distributions (AKA "generalized fns")

motivation:

Q: How do we defn. $\partial_j f$ without assuming f is diff-able?

(A1) If $f \in H^1(\mathbb{R}^n)$, then $\exists!$ "$\partial_j f$" $\in L^2(\mathbb{R}^n)$ s.t. $\widehat{\partial_j f}(p) = 2\pi i p_j \hat{f}(p)$.

(A2) Assume $f : \Omega \longrightarrow V$ for $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, V = fin-dim. inner prod. sp. over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$$\mathcal{D}(\Omega) := \{\varphi : \Omega \longrightarrow \mathbb{R} \mid C^\infty, \text{ cpt support}\} \text{ — "test fns."}$$

EX: For fns. $f \in C^1(\Omega)$ & $g \in C^0(\Omega)$, the following are equivalent:

(1) $\partial_j f = g$

$$\boxed{(2) \quad \int_\Omega \varphi\, g \, dm = -\int_\Omega (\partial_j \varphi) f \quad \forall \varphi \in \mathcal{D}(\Omega).}$$

($\nexists$ bndry terms in integration by parts since $\mathrm{supp}(\varphi)$ cpt)

defn: a fn. $f \in L^1_{loc}(\Omega)$ has weak partial derivative $\partial_j f = g$ (weakly) iff it satisfies condition (2) (assuming also $g \in L^1_{loc}(\Omega)$).

f is weakly diff-able if $\exists$ weak derivs. $\partial_j f$ $\forall j = 1, \dots, n$.

rk: Weak derivs. are def'd only up to equality a.e.

EX: $f : \mathbb{R} \longrightarrow \mathbb{R} : x \longmapsto |x|$ is weakly diff-able: $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

<u>observation</u>: Weak derivatives don't really depend on individual values $f(x)$, but rather on the linear map $\mathcal{D}(\Omega) \longrightarrow V : \varphi \longmapsto \int_\Omega \varphi f \, dm$.

<u>defn</u>: A (vector-valued) <u>distribution</u> on Ω is a $\mathbb{K}$-linear map $\Lambda : \mathcal{D}(\Omega) \longrightarrow V$ that is sequentially continuous, where

$\varphi_k \longrightarrow \varphi_\infty$ in $\mathcal{D}(\Omega)$ iff $\exists$ cpt set $K \subseteq \Omega$ s.t. $\operatorname{supp}(\varphi_k) \subseteq K$ $\forall\, k$ & $\partial^\alpha \varphi_k \xrightarrow{\text{unif}} \partial^\alpha \varphi_\infty$ $\forall$ multi-indices α.

Denote $\mathcal{D}'(\Omega) = \{\text{distributions on } \Omega\}$. <u>Note</u>: if $V = \mathbb{K}$, $\mathcal{D}'(\Omega) =$ dual space of $\mathcal{D}(\Omega)$.

We endow $\mathcal{D}'(\Omega)$ with the locally convex top. induced by the seminorms $\{\|\Lambda\|_\varphi := |\Lambda(\varphi)|\}_{\varphi \in \mathcal{D}(\Omega)}$, i.e.

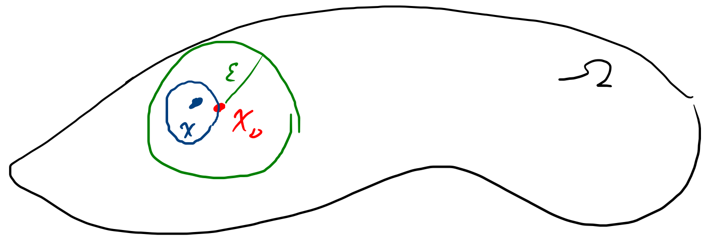
$\Lambda_k \longrightarrow \Lambda$ in $\mathcal{D}'(\Omega)$ iff $\Lambda_k(\varphi) \longrightarrow \Lambda(\varphi)$ $\forall\, \varphi \in \mathcal{D}(\Omega)$.

<u>ex</u>: $\exists$ a natural linear map $L^1_{loc}(\Omega) \longrightarrow \mathcal{D}'(\Omega) : f \longmapsto \Lambda_f$ def'd by $\Lambda_f(\varphi) := \int_\Omega \varphi f \, dm$. <u>EX</u>: $\Lambda_f : \mathcal{D}(\Omega) \longrightarrow V$ is seq. contin.

<u>lemma</u>: The map $f \longmapsto \Lambda_f$ is <u>injective</u>, i.e. if $f \in L^1_{loc}(\Omega)$ satisfies $\int_\Omega \varphi f \, dm = 0$ $\forall\, \varphi \in \mathcal{D}(\Omega)$, then $f = 0$ a.e.

("fns. are determined a.e. by the distribution they def.")

