Recall spectral radius of $T \in \mathcal{L}(X)$: $\quad r(T) := \sup_{\lambda \in \sigma(T)} |\lambda| \geq 0$.

We know $r(T) \leq \|T\|$, not $=$ in general, e.g. $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2)$,

$r(T) = 0 < \|T\|$. OTOH if $T : \mathbb{C}^n \to \mathbb{C}^n$ is self-adjoint,

then $r(T) = \|T\|$.

Thm: For a Hilbert space $\mathcal{H}$ & self-adj. $T \in \mathcal{L}(\mathcal{H})$, $r(T) = \|T\|$

Lemma needed from cpx. an. (see discussion of Cauchy integral formula in Bühler-Salamon)

For a cpx Banach sp. X & holomorphic fn. $\mathbb{C} \overset{open}{\supseteq} \Omega \xrightarrow{f} \mathcal{L}(X)$

$\forall z_0 \in \Omega$, $\exists$ a power series $\sum_{n=0}^{\infty} (z - z_0)^n a_n \quad (a_n \in \mathcal{L}(X))$, that is

equal to f on its circle of convergence $B_R(z_0) \subseteq \mathbb{C}$, where

$$R = \frac{1}{\limsup_{n\to\infty} \|a_n\|^{1/n}} = \sup\{ r > 0 \mid B_r(z_0) \subseteq \Omega \}.$$

Now for $T \in \mathcal{L}(X)$, $R_\lambda(T) = (\lambda - T)^{-1} = [\lambda(1 - \frac{1}{\lambda}T)]^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} T^k$

for $|\lambda| > \|T\|$. Let $z = \frac{1}{\lambda}$, then $z \longmapsto R_\lambda(T)$ is a hol. fn.

of $z \in \mathbb{C}$ for $\{|\lambda| > r(T)\} = \{ |z| < \frac{1}{r(T)} \}$ $\qquad = z \sum_{k=0}^{\infty} z^k T^k$

$\Rightarrow$ radius of convergence $= \frac{1}{r(T)}$, where $r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}$.

Lemma: For $T : \mathcal{H} \to \mathcal{H}$ self-adj., $\|T^2\| = \|T\|^2$.

Pf: In general, $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \|\Lambda_{Tx}\| = \sup_{\|x\|=\|y\|=1} |\Lambda_{Tx}(y)|$

$= \sup_{\|x\|=\|y\|} |\langle Tx, y \rangle|$. This implies,

$\|T^*T\| = \sup_{x,y} |\langle T^*Tx, y\rangle| = \sup_{x,y} |\langle Tx, Ty\rangle| \leq \|T\|^2$.

Pick a seq. $x_n \in \mathcal{H}$ s.t. $\|x_n\| = 1$ s.t. $\|Tx_n\| \to \|T\|$,

then $|\langle Tx_n, Tx_n\rangle| = \|Tx_n\|^2 \to \|T\|^2 \Rightarrow \|T^*T\| = \|T\|^2$.

For T self-adj., $T^*T = T^2$. $\square$

Pf of spectral radius thm: $T : \mathcal{H} \to \mathcal{H}$ self-adj., $r(T) = \limsup_{n\to\infty} \|T^n\|^{1/n}$.

Since $\|T^n\| \leq \|T\|^n$, RHS $\leq \|T\|$. Observe: for $n = 2k$, T^k self-adj.

$\Rightarrow \|T^{2k}\| = \|T^k\|^2 \Rightarrow$ by induction, $\|T^{2^n}\| = \|T\|^{2^n} \Rightarrow \limsup = \|T\|$.

$\square$

Hilbert-Schmidt thm: Assume $A : \mathcal{H} \to \mathcal{H}$ cpct & self-adj. Then $\mathcal{H}$ admits an O-N basis $\{e_\alpha\}_{\alpha \in I}$ s.t. $Ae_\alpha = \lambda_\alpha e_\alpha$ for some $\lambda_\alpha \in \mathbb{R}$.

rk: Riesz-Schauder $\Rightarrow$ $\sigma(A) \setminus \{0\}$ = discrete e-vals of finite mult. $\Rightarrow$ $\{\alpha \in I \mid \lambda_\alpha \neq 0\}$ is at most a countable set $\{\alpha_n \in I \mid n \in \mathbb{N}\}$; if infinite, $\lim_{n\to\infty} \lambda_{\alpha_n} = 0$.

