unfinished business

(1) T Fredholm & K compact $\Rightarrow$ $T+K$ also Fredholm

(2) $\Delta : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega)$ is an iso. for $\Omega \subseteq \mathbb{R}^n$ bdd open.

prop: A normed vec. sp. X is fin-dimensional iff $\overline{B_1(0)} \subseteq X$ is compact.

pf: $\Rightarrow$ follows from Ass. 1.

Assume $B_1(0) =: \mathcal{U}$ has cpct closure.

$\forall\ x \in X$, $x + \frac{1}{2}\mathcal{U}$ is a nbhd of x. $\overline{\mathcal{U}}$ cpct $\Rightarrow$

$$\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \bigcup_{i=1}^{N} \left(x_i + \tfrac{1}{2}\mathcal{U}\right) \quad \text{for some } x_1, \dots, x_N \in \overline{\mathcal{U}}.$$

Let $V := \operatorname{Span}\{x_1, \dots, x_N\} \subseteq X$. claim: $X = V$.

$$\mathcal{U} \subseteq V + \tfrac{1}{2}\mathcal{U} \Rightarrow \tfrac{1}{2}\mathcal{U} \subseteq V + \tfrac{1}{4}\mathcal{U} \Rightarrow \mathcal{U} \subseteq V + V + \tfrac{1}{4}\mathcal{U} = V + \tfrac{1}{4}\mathcal{U}.$$

continue ... $\mathcal{U} \subseteq V + \frac{1}{2^n}\mathcal{U} \quad \forall\ n \in \mathbb{N} \Rightarrow \mathcal{U} \subseteq \overline{V} = V.$

Given $x \in X$, $\varepsilon x \in \mathcal{U} \subseteq V$ for some $\varepsilon > 0 \Rightarrow x \in V$. $\square$

Recall: $T \in \mathcal{L}(X,Y) \rightsquigarrow$ transpose / dual operator $T^* \in \mathcal{L}(Y^*, X^*)$

$$T^*\Lambda := \Lambda \circ T.$$

prop: If $K \in \mathcal{L}(X,Y)$ is cpct, then so is $K^* \in \mathcal{L}(Y^*, X^*)$.

pf: Given $\Lambda_n \in Y^*$ s.t. $\|\Lambda_n\| \le C$, need to find a conv. subseq. of $K^*\Lambda_n \in X^*$.

$M := \overline{K(\overline{B_1(0)} \subseteq X)} \subseteq Y$ is a cpct metric space since K is cpct.

$\Lambda_n|_M : M \to \mathbb{K}$ satisfy $|\Lambda_n(y)| \le C\|y\| \le C \cdot \max_{y \in M} \|y\|$

$$|\Lambda_n(y) - \Lambda_n(y')| \le C\|y - y'\| \quad \forall\, y, y' \in M$$

$\Rightarrow \Lambda_n|_M$ is a unif. bdd equicontinuous seq. of fns on a cpct domain

Arzelà-Ascoli $\Rightarrow$ after restricting to a subseq., $\Lambda_n|_M : M \to \mathbb{K}$ converge uniformly.

Now $K^*\Lambda_n|_{\overline{B_1(0)}} = \Lambda_n \circ K|_{\overline{B_1(0)}}$ converges uniformly $\Rightarrow$ is unif. Cauchy

$\Rightarrow K^*\Lambda_n$ is Cauchy $\Rightarrow$ converges in X^*. $\square$

defn: The annihilator of a subspace $V \subseteq X$ is

$$V^\perp := \{\Lambda \in X^* \mid \Lambda|_V = 0\} \subseteq X^*$$

The pre-annihilator of a subspace $V \subseteq X^*$ is

$${}^\perp V := \{x \in X \mid \Lambda(x) = 0 \quad \forall\, \Lambda \in V\} \subseteq X.$$

rk: $V^\perp$ & ${}^\perp V$ are always closed subspaces.

EX: Assume $V \subseteq X$ is a closed subspace, denote inclusion $i : V \hookrightarrow X$ & quotient proj. $\pi : X \to X/V$.

(1) $i^* : X^* \to V^*$ descends to a Banach sp. iso. $X^*/V^\perp \to V^*$.

