SPECTRUM

linear algebra: $T : \mathbb{C}^n \to \mathbb{C}^n$, $\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invtbl}\}$

funct. an.: Assume X is a cpx. Banach sp., abbreviate for $\lambda \in \mathbb{C}$,

$\lambda := \lambda \cdot \mathrm{Id} \in \mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$.

defn: $\rho(T) := \{\lambda \in \mathbb{C} \mid \lambda - T : X \to X \text{ is invtbl}\}$ = "resolvent set" of T

$\sigma(T) := \mathbb{C} \setminus \rho(T)$ "spectrum" of T

For $\lambda \in \rho(T)$, IMT $\Rightarrow$ the "resolvent" $R_\lambda(T) := (\lambda - T)^{-1}$ is a bdd lin. op. $X \to X$.

$\lambda \in \sigma(T)$ is an eigenvalue of T if $\underbrace{\ker(\lambda - T)}_{\text{"eigenspace"} \ni \text{ eigenvectors}} \neq \{0\}$

Achtung: (1) The set of eigenvalues need not be discrete

(2) Not all $\lambda \in \sigma(T)$ must be eigenvals.

ex: $T : \ell^1 \to \ell^1 : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$ $(x_n \in \mathbb{C})$

$Tx = \lambda x \iff x_{n+1} = \lambda x_n \ \forall n \in \mathbb{N} \iff x_n = x_1 \lambda^{n-1}$

Then $\sum_{n \in \mathbb{N}} |x_n| = |x_1| \sum_{n=1}^{\infty} |\lambda|^{n-1} < \infty \iff |\lambda| < 1$, so $\sigma(T) \supseteq \{|\lambda| < 1\}$.

Identify $(\ell^1)^*$ with ℓ^∞ via the $\mathbb{C}$-lin. iso. $\ell^\infty \to (\ell^1)^* : x \mapsto (x, \cdot)$

where $(x, y) := \sum_{n=1}^{\infty} x_n y_n$. Consider dual operator $T' : \ell^\infty \to \ell^\infty$

def'd by $(T'x, y) = (x, Ty)$, so $\sum_{n \in \mathbb{N}} x_n y_{n+1} = \sum_{n \in \mathbb{N}} (T'x)_n y_n$

$\Rightarrow T' : \ell^\infty \to \ell^\infty : (x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$. This is injective.

Now $T'x = \lambda x \Rightarrow 0 = \lambda x_1 \Rightarrow x_1 = 0$ (assuming $\lambda \neq 0$)

$x_1 = \lambda x_2 \Rightarrow x_2 = 0$

$\dots \Rightarrow x = 0 \Rightarrow \nexists$ eigenvalues.

However, spse $|\lambda| < 1$ & $T x_\lambda = \lambda x_\lambda$, $x_\lambda \in \ell^1 \setminus \{0\}$. Then $\forall y \in \ell^\infty$,

$((\lambda - T')y, x_\lambda) = (y, \lambda x_\lambda) - (y, T x_\lambda) = (y, (\lambda - T) x_\lambda) = 0$, i.e. x_λ defns.

an element in $(\ell^\infty)^*$ that anihilates $\mathrm{im}(\lambda - T') \subseteq \ell^\infty \Rightarrow \mathrm{im}(\lambda - T')$ is

not dense in $\ell^\infty \Rightarrow \lambda \in \sigma(T')$. Proved: $\sigma(T') \supseteq \{|\lambda| < 1\}$.

defn: $\{\lambda \in \sigma(T) \mid \ker(\lambda - T) \neq \{0\}\}$ is called the point spectrum of T (i.e. eigenvalues)

$\{\lambda \in \sigma(T) \mid \operatorname{im}(\lambda - T)$ is not dense in X and $\ker(\lambda - T) = \{0\}\}$ is called the residual spectrum of T.

rk: $\lambda \in \sigma(T)$ may also have $\operatorname{im}(\lambda - T)$ dense but $\subsetneq X$

EX 1 (PSET 10): If this happens, then $\exists$ seq $x_n \in X$ with $\|x_n\| = 1$ $\forall n$ s.t. $(\lambda - T)x_n \longrightarrow 0$ as $n \to \infty$. (We say λ is an "approximate eigenvalue")

EX 2: For $T \in \mathcal{L}(X)$ & dual op. $T' \in \mathcal{L}(X^*)$,

(i) $\lambda \in$ residual sp. of $T \implies \lambda$ is an e-val. of T'.

(ii) λ is an e-val. of $T \implies$ either λ is an e-val of T' or $\lambda \in$ residual sp. of T'.

