Hahn-Banach thm

Q: ~~Do~~ nontrivial contin. lin. fnals $\Lambda : E \to \mathbb{K}$ always exist?

A1: for E a TVS, not always! ex (PSET 2 #4): $L^p([0,1])$ for $0<p<1$ with metric $d(f,g) := \|f-g\|_{L^p}^p$

A2: Yes if E is a Hilbert space, or L^p $(p \geq 1)$, C^k

Q': For which E does there exist $\forall$ $x \in E$ a fnal $\Lambda \in E^*$ s.t. $\Lambda(x) \neq 0$?

To show: True for all LCS E.

Key idea: convexity! Seminorms $\|\cdot\| : E \to \mathbb{R}$ are convex fns: i.e. $\forall$ $x, y \in E$, $\tau \in [0,1]$

$$\|\tau x + (1-\tau) y\| \leq \tau \|x\| + (1-\tau)\|y\|.$$

Recall (PSET 1 #6): For E a LCS, linear fnal $\Lambda : E \to \mathbb{K}$ is contin. iff $\exists$ a contin. seminorm $\|\cdot\|$ on E s.t. $|\Lambda(x)| \leq \|x\|$ $\forall$ $x \in E$.

(For $\Lambda : E \to \mathbb{R}$, suff. to assume $\Lambda(x) \leq \|x\|$ since $-\Lambda(x) = \Lambda(-x) \leq \|-x\| = \|x\|$
$\Rightarrow |\Lambda(x)| \leq \|x\|$.)

thm (Hahn-Banach): Assume E a real V.S., $p : E \to \mathbb{R}$ a convex fn., $V \subseteq E$ a linear subspace & $\Lambda : V \to \mathbb{R}$ is linear and satisfies $\Lambda(x) \leq p(x)$ $\forall$ $x \in V$. Then $\exists$ an extension of Λ to a linear fnal $\Lambda : E \to \mathbb{R}$ s.t. $\Lambda(x) \leq p(x)$ $\forall$ $x \in E$.

cor: For E a normed vec. sp. over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ & a subspace $V \subseteq E$, every $\lambda \in V^*$ can be extended to $\Lambda \in E^*$ s.t. $\Lambda|_V = \lambda$ & $\|\Lambda\| = \|\lambda\|$.

Pf: Case $\mathbb{K} = \mathbb{R}$: $\lambda : V \to \mathbb{R}$ satisfies $\lambda(x) \leq c\|x\|$ $\forall$ $x \in V$, $c = \|\lambda\| \geq 0$.
$p(x) := c\|x\|$ is a convex fn. on E $\overset{HB}{\Longrightarrow}$ $\exists$ $\Lambda : E \to \mathbb{R}$ s.t.
$\Lambda|_V = \lambda$ & $\Lambda(x) \leq c\|x\|$ $\forall$ $x \in E$ $\Rightarrow |\Lambda(x)| \leq c\|x\|$ $\Rightarrow \|\Lambda\| = c$.

Case $\mathbb{K} = \mathbb{C}$: $\mathrm{Re}\,\lambda : V \to \mathbb{R}$ is a $\mathbb{R}$-lin. fnal s.t. $|\mathrm{Re}\,\lambda(x)| \leq |\lambda(x)| \leq c\|x\|$
$\overset{HB}{\Longrightarrow}$ $\exists$ $\mathbb{R}$-lin. $\Phi : E \to \mathbb{R}$ s.t. $\Phi|_V = \mathrm{Re}\,\lambda$ & $|\Phi(x)| \leq c\|x\|$ $\forall$ $x \in E$.
check: $\Lambda(x) := \Phi(x) - i\,\Phi(ix)$ does the trick. $\square$

pf of HB :

Step 1: Lemma 1: If $y \in E \setminus V$, then Λ extends to $V \oplus \mathbb{R}y$ s.t. $\Lambda \le p$.

Pf: Extension of Λ from V to $V \oplus \mathbb{R}y$ is determined by $a := \Lambda(y) \in \mathbb{R}$.

For what $a \in \mathbb{R}$ will Λ satisfy $\underbrace{\Lambda(x+cy) \le p(x+cy)}_{(*)}$ $\forall\, x \in V, c \in \mathbb{R}$?