(2) $\pi^* : (X/V)^* \to X^*$ defines a Banach sp. iso. $(X/V)^* \to V^\perp$.

prop: For any $T \in \mathcal{L}(X,Y)$, $(\operatorname{im} T)^\perp = \ker T^*$ & ${}^\perp(\operatorname{im} T^*) = \ker T$.
If also $\operatorname{im} T$ is closed, then $\operatorname{im} T^*$ is also closed &

$$\operatorname{im} T = {}^\perp(\ker T^*) \quad \& \quad \operatorname{im} T^* = (\ker T)^\perp.$$

pf: 1st 2 are immediate from defns.
Similarly easy: $\operatorname{im} T \subseteq {}^\perp(\ker T^*)$ & $\operatorname{im} T^* \subseteq (\ker T)^\perp$.

claim 1: $\operatorname{im} T$ is dense in ${}^\perp(\ker T^*)$ always. (If $\operatorname{im} T$ closed $\Rightarrow$ $\operatorname{im} T = {}^\perp(\ker T^*)$.)

Spse $\Lambda : {}^\perp(\ker T^*) \to \mathbb{K}$ is a bdd lin. fnal s.t. $\Lambda|_{\operatorname{im} T} = 0$.
HB $\Rightarrow$ can extend to Y & assume $\Lambda \in Y^*$. Then $\Lambda \in (\operatorname{im} T)^\perp = \ker T^*$
$\Rightarrow$ Λ kills everything in ${}^\perp(\ker T^*)$ $\Rightarrow$ original fnal was trivial. ✓

claim 2: If $\operatorname{im} T$ is closed, then $(\ker T)^\perp \subseteq \operatorname{im} T^*$.

Recall: $\operatorname{im} T$ closed $\Rightarrow$ $\|Tx\| \geq c \cdot \inf_{v \in \ker T} \|x+v\| \quad \forall\, x \in X$, a const. $c > 0$ indep. of x.

Spse $\Lambda \in (\ker T)^\perp \subseteq X^*$, so $\Lambda(v) = 0 \quad \forall\, v \in \ker T \Rightarrow$

$$\forall\, x \in X,\ v \in \ker T, \quad \underbrace{|\Lambda(x)| = |\Lambda(x+v)| \leq \|\Lambda\| \cdot \|x+v\|}_{\text{take inf for } v \in \ker T}$$

$\Rightarrow$ $|\Lambda(x)| \leq c' \|Tx\|$ for all $x \in X$, some const. $c' > 0$ indep. of x.

Defn. a fnal. $\lambda_0 : \operatorname{im} T \to \mathbb{K}$ s.t. $\lambda_0(Tx) := \Lambda(x) \quad \forall\, x \in X$
(indep of choices since $\Lambda|_{\ker T} = 0$). We have $|\lambda_0(Tx)| \leq c' \|Tx\|$
$\Rightarrow$ λ_0 is a bdd lin. op. on $\operatorname{im} T$. HB $\rightsquigarrow$ $\exists$ extension $\lambda \in Y^*$,
now $T^*\lambda = \lambda \circ T = \lambda_0 \circ T = \Lambda \Rightarrow \Lambda \in \operatorname{im} T^*$. ☒

Remember: if $\operatorname{im} T$ closed, $(\operatorname{im} T)^\perp = \ker T^*$, ${}^\perp(\operatorname{im} T^*) = \ker T$

$$\operatorname{im} T = {}^\perp(\ker T^*), \quad \operatorname{im} T^* = (\ker T)^\perp.$$

<u>defn</u>: $T : X \to Y$ is <u>semi-Fredholm</u> if $\dim \ker T < \infty$ & $\operatorname{im} T$ is closed.

<u>Lemma</u>: Following conditions are equiv:

(1) T & T^* are semi-Fredholm

(2) T is Fredholm

(3) T^* is Fredholm

Moreover, $\dim \ker T = \dim \operatorname{coker} T^*$ & $\dim \operatorname{coker} T = \dim \ker T^*$

$\Rightarrow \operatorname{ind}(T) = -\operatorname{ind}(T^*)$.

<u>pf that (1) ⇒ (2) & (3)</u>:

$\operatorname{im} T$ closed, $(\ker T)^* \cong X^*/(\ker T)^\perp = X^*/\operatorname{im} T^* = \operatorname{coker} T^*$

$$(\operatorname{coker} T)^* = (Y/\operatorname{im} T)^* \cong (\operatorname{im} T)^\perp = \ker T^*$$

<u>pf that (2) ⇒ (1)</u>: T Fredholm $\Rightarrow$ $\operatorname{im} T$ closed, apply some isos. to conclude T^* is Fredholm. $\square$

<u>Lemma</u>: An operator $T \in \mathcal{L}(X, Y)$ is semi-Fredholm $\Leftrightarrow$ $\exists\, c > 0$ indep. of $x \in X$ & a cpct op. $K : X \to Z$ (some Banach space) s.t.

$$\|x\| \le c\|Tx\| + c\|Kx\| \quad \forall x \in X.$$

<u>pf</u>: <u>⇒</u>: assume T is semi-Fredholm, let $Z := \ker T$ & $K : X \to Z$ a contin. lin. proj.. K has finite rank $\Rightarrow$ cpct.

