Lemma (less obvious than it looks): If $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$ connected &

$f \in L^1_{loc}(\Omega)$ has $\partial_j f \equiv 0$ (weakly) $\forall\, j = 1, \dots, n$, then

$f \equiv$ const a.e.

Pf: $\Lambda_f \in \mathcal{D}'(\Omega)$ has $\partial_j \Lambda_f = \Lambda_{\partial_j f} = 0$, since 0 is a contin. on Ω,

result from TU. $\Rightarrow$ $\Lambda_f = \Lambda_g$ for some $g \in C^1(\Omega)$ s.t. $\partial_j g \equiv 0$

$\Rightarrow$ $g \equiv$ const & $f = g$ a.e. ☒

Sobolev spaces

$\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $\quad 1 \le p \le \infty$, $\quad m \geq 0$ integer.

$W^{m,p}(\Omega) := \{ f \in L^p(\Omega) \mid \exists$ weak derivs. $\partial^\alpha f \in L^p(\Omega) \ \ \forall\ |\alpha| \le m \}$.

ex: $W^{0,p}(\Omega) = L^p(\Omega)$.

$W^{m,p}_{loc}(\Omega) := \{ f \in L^p_{loc}(\Omega) \mid$ " " $L^p_{loc}(\Omega)$ " $\}$

On $W^{m,p}(\Omega)$, def. norm

$$\|f\|_{W^{m,p}} := \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{L^p}.$$

Thm: $W^{m,p}(\Omega)$ is a Banach space.

Pf: $f_j \in W^{m,p}$ Cauchy $\Leftrightarrow$ $\partial^\alpha f_j$ is an L^p-Cauchy seq. $\forall\ |\alpha| \le m$

$\Longrightarrow$ $f_j \xrightarrow{L^p} f$ & $\partial^\alpha f_j \xrightarrow{L^p} g_\alpha$, some $g_\alpha \in L^p$.

to show: $\partial^\alpha f = g_\alpha$ (weakly).

$\varphi \in \mathcal{D}(\Omega)$, $\quad K := \operatorname{supp}(\varphi) \overset{\text{cpct}}{\subseteq} \Omega$, $\quad \partial^\alpha f_j|_K \xrightarrow{L^p} g_\alpha|_K$

$\Rightarrow$ $\partial^\alpha f_j|_K \xrightarrow{L^1} g_\alpha|_K$ $\Rightarrow$ $\varphi\, \partial^\alpha f_j \xrightarrow{L^1} \varphi\, g_\alpha$

$$\Rightarrow \int_\Omega \varphi\, \partial^\alpha f_j \, dm = (-1)^{|\alpha|} \int_\Omega \underbrace{\partial^\alpha \varphi}_{\text{bdd}} \cdot f_j \, dm \longrightarrow (-1)^{|\alpha|} \int_\Omega \partial^\alpha \varphi \cdot f \, dm$$

$$\downarrow$$

$$\int_\Omega \varphi\, g_\alpha \, dm$$

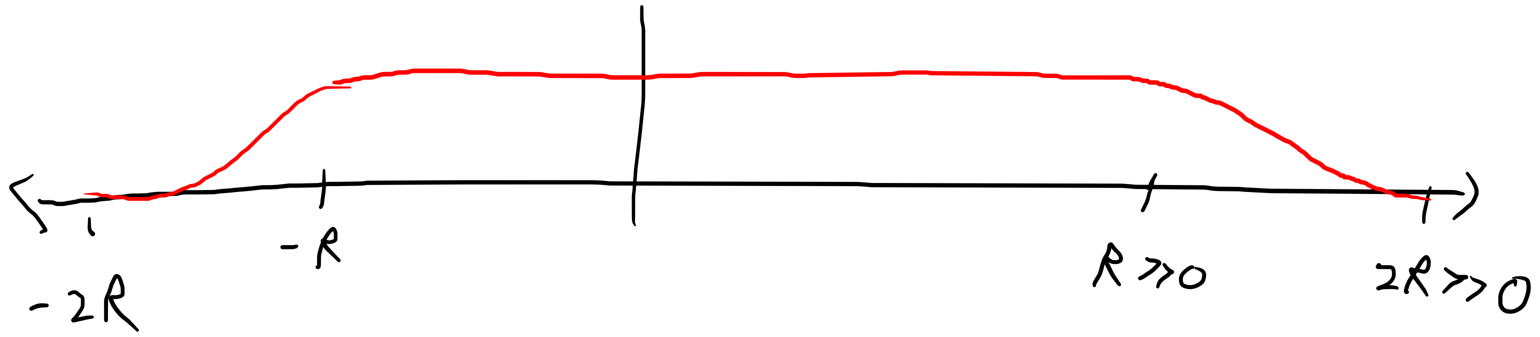
$\Rightarrow$ $\partial^\alpha f = g_\alpha$ weakly.