Pf of thm: $\forall$ e-val. $\lambda \in \mathbb{R}$, choose O-N basis $E_\lambda := \ker(A - \lambda)$

$\rightsquigarrow$ O-N set spanning some closed subspace $E \subseteq \mathcal{H}$, $A(E) \subseteq E$.

Then $v \in E^\perp$, $\Rightarrow$ $\forall\ x \in E$, $\langle x, Av\rangle = \langle \underset{E}{Ax}, v\rangle = 0 \Rightarrow A(E^\perp) \subseteq E^\perp$

$\Rightarrow A|_{E^\perp} \in \mathcal{L}(E^\perp)$ is a cpct self-adj. op. on $E^\perp$ with no eigenvalue.

$\Rightarrow r(A|_{E^\perp}) = 0 = \|A|_{E^\perp}\| \Rightarrow A|_{E^\perp} = 0$, contra! unless $E^\perp = \{0\}$. ☒

ex 1: For $F : \mathbb{T}^n \xrightarrow{L^2} \mathbb{R}$ s.t. $F(x) = F(-x)$, let $T : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n)$, $u \longmapsto F * u$

In terms of Fourier series, $\widehat{Tu}_k = \hat{F}_k \hat{u}_k$ for $k \in \mathbb{Z}^n$,

$F \in L^2(\mathbb{T}^n) \Rightarrow \hat{F} \in \ell^2(\mathbb{Z}^n) \Rightarrow |\hat{F}_k| \longrightarrow 0$ as $|k| \to \infty$,

$$\hat{F}_k = \int_{\mathbb{T}^n} e^{-2\pi i k \cdot x} F(x)\, dx = \int_{\mathbb{T}^n} e^{+2\pi i k\cdot x} F(-x)\, dx = \overline{\hat{F}_k} \Rightarrow \hat{F}_k \in \mathbb{R}$$

$$\Rightarrow \langle u, Tv \rangle_{L^2} = \sum_{k \in \mathbb{Z}^n} \langle \hat{u}_k, \hat{F}_k \hat{v}_k \rangle = \sum_{k \in \mathbb{Z}^n} \langle \hat{F}_k \hat{u}_k, \hat{v}_k \rangle = \langle Tu, v \rangle_{L^2}$$

$\Rightarrow T$ is self-adj. also cpct (see PSET 11).

Fns $u \in L^2(\mathbb{T}^n)$ with only 1 nonzero Fourier coeff. are eigenvectors:

$T(e^{2\pi i k \cdot x}) = \hat{F}_k \cdot e^{2\pi i k \cdot x}$, $\{e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^n}$ is an O-N basis of e-vecs.

ex 2: $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an iso. $\forall\, \Omega \subseteq \mathbb{R}^n$ open + bdd.

Inclusion $H_0^1(\Omega) \xhookrightarrow{j} L^2(\Omega)$ is cpct (see PSET 10).

Let $T := j \Delta^{-1}|_{L^2(\Omega)} : L^2(\Omega) \longrightarrow L^2(\Omega)$; j cpct $\Rightarrow T$ cpct. + injective

EX (via integ. by parts): T is self-adjoint.

$\Rightarrow L^2(\Omega)$ has an O-N basis of e-fns. $\{f_k \in L^2(\Omega)\}_{k \in \mathbb{N}}$,

$T f_k = \mu_k f_k$ for $\mu_k \in \mathbb{R} \setminus \{0\}$, $\lim_{k \to \infty} \mu_k = 0$.

Observe: $T f_k = \mu_k f_k = j \Delta^{-1}(f_k) \Leftrightarrow \mu_k f_k \in H_0^1(\Omega)$ & $\Delta(\mu_k f_k)$

$= \mu_k \Delta f_k = f_k$, i.e. $\Delta f_k = \lambda_k f_k$ for $\lambda_k := \frac{1}{\mu_k} \in \mathbb{R} \setminus \{0\}$ s.t. $|\lambda_k| \to \infty$.

Cor: $\forall$ open bdd $\Omega \subseteq \mathbb{R}^n$, $L^2(\Omega)$ admits an O-N basis consisting of "eigenfunctions" of Δ of class $H_0^1(\Omega)$, with e-vals. that accumulate only at ∞. $\boxtimes$

<u>thm</u> (singular value decomposition): $A : \mathcal{H} \to \mathcal{H}$ cpct, $\dim \mathcal{H} = \infty$,

then $\exists$ (not necessarily complete) O-N sets $\{\varphi_n\}_{n \in \mathbb{N}}$, $\{\psi_n\}_{n \in \mathbb{N}}$

& numbers $\{\lambda_n \geq 0\}_{n \in \mathbb{N}}$ s.t. $\lambda_n \to 0$ & $Af = \sum_{n=1}^{\infty} \lambda_n \langle \psi_n, f \rangle \varphi_n$.