<u>pf</u>: Suppose $\int_\Omega \varphi f \, dm = 0$ $\forall\, \varphi \in \mathcal{D}(\Omega)$. Fix $x_0 \in \Omega$ & $\varepsilon > 0$ small s.t. $\overline{B_\varepsilon(x_0)} \subseteq \Omega$.



Let $g := \begin{cases} f & \text{on } B_\varepsilon(x_0) \\ 0 & \text{elsewhere} \end{cases}$,

so $g \in L^1(\mathbb{R}^n)$. Choose approx. id. ρ_j w/ shrinking support, WLOG $\operatorname{supp}(\rho_j) \subseteq B_{\varepsilon/2}(0)$ $\forall\, j$. Then $\rho_j(x - \cdot)$ for $x \in B_{\varepsilon/2}(x_0)$ has support in $B_\varepsilon(x_0) \Rightarrow$ extends to a fn. in $\mathcal{D}(\Omega)$, vanishing outside $B_\varepsilon(x_0)$. Let $g_j := \rho_j * g$, so $g_j(x) = \int_{\mathbb{R}^n} \rho_j(x-y)\, g(y)\, dy$

$= 0$ $\forall\, x$ by assumption. Now $\underset{= 0}{g_j} \xrightarrow{L^1} g \Rightarrow g = 0$ a.e.

$\Rightarrow f = 0$ a.e. ☑

ex: The "Dirac δ-fn." is $\delta \in \mathcal{D}'(\mathbb{R}^n)$ def'd by $\delta(\varphi) := \varphi(0)$.

For $x \in \Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $\delta_x \in \mathcal{D}'(\Omega)$, $\delta_x(\varphi) := \varphi(x)$.

(physicists' notation: $\delta_x = \delta(\cdot - x)$.)

notation: $(\Lambda, \varphi) := \Lambda(\varphi)$ for $\Lambda \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$

$f \in L^1_{loc}(\Omega)$, $(f, \varphi) := \Lambda_f(\varphi)$.

defn: For $\Lambda \in \mathcal{D}'(\Omega)$, $\partial_j \Lambda \in \mathcal{D}'(\Omega)$ def'd by $(\partial_j \Lambda, \varphi) := -(\Lambda, \partial_j \varphi)$.

$\partial_j \Lambda$ = "distributional partial deriv. of Λ"

Now $f \in L^1_{loc}(\Omega)$ has $\partial_j f = g$ (weakly) iff $\partial_j \Lambda_f = \Lambda_g$.

ex: $(\partial_j \delta, \varphi) = -(\delta, \partial_j \varphi) = -\partial_j \varphi(0)$.

For a multi-index α, similarly, $(\partial^\alpha \Lambda, \varphi) = (-1)^{|\alpha|}(\Lambda, \partial^\alpha \varphi)$.

ex: $(\partial^\alpha \delta, \varphi) = (-1)^{|\alpha|} \partial^\alpha \varphi(0)$.

products:

$f \in \mathcal{D}'(\Omega)$, $\varphi : \Omega \xrightarrow{C^\infty} \mathbb{K}$ $\rightsquigarrow$ $\varphi f \in \mathcal{D}'(\Omega)$ def'd by

$(\varphi f, \psi) := (f, \varphi\psi)$. Makes sense since $\psi \in \mathcal{D}(\Omega)$ & $\varphi \in C^\infty(\Omega) \Rightarrow \varphi\psi \in \mathcal{D}(\Omega)$.

EX: For $\varphi \in C^\infty(\Omega)$ & $f \in \mathcal{D}'(\Omega)$,

$$\partial_j(\varphi f) = (\partial_j \varphi) f + \varphi (\partial_j f)$$

classical deriv. distributional deriv.

convolution:

observe: $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $f \in L^1_{loc}(\mathbb{R}^n) \Rightarrow$

$$(\varphi * f)(x) = \int_{\mathbb{R}^n} \varphi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \sigma\varphi(y-x) f(y)\, dy$$

(def: $\sigma : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^n) : \ \sigma\psi(x) := \psi(-x)$)
$v \in \mathbb{R}^n$, $\tau_v : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}(\mathbb{R}^n) : \ \tau_v\psi(x) := \psi(x+v)$)

$$= \int_{\mathbb{R}^n} \tau_{-x}\sigma\varphi(y) f(y)\, dy = (\Lambda_f, \tau_{-x}\sigma\varphi).$$

def: For $\varphi \in \mathcal{D}(\Omega)$ & $\Lambda \in \mathcal{D}'(\Omega)$, $\varphi * \Lambda : \Omega' \longrightarrow V$ is def'd by