EX 3: For $\mathcal{H}$ a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ & adjoint $T^* \in \mathcal{L}(\mathcal{H})$
$(\langle x, Ty\rangle = \langle T^* x, y\rangle \quad \forall\, x, y \in \mathcal{H})$

$\sigma(T^*) = \{\bar{\lambda} \in \mathbb{C} \mid \lambda \in \sigma(T)\}$.

thm: $T \in \mathcal{L}(\mathcal{H})$ self-adjoint (i.e. $T = T^*$) $\implies \sigma(T) \subseteq \mathbb{R}$, $\nexists$ residual spec., & e-vectors for distinct e-vals. are orthogonal.

pf: $\lambda, \mu \in \mathbb{R} \implies \|[T - (\lambda + i\mu)]x\|^2 = \langle (T-\lambda)x - i\mu x, (T-\lambda)x - i\mu x\rangle$

$$= \|(T-\lambda)x\|^2 + \mu^2\|x\|^2 \underbrace{- i\mu\langle (T-\lambda)x, x\rangle + i\mu\langle x, (T-\lambda)x\rangle}_{=0}$$

In particular, $\|[T - (\lambda + i\mu)]x\| \geq \mu\|x\|$. $\implies$ If $\mu \neq 0$, then $T - (\lambda + i\mu)$ is inj. w/ closed image.

Then if $\lambda + i\mu \in \sigma(T) \implies \lambda + i\mu \in$ residual spectrum $\overset{(EX 2)}{\implies} \lambda + i\mu$ is an e-val. $T': \mathcal{H}^* \to \mathcal{H}^*$ $\overset{(EX)}{\implies} \lambda - i\mu$ is an e-val. of $T^* = T$, contra!

$\implies \sigma(T) \subseteq \mathbb{R}$. If $\lambda \in$ residual sp., $\implies \lambda$ is an e-val. of $T^* = T$, contra!

$\implies \nexists$ residual spectrum.

If $Tv = \lambda v$, $Tw = \mu w$ & $\lambda \neq \mu$, then $(\lambda - \mu)\langle v, w\rangle = \langle \lambda v, w\rangle - \langle v, \mu w\rangle$
$= \langle Tv, w\rangle - \langle v, Tw\rangle = 0 \implies \langle v, w\rangle = 0$. ∎

thm: $\rho(T) \subseteq \mathbb{C}$ is open, & $\rho(T) \longrightarrow \mathcal{L}(X) : \lambda \longmapsto R_\lambda(T) = (\lambda - T)^{-1}$

is a complex analytic fn.

pf: Spse $\lambda_0 \in \rho(T)$, so $\lambda_0 - T$ is invtbl.

Recall: $\|A\| < 1 \Rightarrow (\mathrm{Id} + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k$, so if A_0 invtbl & $\|B\|$ small,

Then $(A_0 + B)^{-1} = (\mathrm{Id} + A_0^{-1}B)^{-1} A_0^{-1} = \sum_{k=0}^{\infty} (-1)^k (A_0^{-1}B)^k A_0^{-1}$ $\left(\text{valid if } \|B\| < \frac{1}{\|A_0^{-1}\|}\right)$

$\Rightarrow$ for $\mu \in \mathbb{C}$ small, $R_{\lambda_0+\mu}(T) = [(\lambda_0 - T) + \mu]^{-1}$

$= \sum_{k=0}^{\infty} (-1)^k (R_{\lambda_0}(T)\mu)^k R_{\lambda_0}(T) = \sum_{k=0}^{\infty} (-1)^k \mu^k R_{\lambda_0}(T)^{k+1}$

valid whenever $|\mu| < \frac{1}{\|R_{\lambda_0}(T)\|}$. ∎

cor: $\sigma(T) \neq \emptyset$.

pf: If $\sigma(T) = \emptyset$, then $\mathbb{C} \longrightarrow \mathcal{L}(H) : \lambda \longmapsto R_\lambda(T)$ is a global analytic fn.

s.t. $R_\lambda(T) = (\lambda - T)^{-1} = \frac{1}{\lambda}(\mathrm{Id} - \frac{1}{\lambda}T)^{-1} \longrightarrow 0$ as $|\lambda| \to \infty$.

$\Rightarrow \forall \Lambda \in \mathcal{L}(X)^*$, $\lambda \longmapsto \Lambda(R_\lambda(T))$ is a globally bdd analytic fn. $\mathbb{C} \to \mathbb{C}$

$\Rightarrow$ it is constant $\Rightarrow$ it is 0, contra! ∎

defn: $r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \geq 0$ is the spectral radius of T.

observe: $(\lambda - T) = \lambda(\mathrm{Id} - \frac{1}{\lambda}T)$ invtbl whenever $|\lambda| > \|T\| \Rightarrow r(T) \leq \|T\|$.

compact operators

<u>Riesz-Schauder thm</u>: If $K : X \to X$ is cpt, then $\sigma(K) \subseteq \mathbb{C}$ has no limit pts. except possibly 0, & every $\lambda \in \sigma(K) \setminus \{0\}$ is an eigenvalue of finite multiplicity (i.e. $\dim \ker(\lambda - T) < \infty$).

