Fredholm operators (ref: - Bühler - Salamon; - notes on website)

X, Y Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

defn: $T \in \mathcal{L}(X,Y)$ is a Fredholm operator if

$\dim \ker T < \infty$ and $\underbrace{\dim \operatorname{coker} T}_{\text{"codim im } T\text{"}} < \infty$

where $\operatorname{coker} T := Y/\operatorname{im} T$.

The Fredholm index of T is $\operatorname{ind}(T) := \dim \ker T - \dim \operatorname{coker} T \in \mathbb{Z}$.

ex: $A : \mathbb{K}^n \to \mathbb{K}^m$ linear is Fredholm: A descends to an iso

$\mathbb{K}^n/\ker A \xrightarrow{\cong} \operatorname{im} A \Rightarrow \operatorname{ind}(A) = \dim \ker A - (m - \dim \operatorname{im} A)$

$= n \underbrace{- (n - \dim \ker A) + \dim \operatorname{im} A}_{=0} - m$

$= n - m.$

ex: $\frac{d}{dx} : C^k(S^1) \to C^{k-1}(S^1)$ for any $k \in \mathbb{N}$. (for fns. $f : S^1 \to V$, $\dim V < \infty$)

$\ker\left(\frac{d}{dx}\right) = \{\text{const. fns. } f : S^1 \to V\} \Rightarrow \dim = V.$

Given $g \in C^{k-1}(S^1)$, $f(x) := \text{const.} + \int_0^x g(t)\,dt$ is a periodic fn.

iff $\int_0^1 g(t)\,dt = 0 \Rightarrow \operatorname{im}\left(\frac{d}{dx}\right) = \{ g \in C^{k-1}(S^1) \mid \int_{S^1} g\, dm = 0 \}$

that has $\underset{= \dim V}{\text{codim}}$ $\Rightarrow \operatorname{ind}\left(\frac{d}{dt}\right) = \dim V - \dim V = 0.$

ex': Similarly, $\frac{d}{dx} : C^{k,\alpha}(S^1) \to C^{k-1,\alpha}(S^1)$ for $\alpha \in (0,1]$,

$H^{s+1}(S^1) \to H^s(S^1)$ for $s \geq 0$

have index 0.

ex'': $\frac{d}{dx} : C^k([0,1]) \to C^{k-1}([0,1])$ is surj. with $\dim \ker = \dim V$

$\Rightarrow \operatorname{ind}\left(\frac{d}{dt}\right) = \dim V.$

$\frac{d}{dx} : \{ f \in C^k([0,1]) \mid f(0) = f(1) = 0 \} \to C^{k-1}([0,1])$ is inj.

with image of codim $= \dim V \Rightarrow \operatorname{ind}\left(\frac{d}{dt}\right) = -\dim V.$

non-ex1: $\partial_j : \begin{cases} C^k(\mathbb{T}^n) \to C^{k-1}(\mathbb{T}^n) \\ H^{s+1}(\mathbb{T}^n) \to H^s(\mathbb{T}^n) \end{cases}$ is not Fredholm if $n \geq 2$

$\dim \ker \partial_j = \infty$

non-ex.2: $\Delta = \sum_{j=1}^n \partial_j^2 : \begin{cases} C^{k+2}(\Omega) \to C^k(\Omega) & \text{any } k \geq 0 \\ W^{k+2,p}(\Omega) \to W^{k,p}(\Omega) & \text{any } k \geq 0,\ 1 \leq p \leq \infty \end{cases}$ $\Omega \overset{\text{open}}{\underset{\text{bdd}}{\subseteq}} \mathbb{R}^n$

bdd but not Fredholm; ker is ∞-dim.

ex: $\Delta : H^{s+2}(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$ is Fredholm, $\operatorname{ind}\Delta = 0$.

$u \in H^{s+2}(\mathbb{T}^n)$, $\Delta u = f \in H^s(\mathbb{T}^n)$

$\Rightarrow \hat{f}_k = (\widehat{\Delta u})_k = \sum_{j=1}^n \widehat{\partial_j^2 u}_k = -4\pi^2 |k|^2 \hat{u}_k$

$f \equiv 0 \iff \hat{f}_k = 0 \;\forall\, k \in \mathbb{Z}^n \iff \hat{u}_k = 0 \;\forall\, k \in \mathbb{Z}^n \setminus \{0\}$

$\Rightarrow \ker \Delta = \{u \in H^{s+2}(\mathbb{T}^n) \mid \hat{u}_k = 0 \;\forall\, k \neq 0\} = \{u : \mathbb{T}^n \to V \text{ const}\}$.

