examples of Banach spaces

convention: All fns. take values in some fixed fin-dim. inner product space $(V, \langle\, ,\rangle)$ over $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$.

(1) $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $m \geq 0$ integer, for C^m-fns $f : \Omega \to V$, defn. the

C^m-norm $\quad \|f\|_{C^m} := \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha f(x)|$,

sum over all multi-indices $\alpha = (\underbrace{\alpha_1, \dots, \alpha_n}_{\text{integers} \geq 0})$ of order

$|\alpha| := \sum_{j=1}^n \alpha_j$, $\quad \partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$

$C_b^m(\Omega) := \{ f : \Omega \xrightarrow{C^m} V \mid \|f\|_{C^m} < \infty \}$ is now a normed vec. sp.

A seq. $f_k \to f$ in $C_b^m(\Omega) \iff \|f_k - f\|_{C^m} \to 0$ as $k \to \infty$

$\iff \partial^\alpha f_k \to \partial^\alpha f$ unif. on Ω $\forall\ |\alpha| \leq m$.

thm: $C_b^m(\Omega)$ is complete.

pf: $f_k \in C_b^m(\Omega)$ Cauchy $\iff \forall\ |\alpha| \leq m$, $\partial^\alpha f_k$ is unif. Cauchy

$\overset{\text{(Ana. 1)}}{\Longrightarrow} \partial^\alpha f_k \xrightarrow{\text{unif}} g_\alpha \in C_b^0(\Omega)$ $\forall\ |\alpha| \leq m$, g_α some fn.

In case $\alpha_0 := (0, \dots, 0)$, write $f := g_{\alpha_0}$, so $f_k \xrightarrow{\text{unif}} f$.

To show: $\partial^\alpha f = g_\alpha \quad \forall\ |\alpha| \leq m$.

case $|\alpha| = 1$: to show if $\partial_j f_k \xrightarrow{\text{unif}} g_j$, then $\partial_j f = g_j$.

$\forall\ x \in \Omega$, pick $h \in \mathbb{R}$ suff. close to 0, $e_j :=$ jth basis vector of $\mathbb{R}^n$,

then $f_k(x + he_j) = f_k(x) + h \int_0^1 \partial_j f_k(x + the_j)\, dt$

$\downarrow \qquad \downarrow \qquad \downarrow$ (due to unif. conv. of $\partial_j f_k \to g_j$)

$f(x + he_j) = f(x) + h \int_0^1 g_j(x + the_j)\, dt \implies \partial_j f = g_j$.

Rest by induction on $|\alpha|$. $\square$

(2) (X,μ) measure space, $1 \le p \le \infty$,

$L^p(X) = L^p(X,\mu) := \{ f : X \to V \text{ measurable} \mid \|f\|_{L^p} < \infty \} / \sim$

where $f \sim g$ iff $f = g$ a.e.,

$\|f\|_{L^p} := \left(\int_X |f|^p \, d\mu \right)^{1/p}$ for $p < \infty$

$\|f\|_{L^\infty} := \operatorname{ess\,sup} |f| := \inf \{ c \ge 0 \mid |f| \le c \text{ a.e.} \}$.

<u>non-examples</u>

(3) $C^\infty_b(\Omega) := \{ f : \Omega \to V \text{ smooth} \mid \partial^\alpha f \text{ bdd } \forall \alpha \}$,

$f_k \xrightarrow{C^\infty} f$ means $\partial^\alpha f_k \xrightarrow{\text{unif}} \partial^\alpha f \quad \forall \alpha \iff$

$f_k \xrightarrow{C^m} f \quad \forall m \ge 0$, i.e. $\|f - f_k\|_{C^m} \to 0$ as $k \to \infty \quad \forall m$.

(4) $0 \le m \le \infty$, $C^m_{loc}(\Omega) = C^m(\Omega) := \{ f : \Omega \to V \text{ of class } C^m \}$,

$f_k \xrightarrow{C^m_{loc}} f$ means $\partial^\alpha f_k \xrightarrow{\text{unif}} \partial^\alpha f$ on all compact subsets $\forall |\alpha| \le m$

$\iff \forall j \le m, \ \forall K \subseteq \Omega$ cpt, $f_k|_K \xrightarrow{C^j} f|_K$, i.e.

$\|f - f_k\|_{C^j(K)} := \|(f - f_k)|_K\|_{C^j} \to 0$ as $k \to \infty$.

<u>trouble</u>: Convergence in (3) & (4) cannot be described in terms of any single norm. In (4), $\|f\|_{C^j(K)} = 0 \not\Rightarrow f \equiv 0$, so $\|\cdot\|_{C^j(K)}$ are not norms.