$(*)$ holds already for $c=0$. Assume $c>0$, then

$$\Lambda(x+cy) = \Lambda(x) + ca \le p(x+cy) \iff \boxed{a \le \frac{1}{c}\left[p(x+cy) - \Lambda(x)\right] \ \forall x \in V, c>0} \quad (1)$$

$$\Lambda(x-cy) = \Lambda(x) - ca \le p(x-cy) \iff \boxed{a \ge \frac{1}{c}\left[\Lambda(x) - p(x-cy)\right] \ \forall x \in V, c>0} \quad (2)$$

claim ($\Rightarrow$ Lemma 1): $\forall\ x, x' \in V$ & $\alpha, \beta > 0$,

$$\frac{1}{\alpha}\left[\Lambda(x) - p(x-\alpha y)\right] \le \frac{1}{\beta}\left[p(x'+\beta y) - \Lambda(x')\right].$$

(Then choose any $a \in [\sup(\text{LHS}), \inf(\text{RHS})]$, done.)

pf of claim: $\iff \beta\Lambda(x) - \beta p(x-\alpha y) \le \alpha p(x'+\beta y) - \alpha\Lambda(x') \iff$

$$\beta\Lambda(x) + \alpha\Lambda(x') = \Lambda(\beta x + \alpha x') \le \boxed{\alpha p(x'+\beta y) + \beta p(x-\alpha y)}. \quad (3)$$

Observe: $\Lambda(\beta x + \alpha x') = (\alpha+\beta)\Lambda\left(\frac{\beta}{\alpha+\beta}x + \frac{\alpha}{\alpha+\beta}x'\right) \le (\alpha+\beta)p\left(\frac{\beta}{\alpha+\beta}x + \frac{\alpha}{\alpha+\beta}x'\right)$

$$= (\alpha+\beta)\,p\left(\frac{\beta}{\alpha+\beta}(x-\alpha y) + \frac{\alpha}{\alpha+\beta}(x'+\beta y)\right)$$

$$\overset{\text{convexity}}{\le} (\alpha+\beta)\left[\frac{\beta}{\alpha+\beta}p(x-\alpha y) + \frac{\alpha}{\alpha+\beta}p(x'+\beta y)\right] = (3).$$

☑

Step 2: Def a partially ordered set $(S, \prec)$:

$$S := \{(V', \Lambda') \mid V' \subseteq E \text{ a subspace containing } V,\ \Lambda' : V' \to \mathbb{R} \text{ linear s.t. } \Lambda' \le p \ \&\ \Lambda'|_V = \Lambda\}$$

$(X_1, \Lambda_1) \prec (X_2, \Lambda_2)$ iff $X_1 \subseteq X_2$ & $\Lambda_2|_{X_1} = \Lambda_1$.

$S \neq \emptyset$ since $(V, \Lambda) \in S$; $S_0 \subseteq S$ a totally ordered subset, $\exists$ an upper bound $(X', \Lambda') \in S$ with $X' := \bigcup_{(X_0, \Lambda_0) \in S_0} X_0$.

Zorn's lemma $\Rightarrow$ S has a maximal element $(V_\infty, \Lambda_\infty) \in S$

If $V_\infty \neq E$, Lemma 1 $\Rightarrow$ can extend Λ_∞ further $\Rightarrow$ $(V_\infty, \Lambda_\infty)$ not maximal.

$\Rightarrow$ Λ_∞ is the desired extension to E. ☑

Applications of HB

(1) For X a normed vec. sp., the canonical map $J : X \to X^{**}$ (recall $(Jx)(\Lambda) = \Lambda(x)$)
is injective & isometric: $\|Jx\| = \|x\| \quad \forall\ x \in X$.

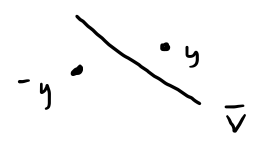
Pf: Given $x \neq 0 \in X$, defn. linear $\Lambda : \mathbb{K}x \to \mathbb{K}$ s.t. $\Lambda(x) = \|x\|$,
$\|\Lambda\| = 1$, then HB $\rightsquigarrow$ extension $\Lambda \in X^*$ s.t. $\|\Lambda\| = 1$ (i.e. $|\Lambda(y)| \le \|y\|$ $\forall\ y \in X$)
then $|Jx(\Lambda)| = |\Lambda(x)| = \|x\| \cdot \|\Lambda\| \implies \|Jx\| \geq \|x\|$.
(We already know $\|Jx\| \le \|x\|$.) ☒

(2) For X a LCS, the weak top. on X is Hausdorff ($\Leftrightarrow$ if $x_n \to x$ & $x_n \to y$ then $x = y$ $\iff$ $\forall\ x \neq 0 \in X$, $\exists\ \Lambda \in X^*$ s.t. $\Lambda(x) \neq 0$.)