$X \to Y \oplus Z : x \mapsto (Tx, Kx)$ is an inj. lin. map with image $= \operatorname{im} T \oplus Z \subseteq Y \oplus Z \Rightarrow$ closed.

EX 4 from TAKEHOME $\Rightarrow$ $\exists$ estimate $\|(Tx, Kx)\| = \|Tx\| + \|Kx\| \ge c\|x\|$.

<u>⇐</u>: Assuming the estimate, <u>claim</u>: unit ball in $\ker T$ is cpct.

Assume $x_n \in \ker T$ s.t. $\|x_n\| \le 1$. K cpct $\Rightarrow$ after restricting to a subseq., Kx_n conv. in Z $\Rightarrow$ Kx_n is Cauchy. Now $\forall\, m, n$ large,

$$\|x_n - x_m\| \le c\|\underbrace{Tx_n - Tx_m}_{=0}\| + c\|\underbrace{Kx_n - Kx_m}_{\text{small}}\| \Rightarrow x_n \text{ is Cauchy in } X \Rightarrow \text{converges.}$$

$\Rightarrow \dim \ker T < \infty$.

<u>claim</u>: $\operatorname{im} T$ is closed. $\dim \ker T < \infty \Rightarrow X = \ker T \oplus V$ for some closed $V \subseteq X$.

then $\operatorname{im} T = \operatorname{im}(T|_V)$, but $T|_V : V \to Y$ is inj.

Now spse $x_n \in V$ s.t. $Tx_n \to y \in Y$.

<u>subclaim</u> 1: x_n is bdd. If not, after restricting to a subseq., can assume $\|x_n\| \to \infty$. Let $z_n := \frac{x_n}{\|x_n\|}$, then $Tz_n = \frac{1}{\|x_n\|} Tx_n \to 0$.

For a further subseq., WLOG Kz_n converges $\Rightarrow$ is Cauchy, $\Rightarrow$

$$\|z_n - z_m\| \le c\|\underbrace{Tz_n - Tz_m}_{\text{small}}\| + c\|\underbrace{Kz_n - Kz_m}_{\text{small}}\| \Rightarrow z_n \text{ is Cauchy,}$$

write $z_n \to z_\infty \in X$. $\|z_n\| = 1\ \forall n \Rightarrow \|z_\infty\| = 1 \Rightarrow z_\infty \ne 0$ & $z_\infty \in V$

& $Tz_\infty = 0$, but $V \cap \ker T = \{0\}$ contradiction!

<u>subclaim</u> 2: A subseq. of x_n converges ($\Rightarrow y \in \operatorname{im} T \Rightarrow \operatorname{im} T$ closed).

Same trick: for a subseq. Kx_n converges $\Rightarrow$ Cauchy, Tx_n also Cauchy $\Rightarrow x_n$ is Cauchy (due to estimate). $\square$

pf that T Fredholm & K cpct $\Rightarrow$ $T+K$ Fredholm:

T is semi-Fredholm $\Rightarrow$ $\exists$ estimate $\|x\| \le c\|Tx\| + c\|K_0 x\|$ for some $c>0$ & cpct op. $K_0 : X \to Z$.

$\Rightarrow$ $\|x\| \le c\|(T+K)x\| + c\|Kx\| + c\|K_0 x\| = c\|(T+K)x\| + c\|K_1 x\|$

for a cpct op. $K_1 : X \to X \oplus Z : x \longmapsto (Kx, K_0 x)$

$\Rightarrow$ $T+K$ is semi-Fredholm.

T^* is also semi-Fredholm, K^* is also cpct $\Rightarrow$ $T^* + K^* = (T+K)^*$ is also semi-Fredholm $\Rightarrow$ $T+K$ is Fredholm. $\blacksquare$

sketch pf that $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an iso.

(1) The inclusion $j : H^1_0(\Omega) \hookrightarrow H^{-1}(\Omega)$ is cpct.

(2) $\Phi := j - \frac{1}{4\pi^2}\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an isomorphism.

$\Rightarrow$ $\Delta = 4\pi^2 \Phi + 4\pi^2 j = \underbrace{\text{iso.}}_{\text{Fred. ind}=0} + \text{cpct}$ $\Rightarrow$ is Fredholm w/ index 0.

(3) Δ is injective $\Rightarrow$ also surjective.