<u>pf</u>: $\lambda \in \mathbb{C} \setminus \{0\} \Rightarrow \lambda : X \to X$ is an iso. $\Rightarrow$ Fredholm w/ index $= 0$ $\Rightarrow$ $\lambda - K$ is also Fredholm w/ index 0, so inj iff surj.

$\Rightarrow \lambda \in \sigma(K)$ iff $0 < \dim \ker(\lambda - K) < \infty$.

<u>remaining to prove</u>: if $\lambda_0 \in \sigma(K) \setminus \{0\}$, $\exists$ a nbhd $\lambda_0 \in \mathcal{U} \subseteq \mathbb{C}$ s.t. $\sigma(K) \cap \mathcal{U} = \{\lambda_0\}$. Write $T_\mu := \lambda_0 + \mu - K = (\lambda_0 - K) + \mu = T_0 + \mu$ for $\mu \in \mathbb{C}$ with $|\mu|$ small. Choose splitting $X = V \oplus \ker T_0 = \operatorname{im} T_0 \oplus C$ of closed subspaces ($\operatorname{ind}(T_0) = 0 \Rightarrow \dim C = \dim \ker T_0 < \infty$).

Write $T_0 = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$ & $\mathrm{Id} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : V \oplus \ker T_0 \to \operatorname{im} T_0 \oplus C$,

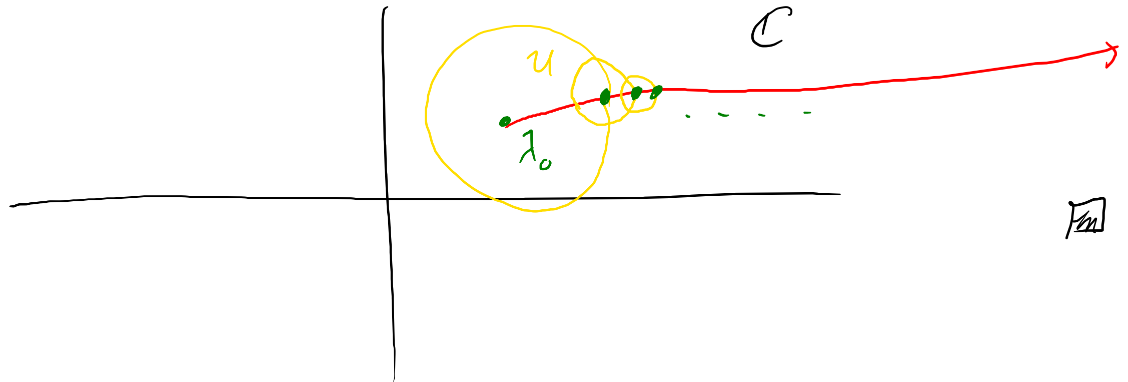
so $T_\mu = T_0 + \mu \cdot \mathrm{Id} = \begin{pmatrix} A_0 + \mu\alpha & \mu\beta \\ \mu\gamma & \mu\delta \end{pmatrix}$. $A_0 : V \to \operatorname{im} T_0$ is invtbl.

Let $\mathcal{U} := \{\lambda_0 + \mu \in \mathbb{C} \mid |\mu|$ small enough s.t. $A_0 + \mu\alpha$ is still invtbl$\}$.

<u>Recall</u> from pf. that $\operatorname{ind} : \operatorname{Fred}(X, Y) \to \mathbb{Z}$ is continuous:

T_μ is conjugate to $\begin{pmatrix} A_0 + \mu\alpha & 0 \\ 0 & \mu\delta - \mu\gamma(A_0 + \mu\alpha)^{-1}\mu\beta \end{pmatrix}$

Then T_μ invtbl $\Leftrightarrow$ $\Phi(\mu) := \mu\delta - \mu^2\gamma(A_0 + \mu\alpha)^{-1}$ is an invtbl map $\ker T_0 \to C$. Choose an iso. btwn C & $\ker T_0$; use this to write the fn. $\mathcal{U} \to \mathbb{C} : \lambda_0 + \mu \mapsto \det \Phi(\mu)$. This fn. is analytic & vanishes at $\mu = 0$ since λ_0 is an e-val; its zero-set is $\{\lambda \in \mathcal{U} \mid \lambda \in \sigma(K)\}$. If λ_0 is not an isolated pt. in $\sigma(K)$, fn $\equiv 0$ $\Rightarrow$ $\mathcal{U} \subseteq \sigma(K)$. Topological argument $\Rightarrow$ $\sigma(K)$ is unbdd in $\mathbb{C}$, contra!



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