(3) thm (cf. PSET 4 #1 for L^p, $1 < p < \infty$):
On any Banach sp. E, a subspace $V \subseteq E$ is dense $\iff$ all $\Lambda \in E^*$ s.t. $\Lambda|_V = 0$ are trivial.

Pf: $\Rightarrow$: Obvious by uniqueness of contin. extensions from dense subspaces.
$\Leftarrow$: If V not dense, $\bar{V} \subsetneq E$, need to find nontrivial $\Lambda \in E^*$ s.t. $\Lambda|_{\bar{V}} = 0$.
$\exists\ y \in E \setminus \bar{V}$ & a lin. fnal $\Lambda : \bar{V} \oplus \mathbb{K}y \to \mathbb{K}$ s.t. $\Lambda|_{\bar{V}} = 0$
but $\Lambda(y) \neq 0$.

claim 1: $\operatorname{dist}(\pm y, \bar{V}) > 0$. pf: If not, then $\exists$ seq. $x_n \in \bar{V}$ s.t.



$\|\pm y - x_n\| \to 0$, i.e. $x_n \to \pm y$ $\Rightarrow$ since $\bar{V}$ is closed,
$\Rightarrow \pm y \in \bar{V}$, contra!

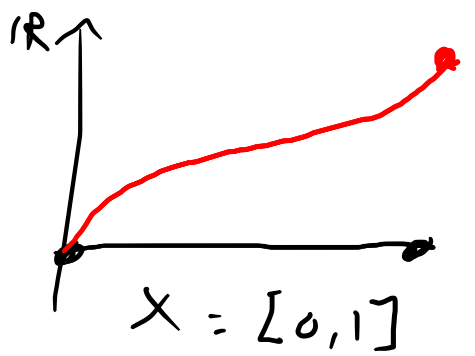
claim 2: $\Lambda : \bar{V} \oplus \mathbb{K}y \to \mathbb{K}$ is bdd. pf: Claim 1 $\Rightarrow$ $\exists\ \delta > 0$ s.t.
$\|x + y\| \geq \delta \quad \forall\ x \in \bar{V}$. Then $\forall\ x \in \bar{V}$ & $c \in \mathbb{K}$,

$$\frac{|\Lambda(x + cy)|}{\|x + cy\|} = \frac{|c| \cdot |\Lambda(y)|}{|c| \cdot \|\frac{x}{c} + y\|} \le \frac{|\Lambda(y)|}{\delta}.$$

HB $\rightsquigarrow$ extension $\Lambda : E \to \mathbb{K}$ s.t. $\Lambda(y) \neq 0$ but $\Lambda|_V = 0$. ☒

(4) On all "reasonable" measure spaces (X,μ), $\exists\ \Lambda \in (L^\infty(X))^*$ not representable by any fn. in $L^1(X)$.

Pf: Choose $f: X \to \mathbb{R}$ s.t. $|f| < \|f\|_{L^\infty} < \infty$ e.g.



PSET 4 #2 $\Rightarrow \forall\ g \in L^1(X) \setminus \{0\}$, $|\Lambda_g(f)| = \left|\int_X g f \, d\mu\right| < \|g\|_{L^1} \cdot \|f\|_{L^\infty}$

$= \|\Lambda_g\| \cdot \|f\|_{L^\infty}$.

But HB $\Rightarrow \exists\ \Lambda \in (L^\infty(X))^*$ s.t. $\|\Lambda\| = 1$

& $|\Lambda(f)| = \|f\|_{L^\infty}$, not $< \|\Lambda\| \cdot \|f\|_{L^\infty}$, $\Rightarrow \Lambda \neq \Lambda_g\ \forall\ g \in L^1$. $\boxtimes$

(5) Banach sp. E is reflexive $\iff E^*$ is reflexive. ($\Rightarrow L^\infty$ not reflexive)

(6) thm: For a Banach sp. E & fin.-dim. subspace $V \subseteq E$,

$\exists$ a closed subspace $W \subseteq E$ s.t. $V \oplus W = E$ (i.e. $V + W = E$ & $V \cap W = \{0\}$)

Pf: Idea: defn. $W := \ker \Pi$ for a bdd linear projection $\Pi : E \to V$ (i.e. $\Pi(x) = x\ \forall\ x \in V$). Choose a basis $v_1, \dots, v_n \in V \overset{HB}{\rightsquigarrow}$

$\Lambda_1, \dots, \Lambda_n \in E^*$ s.t. $\Lambda_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Let $\Pi(x) := \sum_{i=1}^{n} \Lambda_i(x) v_i$. $\boxtimes$