$(\varphi * \Lambda)(x) := \Lambda(\tau_{-x}\sigma\varphi)$ for $x \in \Omega' := \{ y \in \mathbb{R}^n \mid -\operatorname{supp}(\varphi) + y \in \Omega \}$

also: $\Lambda * \varphi := \varphi * \Lambda$.

thm: For $\Lambda \in \mathcal{D}'(\Omega)$ & $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the fn. $\varphi * \Lambda : \Omega' \longrightarrow V$ def'd as above is smooth & satisfies

$$\partial^\alpha(\varphi * \Lambda) = (\partial^\alpha\varphi) * \Lambda = \varphi * (\partial^\alpha\Lambda) \qquad \forall \text{ multiindic } \alpha.$$

beginning of pf. To show $\varphi * \Lambda$ is contin:

assume $x_j \longrightarrow x$, then $(\varphi * \Lambda)(x_j) = \Lambda(\tau_{-x_j}\sigma\varphi) \longrightarrow \Lambda(\tau_{-x}\sigma\varphi)$
$= (\varphi * \Lambda)(x)$ if we can prove $\tau_{-x_j}\psi \xrightarrow{\mathcal{D}(\mathbb{R}^n)} \tau_{-x}\psi \quad \forall \psi \in \mathcal{D}(\mathbb{R}^n)$.

$\exists$ cpt $K \subseteq \mathbb{R}^n$ s.t. $\operatorname{supp}(\tau_{-x_j}\psi) \subseteq K \quad \forall j$.

$\partial^\alpha(\tau_{-x_j}\psi) = \tau_{-x_j}\partial^\alpha\psi \xrightarrow{\text{unif.}} \tau_{-x}\partial^\alpha\psi = \partial^\alpha(\tau_{-x}\psi)$

since $\partial^\alpha\psi$ is unif. contin. ☑

ex: $(\varphi * \delta)(x) = \delta(\tau_{-x}\sigma\varphi) = \tau_{-x}\sigma\varphi(0) = \sigma\varphi(-x) = \varphi(x) = \delta(\varphi)$,
i.e. $\varphi * \delta = \varphi = \delta * \varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$.

rk: a seq. $\rho_j : \mathbb{R}^n \xrightarrow{C^\infty} [0,\infty)$ is an approx. id. iff $\rho_j \xrightarrow{\mathcal{D}'(\mathbb{R}^n)} \delta$.

thm ("density of C^∞ in $\mathcal{D}'$"): Spse $\Lambda \in \mathcal{D}'(\Omega)$ & ρ_j is an approx. id. w/ shrinking support. Then $\Lambda_j := \rho_j * \Lambda : \Omega_j \longrightarrow V$ are smooth fns. s.t. $\bigcup_{j \in \mathbb{N}} \Omega_j = \Omega$ & for any $\varphi \in \mathcal{D}(\Omega)$,

$$(\Lambda_j, \varphi) \longrightarrow (\Lambda, \varphi) \text{ as } j \to \infty.$$

cor (using cutoff fn.): $C_0^\infty(\Omega)$ is dense in $\mathcal{D}'(\Omega)$.

thm. For $\Lambda \in \mathcal{D}'(\Omega)$ & integers $k, m \geq 0$, following are equivalent:
(1) $\Lambda = \Lambda_f$ for some $f \in C^{k+m}(\Omega)$
(2) $\partial^\alpha\Lambda = \Lambda_{g_\alpha}$ for some $g_\alpha \in C^k(\Omega)$ $\forall$ multi-indices α w/ $|\alpha| = m$.

e.g. if $\partial_j\Lambda \in C^0 \quad \forall j = 1, \dots, n$, $\Rightarrow \Lambda \in C^1$.

pf sketch: $\partial_j\Lambda = g_j \in C^0(\Omega)$. let $f_k := \rho_k * \Lambda \in C^\infty$ for an approx. id. ρ_k. Then $\partial_j f_k = \rho_k * \partial_j\Lambda = \rho_k * g_j \xrightarrow{C^0_{loc}} g_j$ as $k \to \infty$. Since $f_k \in C^1$, can pick $x_0 \in \Omega$ & for $h \in \mathbb{R}^n$ small, write

$$f_k(x_0 + h) = f_k(x_0) + \sum_{j=1}^n h_j \int_0^1 \underbrace{\partial_j f_k(x_0 + th)}_{\rho_k * g_j \xrightarrow{C^0_{loc}} g_j} dt$$

$\downarrow$ something since $f_k \xrightarrow{\mathcal{D}'} \Lambda$.

$\downarrow$ $\displaystyle\sum_{j=1}^n h_j \int_0^1 g_j(x_0 + th)\, dt$

$\Rightarrow f_k(x_0+h)$ also converges, $\Rightarrow f_k$ conv. in C^0_{loc} to a fn. that satisfies FTC with derivative of class C^0
$\Rightarrow$ limit $\in C^1$. ☑