☒

Lemma: $W^{m,p}(\Omega) \cap C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$ for $p < \infty$.

pf: Convolution w/ approx. id. $\boxtimes$

EX (by multiplying by gently decaying cutoff fns.): $C_0^\infty(\mathbb{R}^n)$ dense in $W^{m,p}(\mathbb{R}^n)$ $\forall\, p < \infty$.



rk: For $\Omega \subsetneq \mathbb{R}^n$, $C_0^\infty(\Omega)$ is not dense in $W^{m,p}(\Omega)$ generally.

$W_0^{m,p}(\Omega) :=$ closure of $C_0^\infty(\Omega)$, a closed subspace of $W^{m,p}(\Omega)$.

prop: $\forall$ integers $m \geq 0$, $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$.

pf: $\mathcal{S}(\mathbb{R}^n)$ is dense in both & both norms are equivalent on $\mathcal{S}(\mathbb{R}^n)$. $\boxtimes$

Sobolev embedding thm

Spse $k \in \mathbb{N}$, $1 \leq p < \infty$ satisfy $kp > n$ & $0 < k - \frac{n}{p} \leq 1$, & $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$ is "reasonable". Then $\forall$ integers $m \geq 0$ & $\alpha \in (0,1)$ s.t. $\alpha \leq k - \frac{n}{p}$, $\exists$ contin. inclusion $W^{k+m,p}(\Omega) \hookrightarrow C^{m,\alpha}(\Omega)$.

"Fns of class $W^{k,p}$ have $k - \frac{n}{p}$ contin. derivs."

pf: Take a PDE course. $\boxtimes$

Fourier transform : assume fn. take values in a cpx V.S. V.

observation: for $f \in L^1(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$(\mathcal{F} f, \varphi) = \int_{\mathbb{R}^n} \varphi(p) \left(\int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx \right) dp = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-2\pi i p\cdot x} \varphi(p)\, dp \right) f(x)\, dx$$

$$= (f, \mathcal{F}\varphi).$$

This suggests defn: for $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$, $\mathcal{F}\Lambda(\varphi) := \Lambda(\mathcal{F}\varphi)$.

problem : $\mathcal{F}\varphi \notin \mathcal{D}(\mathbb{R}^n)$, in general.

solution : replace $\mathcal{D}(\mathbb{R}^n)$ with $\mathcal{S}(\mathbb{R}^n)$.

defn : a tempered distribution (valued in V) is linear map $\Lambda : \mathcal{S}(\mathbb{R}^n) \to V$ that is sequentially contin., where $\varphi_k \xrightarrow{\mathcal{S}(\mathbb{R}^n)} \varphi_\infty$ iff $\forall$ multi-indices α, β, $x^\alpha \partial^\beta \varphi_k \xrightarrow{\text{unif}} x^\alpha \partial^\beta \varphi_\infty$. $\mathcal{S}'(\mathbb{R}^n) := \{\text{tempered distr.}\}$ is endowed with the weak*-top.

rk : $L^1_{loc} \not\subset \mathcal{S}'(\mathbb{R}^n)$, e.g. $f(x) := e^{x^2}$ is in $L^1_{loc}(\mathbb{R})$, but φf is not in $L^1(\mathbb{R})$ for every $\varphi \in \mathcal{S}(\mathbb{R})$, e.g. $\varphi(x) = e^{-x^2}$.

EX : (1) Every fn. f on $\mathbb{R}^n$ that is "polynomially bdd", i.e.

$$|f(x)| \le C(1+|x|)^k \quad \text{for some } C > 0 \text{ \& } k \in \mathbb{N},$$

defn. a tempered dist. by $\Lambda_f(\varphi) = \int_{\mathbb{R}^n} \varphi f \, dm$.

(2) all $f \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$ similarly defn. $\Lambda_f \in \mathcal{S}'(\mathbb{R}^n)$.

facts about $\mathcal{S}'(\mathbb{R}^n)$

(1) $\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a contin. linear map def'd same as before, i.e. $\partial^\alpha \Lambda(\varphi) := (-1)^{|\alpha|} \Lambda(\partial^\alpha \varphi)$ (ok since $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ well-def'd & contin.)

(2) $\varphi \Lambda \in \mathcal{S}'(\mathbb{R}^n)$ is well-def'd for $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$ & $\varphi \in C^\infty(\mathbb{R}^n)$ s.t. $\partial^\alpha \varphi$ polyn. bdd. $\forall \alpha$.

(3) $\varphi \in \mathcal{S}(\mathbb{R}^n)$ & $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$ $\rightsquigarrow$ $\varphi * \Lambda : \mathbb{R}^n \to V$ is smooth s.t.

$$\partial^\alpha(\varphi * \Lambda) = (\partial^\alpha \varphi) * \Lambda = \varphi * \partial^\alpha \Lambda.$$

(4) $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$.

defn: For $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$, $\quad \mathcal{F}\Lambda \in \mathcal{S}'(\mathbb{R}^n)$ is $\quad (\mathcal{F}\Lambda, \varphi) := (\Lambda, \mathcal{F}\varphi)$.