(*) <u>recall</u>: In a metric space M, $x_n \to x$ iff $\forall$ neighborhoods $U \subseteq M$ of x (i.e. sets containing some open set containing x), $x_n \in U \quad \forall n$ suff. large.

A map $f : M \to M'$ is continuous iff $\forall U \subseteq M'$ open, $f^{-1}(U)$ is open in M.

<u>insight</u>: To discuss continuity & convergence, one doesn't need a metric if one knows what an open set is.

<u>def</u>: A <u>topology</u> on a set X is a collection $\mathcal{T}$ of subsets of X (the "open" sets) s.t.:

(i) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$ (ii) Arbitrary unions of sets from $\mathcal{T}$ are also in $\mathcal{T}$

(iii) $\forall A \in \mathcal{T}, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$. ($\Rightarrow$ also all finite intersections)

$(X, \mathcal{T})$ is now a <u>topological space</u>.

For top. spaces X & Y, continuity of maps $f : X \to Y$ def'd via (*).
Similarly, convergence of a seq. $x_n \to x$ in a top. space.

$f: X \to Y$ is a <u>homeomorphism</u> if it is bijective & both f & f^{-1} are continuous. ($\Rightarrow$ in particular, $\forall\, U \subseteq X$ open, $f(U) \subseteq Y$ also open.)

<u>ex</u>: Any metric sp. (e.g. normed vec. spaces) is also a top. sp. with usual notion of open sets.

<u>ex</u>: X, Y top. spaces $\rightsquigarrow$ $X \times Y$ inherits a natural <u>product top.</u>: the smallest top. containing $U \times V \subseteq X \times Y$ $\forall\, U \overset{open}{\subseteq} X$, $V \overset{open}{\subseteq} Y$.

<u>useful thm</u>: Contin. maps $f: X \to Y$ are also <u>sequentially contin.</u>, i.e. $\forall$ convergent seqs. $x_n \to x$ in X, it follows that $f(x_n) \to f(x)$. ☒

<u>Achtung</u>: converse is false in general top. sp. (but true in metric spaces).

<u>defn</u>: A <u>topological vector space</u> (TVS) over $\mathbb{K}$ is a vec. space X endowed with a top. s.t. the maps

$$X \times X \to X : (x,y) \mapsto x+y$$
$$\mathbb{K} \times X \to X : (\lambda, x) \mapsto \lambda x$$

are both contin.

<u>ex</u>: Normed vec. spaces are TVSs

<u>ex (thm)</u>: Every fin-dim. vec. sp. admits a <u>unique</u> topology that makes it a TVS.

<u>defn</u>: A <u>seminorm</u> on a vec. sp X is a fn. $\|\cdot\| : X \to [0,\infty)$ s.t.

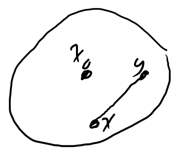
(i) $\|\lambda x\| = |\lambda| \cdot \|x\|$ $\forall\, \lambda \in \mathbb{K}$, $x \in X$,

(ii) $\|x+y\| \leq \|x\| + \|y\|$ $\forall\, x, y \in X$.

(Allow $\|x\| = 0$ even if $x \neq 0$.)

<u>defn</u>: A <u>locally convex space</u> (LCS) is a vec. sp. X endowed w/ the smallest top. containing all sets of the form $B^\alpha_R(x_0) := \{x \in X \mid \|x - x_0\|_\alpha < R\}$ $\forall$ possible $x_0 \in X$, $R > 0$ & some given family of seminorms $\{\|\cdot\|_\alpha : X \to [0,\infty)\}_{\alpha \in I}$ s.t.

$$\|x\|_\alpha = 0 \ \forall\, \alpha \in I \Rightarrow x = 0.$$

<u>rk</u>: $B^\alpha_R(x_0)$ is a convex set:  $x, y \in B^\alpha_R(x_0) \Rightarrow tx + (1-t)y \in B^\alpha_R(x_0)$ $\forall\, t \in [0,1]$.

<u>prop</u>: For X a LCS w/ top. generated by the family of seminorms $\{\|\cdot\|_\alpha\}_{\alpha \in I}$:

(1) X is a TVS (PSET 1)

(2) A seq $x_n \to x$ in X $\Leftrightarrow$ $\forall\, \alpha \in I$, $\|x - x_n\|_\alpha \to 0$ as $n \to \infty$.

(3) A set $U \subseteq X$ is open $\Leftrightarrow$ $\forall\, x_0 \in U$, $\exists$ a finite subset $I_0 \subseteq I$ & $\varepsilon_i > 0$ for $i \in I_0$ s.t. $\{x \in X \mid \|x - x_0\|_{\alpha_i} < \varepsilon_i \ \forall\, i \in I_0\} \subseteq U$.