Given f, can solve $\Delta u = f$ for u by writing $\hat{u}_k := -\frac{1}{4\pi^2 |k|^2} \hat{f}_k$,

works iff $\hat{f}_0 = 0$,

$\Rightarrow \operatorname{im}\Delta = \{f \in H^s(\mathbb{T}^n) \mid \hat{f}_0 = 0\}$ has codim $= \dim V$

$\Rightarrow \operatorname{ind}\Delta = \dim V - \dim V = 0$.

non-ex: $L := \partial_t^2 - \partial_x^2 : H^{s+2}(\mathbb{T}^2) \to H^s(\mathbb{T}^2)$ has $\dim \ker L = \infty$.

$\Rightarrow$ not Fredholm.

thm ("Poisson eqn. w/ vanishing bdry cond. has unique sol."):
The Laplace operator $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ for any bdd $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^n$ is a Banach space iso. ($\Rightarrow$ is Fredholm w/ index 0).

defn: For $s \in \mathbb{R}$, $H^s(\mathbb{R}^n) := \{\Lambda \in \mathcal{S}'(\mathbb{R}^n) \mid (1+|p|^2)^{s/2}\hat{\Lambda} \text{ is a fn. in } L^2(\mathbb{R}^n)\}$

$$\|\Lambda\|_{H^s} := \|(1+|p|^2)^{s/2}\hat{\Lambda}\|_{L^2}.$$

EX: For $s \geq 0$, $\exists$ natural $\mathbb{R}$-linear iso. $H^{-s}(\mathbb{R}^n) \longrightarrow (H^s(\mathbb{R}^n))^*$:

$\Lambda \longmapsto \langle \Lambda, \cdot \rangle$ defined by $\langle \Lambda, f \rangle := \langle (1+|p|^2)^{-s/2}\hat{\Lambda}, (1+|p|^2)^{s/2}\hat{f} \rangle_{L^2}$

note: For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $\langle \varphi, \psi \rangle = \int_{\mathbb{R}^n} \langle (1+|p|^2)^{-s/2}\hat{\varphi}(p), (1+|p|^2)^{s/2}\hat{\psi}(p) \rangle \, dp$

$= \langle \hat{\varphi}, \hat{\psi} \rangle_{L^2} = \langle \varphi, \psi \rangle_{L^2}$.

EX: $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ $\forall\, s \in \mathbb{R}^n$. (proved already for $s \geq 0$.)

defn: For a bdd open subset $\Omega \subseteq \mathbb{R}^n$, $H_0^s(\Omega) :=$ closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$.

not quite right, see typed notes for correction main results on Fredholm ops

thm 1: $\mathrm{Fred}(X,Y) := \{T \in \mathcal{L}(X,Y) \mid T \text{ is Fredholm}\}$ is an open subset of $\mathcal{L}(X,Y)$ & $\mathrm{ind} : \mathrm{Fred}(X,Y) \to \mathbb{Z}$ is continuous (i.e. locally constant).

cor: For any contin. family of Fredholm ops. $\{T_s\}_{s \in [0,1]}$, $\mathrm{ind}(T_s)$ is indep. of s. ☐

thm 2: If $T \in \mathcal{L}(X,Y)$ is Fredholm & $K \in \mathcal{L}(X,Y)$ is cpct, then $T+K$ is also Fredholm.

Now $\{T_s := T + sK\}_{s \in [0,1]}$ is a contin. fam. of Fredholm ops. $\Rightarrow$ $\boxed{\mathrm{ind}(T) = \mathrm{ind}(T+K)}$.

cor ("Fredholm alternative"): For any cpct $K : X \to X$, exactly one of the following is true:

(i) The eqn. $x - Kx = 0$ has a nontrivial fin.-dim. space of sols.

(ii) The eqn. $x - Kx = y$ has a unique sol. for each $y \in X$.

pf: Let $T := \mathrm{Id} - K : X \to X$. Id is Fredholm w/ index 0

$\Rightarrow$ $\mathrm{ind}(T) = 0$ $\Rightarrow$ $\dim \ker T = \operatorname{codim} \operatorname{im} T < \infty$.

(i) $\Leftrightarrow$ $\dim \ker T > 0$. Otherwise T is inj & surj. ☐

EX: (1) (see Take-home): $T \in \mathcal{L}(X,Y)$ injective has closed image iff

$$\|x\| \le c\|Tx\| \quad \text{for some } c>0 \quad \forall x \in X.$$

(2) Any $T \in \mathcal{L}(X,Y)$ descends to a bdd lin. op. $X/_{\ker T} \longrightarrow Y$.

$\Rightarrow$ For $T \in \mathcal{L}(X,Y)$, $\operatorname{im} T$ is closed $\Leftrightarrow$ $X/_{\ker T} \xrightarrow{T} Y$ has closed image

$\Leftrightarrow$ $\exists\ c>0$ s.t. $\|\underset{X/\ker T}{[x]}\| \le c\|Tx\|$, i.e. $\boxed{\inf_{v \in \ker T} \|x+v\| \le c\|Tx\|}$.

