 x^2}\, dx\right) \cdot \dots \cdot \left(\int_{\mathbb{R}} e^{-2\pi i p_n x} e^{-a^2 x^2}\, dx\right)$$

$\Rightarrow$ sufficient to prove that for $f(x) = e^{-a^2 x^2}$ on $\mathbb{R}$, $\hat{f}(p) = \frac{\sqrt{\pi}}{a} e^{-(\pi/a)^2 p^2}$.

For $p = 0$, $\hat{f}(0) = \int_{\mathbb{R}} e^{-a^2 x^2}\, dx = \frac{\sqrt{\pi}}{a}$.

$$\hat{f}'(p) = -2\pi i\, \widehat{xf}(p) = -2\pi i \int_{\mathbb{R}} e^{-2\pi i p x}\, x e^{-a^2 x^2}\, dx$$

$$= +\frac{2\pi i}{2a^2} \int_{\mathbb{R}} e^{-2\pi i p x} \frac{d}{dx}\left(e^{-a^2 x^2}\right) dx = -\frac{\pi i}{a^2} \int_{\mathbb{R}} \frac{d}{dx}\left(e^{-2\pi i p x}\right) e^{-a^2 x^2}\, dx$$

$$= -\frac{2\pi^2}{a^2}\, p \int_{\mathbb{R}} e^{-2\pi i p x} e^{-a^2 x^2}\, dx = -\frac{2\pi^2}{a^2}\, p\, \hat{f}(p).$$

$\Rightarrow \hat{f}$ is the ! sol. to the IVP $\begin{cases} \frac{d\hat{f}}{dp} = -\frac{2\pi^2}{a^2} p \hat{f} \\ \hat{f}(0) = \frac{\sqrt{\pi}}{a} \end{cases}$.

That is satisfied by $\frac{\sqrt{\pi}}{a} e^{-(\frac{\pi}{a})^2 p^2}$. ☒

__cor:__ For Gaussian f, $\mathcal{F}\mathcal{F}^* f = \mathcal{F}^* \mathcal{F} f = f$. ☒

approx identity:

Lemma: If $\rho : \mathbb{R}^n \xrightarrow{C^\infty} [0,\infty)$ s.t. $\int_{\mathbb{R}^n} \rho \, dm = 1$, then for $\rho_j(x) := j^n \rho(jx)$,

$$\int_{\mathbb{R}^n} \rho_j \varphi \, dm \xrightarrow{j\to\infty} \varphi(0)$$ for any bdd fn. φ contin. at 0.

pf: $\int_{\mathbb{R}^n} \rho_j(x)\varphi(x)\,dx = j^n \int_{\mathbb{R}^n} \rho(jx)\varphi(x)\,dx = \int_{\mathbb{R}^n} \rho(x)\varphi(x/j)\,dx$

$\xrightarrow{\text{dom. conv.}} \int_{\mathbb{R}^n} \rho(x)\varphi(0)\,dx = \varphi(0)$. $\square$

Lemma: The fn. ρ in previous lemma can be chosen s.t. $\mathcal{F}\mathcal{F}^* \rho_j = \mathcal{F}^*\mathcal{F}\rho_j = \rho_j$ $\forall j$.

pf: Gaussian! $\square$

pf that $\mathcal{F}^*\mathcal{F} f = f$ for $f \in \mathcal{S}(\mathbb{R}^n)$:

Consider $F : \mathbb{R}^n \times \mathbb{R}^n \to V$, $F(y,p) = e^{2\pi i p\cdot x} e^{-2\pi i p \cdot y} \hat{\rho}_j(p) f(y)$

(any $x \in \mathbb{R}^n$). $\hat{\rho}_j \in \mathcal{S} \subseteq L^1 \implies f \in \mathcal{S} \subseteq L^1 \implies F \in L^1$.

Fubini $\implies$

$$\underbrace{\int_{\mathbb{R}^n} e^{2\pi i p\cdot x}\hat{\rho}_j(p)\Big(\underbrace{\int_{\mathbb{R}^n} e^{-2\pi i p\cdot y} f(y)\,dy}_{\hat{f}(p)}\Big)dp}_{\int_{\mathbb{R}^n} e^{2\pi i p\cdot x}\underbrace{\hat{\rho}_j(p)}_{\int e^{-2\pi i p\cdot x}\rho_j(x)\,dx \xrightarrow{j\to\infty} 1}\hat{f}(p)\,dp} = \underbrace{\int_{\mathbb{R}^n}\Big(\underbrace{\int_{\mathbb{R}^n} e^{2\pi i p\cdot(x-y)}\hat{\rho}_j(p)\,dp}_{\rho_j(x-y)}\Big) f(y)\,dy}_{(\rho_j * f)(x) \xrightarrow{j\to\infty} f(x)}$$

$\big\downarrow$ $j\to\infty$ (dom. conv.)

$\int_{\mathbb{R}^n} e^{2\pi i p\cdot x}\hat{f}(p)\,dp = (\mathcal{F}^*\hat{f})(x)$. $\square$

Lemma: $\forall f, g \in \mathcal{S}(\mathbb{R}^n)$, $\langle f, \mathcal{F} g\rangle_{L^2} = \langle \mathcal{F}^* f, g\rangle_{L^2}$.

pf: Fubini.

cor (Plancherel): $\langle \hat{f}, \hat{g}\rangle_{L^2} = \langle \mathcal{F} f, \mathcal{F} g\rangle_{L^2} = \langle \mathcal{F}^*\mathcal{F} f, g\rangle_{L^2} = \langle f, g\rangle_{L^2}$. $\square$