$\qquad \mathcal{F}^*\Lambda \qquad (\mathcal{F}^*\Lambda, \varphi) := (\Lambda, \mathcal{F}^*\varphi)$.

easy to check: $\mathcal{F}, \mathcal{F}^*: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ are contin. lin. maps s.t. $\mathcal{F}^* = \mathcal{F}^{-1}$.

ex: $(\mathcal{F}\delta, \varphi) = (\delta, \mathcal{F}\varphi) = \hat{\varphi}(0) = \int_{\mathbb{R}^n} e^{-2\pi i 0\cdot x} \varphi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, dx = (1, \varphi)$

$\Rightarrow \mathcal{F}\delta = 1$, sim. $\mathcal{F}^*\delta = 1 \Rightarrow \mathcal{F}(1) = \mathcal{F}^*(1) = \delta$.

EX: Are previous relations btwn $\mathcal{F}, \mathcal{F}^*$ & ∂^α, e.g. $\mathcal{F}\partial^\alpha f = (2\pi i p)^\alpha \mathcal{F} f$ are all valid on $\mathcal{S}'(\mathbb{R}^n)$.

EX: $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\Lambda \in \mathcal{S}'(\mathbb{R}^n)$, $\quad \mathcal{F}(\varphi * \Lambda) = \mathcal{F}\varphi \cdot \mathcal{F}\Lambda$.

observation: $\mathcal{F}$ & $\mathcal{F}^*$ interchange decay conditions with regularity.

e.g. $\quad L^1 \xrightarrow{\mathcal{F}, \mathcal{F}^*} C^0_b$

$\{ f \mid |f(x)| \le \frac{1}{1+|x|} g$ for some $g \in L^1 \} \xrightarrow{\mathcal{F}, \mathcal{F}^*} C^1_b$

$\mathcal{S} \xrightarrow{\mathcal{F}, \mathcal{F}^*} \mathcal{S}$

Q: What if we assume best possible decay but no regularity?

defn: The support of $\Lambda \in \mathcal{D}'(\Omega)$ is the smallest closed subset $K \subseteq \Omega$ s.t. $\Lambda(\varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$ s.t. $\operatorname{supp}(\varphi) \cap K = \emptyset$.

Lemma: The following conds. on $\Lambda \in \mathcal{D}'(\Omega)$ are equivalent:

(1) Λ has cpct support

(2) $\Lambda = f\Lambda'$ for some $f \in C^\infty_c(\Omega)$ & $\Lambda' \in \mathcal{D}'(\Omega)$

(3) Λ extends to a contin. linear map on $C^\infty(\Omega)$, contin. w.r.t. the C^∞_{loc}-top.

pf that (2) $\Rightarrow$ (3): Defn for $\Lambda = f\Lambda'$ & $\psi \in C^\infty(\Omega)$,

$\Lambda(\psi) := \Lambda'(f\psi)$, makes sense b.c. $f \in C^\infty_c \Rightarrow f\psi \in \mathcal{D}(\Omega)$.

If $\psi_k \xrightarrow{C^\infty_{loc}} \psi_\infty$, $\quad \Lambda(\psi_k) = \Lambda'(f\psi_k) \longrightarrow \Lambda'(f\psi_\infty) = \Lambda(\psi_\infty)$

since $f\psi_k \xrightarrow{\mathcal{D}(\Omega)} f\psi_\infty$ & Λ' is contin on $\mathcal{D}(\Omega)$. ☐

cor: Every $\Lambda \in \mathcal{D}(\mathbb{R}^n)$ with cpct supp. is also in $\mathcal{S}'(\mathbb{R}^n)$.

observation: Spse $f \in L^1_{loc}(\mathbb{R}^n)$ has cpct supp. Then $f \in L^1(\mathbb{R}^n)$,

$$\hat{f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\, dx = \Lambda_f\Big(\underbrace{e^{-2\pi i p \cdot x}}_{\in C^\infty(\mathbb{R}^n)}\Big)$$

makes sense b.c. $\Lambda_f \in \mathcal{D}'(\mathbb{R}^n)$ has cpct supp.

$\hat{f}$ is contin.

$$\partial_j \hat{f}(p) = \int e^{-2\pi i p \cdot x} \underbrace{(-2\pi i x_j) f(x)}_{f_j(x)}\, dx = \Lambda_{f_j}\left(e^{-2\pi i p \cdot x}\right)$$

makes sense since $f_j \in L^1$ w/ cpct supp.

$\Rightarrow \Lambda_{f_j}$ has cpct supp.

$\Rightarrow \hat{f}$ is in C^1. Keep going... $\hat{f} \in C^\infty$.

<u>thm</u>: For any $\Lambda \in \mathcal{D}'(\mathbb{R}^n)$ w/ cpct support, $\mathcal{F}\Lambda$ & $\mathcal{F}^*\Lambda$ are smooth fn. given by

$$(\mathcal{F}\Lambda)(p) = \Lambda\left(x \mapsto e^{-2\pi i p \cdot x}\right)$$

$$(\mathcal{F}^*\Lambda)(p) = \Lambda\left(x \mapsto e^{2\pi i p \cdot x}\right).$$