<u>Lemma</u>: If $T \in \mathcal{L}(X,Y)$ has $\dim \operatorname{coker} < \infty$, then $\operatorname{im} T$ is closed.

Pf: Choose $w_1, \dots, w_n \in Y$ s.t. $[w_1], \dots, [w_n]$ form a basis of $Y/\operatorname{im} T$.

Def. inj. map $\Phi : \mathbb{K}^n \hookrightarrow Y : (\lambda_1, \dots, \lambda_n) \longmapsto \sum_{j=1}^{n} \lambda_j w_j$.

$\rightsquigarrow$ surj. op. $\Psi : X \oplus \mathbb{R}^n \longrightarrow Y : (x,z) \longmapsto Tx + \Phi z$.

$\Psi(x,z) = 0 \Leftrightarrow Tx = 0 = \Phi z \Rightarrow z = 0$ & $x \in \ker T$, i.e.

$\ker \Psi = \ker T \oplus \{0\} \subseteq X \oplus \mathbb{R}^n$. Ψ has closed image $\Rightarrow$

$\exists\ c>0$ s.t. $\inf_{v \in \ker T} (\|x+v\| + \|z\|) \le c\|\Psi(x,z)\|$

$\forall\ x \in X,\ z \in \mathbb{R}^n$. Set $z=0 \Rightarrow \inf_{v \in \ker T} \|x+v\| \le c\|Tx\| \Rightarrow \operatorname{im} T$ closed. $\square$

pf of thm 1: Assume $T_0 \in \mathrm{Fred}(X,Y)$.

Let $K := \ker T_0 \subseteq X$: $\dim K < \infty \Rightarrow \exists$ closed subspace $V \subseteq X$ s.t. $X = V \oplus K$.

Let $W := \operatorname{im} T_0$: $\operatorname{codim} W < \infty \Rightarrow \exists$ closed subspace $C \subseteq Y$ s.t. $Y = W \oplus C$.

Any $T \in \mathcal{L}(X,Y)$ can be written as

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : V \oplus K \to W \oplus C. \quad \text{so} \quad \begin{array}{ll} A \in \mathcal{L}(V,W), & B \in \mathcal{L}(K,W) \\ C \in \mathcal{L}(V,C), & D \in \mathcal{L}(K,C). \end{array}$$

(note: $\dim K, \dim C < \infty$).

$$T_0 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where } A_0 \in \mathcal{L}(V,W) \text{ is an iso.}$$

IMT $\Rightarrow$ all small perturbations of A_0 in $\mathcal{L}(V,W)$ are also isos.

$\Rightarrow \mathcal{U} := \{ T \in \mathcal{L}(X,Y) \mid A : V \to W \text{ is invertible} \}$ is an open set containing T_0.

<u>claim</u>: Every $T \in \mathcal{U}$ is Fredholm, $\dim \ker T \le \dim \ker T_0$, $\dim \operatorname{coker} T \le \dim \operatorname{coker} T_0$, $\operatorname{ind}(T) = \operatorname{ind}(T_0)$.

Def'n Banach space isos. $\Phi : X \to X$ $(X = V \oplus K)$, $\Psi : Y \to Y$ $(Y = W \oplus C)$,

$$\Phi = \begin{pmatrix} \mathrm{Id} & -A^{-1}B \\ 0 & \mathrm{Id} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \mathrm{Id} & 0 \\ -CA^{-1} & \mathrm{Id} \end{pmatrix}$$

$$\Rightarrow T \text{ is conjugate to } T' := \Psi T \Phi = \begin{pmatrix} A & 0 \\ 0 & T^{\mathrm{red}} \end{pmatrix} : V \oplus K \to W \oplus C$$

where $T^{\mathrm{red}} := D - CA^{-1}B : K \to C$

$\left.\begin{array}{l}\dim \ker T = \dim \ker T' = \dim(\{0\} \oplus \ker T^{\mathrm{red}}) = \dim \ker T^{\mathrm{red}}\end{array}\right\} \Leftarrow A$ is invertible.

$\dim \operatorname{coker} T = \dim \operatorname{coker} T' = \dim \operatorname{coker} T^{\mathrm{red}}$

$$\begin{aligned} \operatorname{ind}(T) &= \dim \ker T^{\mathrm{red}} - \dim \operatorname{coker} T^{\mathrm{red}} = \operatorname{ind}(T^{\mathrm{red}}) = \operatorname{ind}(K \xrightarrow{0} C) \\ &= \dim K - \dim C = \dim \ker T_0 - \dim \operatorname{coker} T_0 = \operatorname{ind}(T_0). \end{aligned}$$

$\boxtimes$