(<u>rk</u>: All operators of this form can be approximated by ops. w/ finite rank $\Rightarrow$ they are cpct.)

Pf: $A^*A : \mathcal{H} \to \mathcal{H}$ is cpct & self-adj. $\Rightarrow \exists$ O-N set $\{\psi_n\}_{n \in \mathbb{N}}$ s.t.

$A^*A \psi_n = \mu_n \psi_n$ for some $\mu_n \in \mathbb{R} \setminus \{0\}$ & $A^*A\big|_{\mathrm{Span}\{\psi_n\}^\perp} = 0$

$\Rightarrow A\big|_{\mathrm{Span}\{\psi_n\}^\perp} = 0$. $\quad \mu_n = \langle \psi_n, A^*A\psi_n \rangle = \|A\psi_n\|^2 > 0$

$\Rightarrow$ can dfn. $\lambda_n := \sqrt{\mu_n} > 0$. Set $\varphi_n := \frac{1}{\lambda_n} A\psi_n$.

For $m, n \in \mathbb{N}$, $\langle \varphi_m, \varphi_n \rangle = \frac{1}{\lambda_m \lambda_n} \langle A\psi_m, A\psi_n \rangle = \frac{1}{\lambda_m \lambda_n} \langle \psi_m, A^*A\psi_n \rangle$

$= \frac{\mu_n}{\lambda_m \lambda_n} \langle \psi_m, \psi_n \rangle = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \Rightarrow \{\varphi_n\}$ is O-N.

Choose $f \in \mathcal{H}$ & check $\langle \varphi_n, Af \rangle = \lambda_n \langle \psi_n, f \rangle$. $\boxed{//}$

self-adjoint (but not cpt) operators

<u>ex</u>: $I := [a,b] \subseteq \mathbb{R}$, $\quad T : L^2(I) \to L^2(I)$, $\quad Tf(x) := x f(x)$.

This is bdd on L^2 since $[a,b] \to \mathbb{R} : x \mapsto x$ is a bdd fn.

$$\langle f, Tg \rangle_{L^2} = \int_a^b \langle f(x), x g(x) \rangle \, dx = \int_a^b \langle x f(x), g(x) \rangle \, dx = \langle Tf, g \rangle_{L^2}$$

$\Rightarrow$ T is self-adj.

<u>Eigenvals</u>? $(T - \lambda) f = 0 \iff (x - \lambda) f(x) = 0$ for almost every $x \in [a,b]$

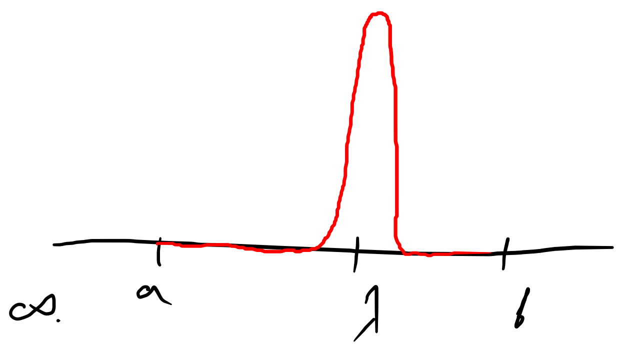
$\iff f = 0$ a.e. $\Rightarrow \nexists$ eigenvals!

<u>Approximate e-vals?</u> For $\lambda \in [a,b]$, choose seq. $f_n : [a,b] \to [0,\infty)$

s.t. $\|f_n\|_{L^2} = 1$ but $\operatorname{supp}(f_n) \subseteq (\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$.

Then $\|(T - \lambda) f_n\|_{L^2}^2 = \int_a^b (x - \lambda)^2 |f_n(x)|^2 \, dx$

$$= \int_{\lambda - \frac{1}{n}}^{\lambda + \frac{1}{n}} (x - \lambda)^2 |f_n(x)|^2 \, dx \le \frac{1}{n^2} \|f_n\|_{L^2}^2 = \frac{1}{n^2} \to 0 \text{ as } n \to \infty.$$



$\Rightarrow [a,b] \subseteq \sigma(T)$.

<u>Q</u>: How to "diagonalize" an op. with no e-vecs?

<u>A</u>: T is already "diagonal".

<u>spectral thm for bdd self-adj. ops.</u>: $T \in \mathcal{L}(\mathcal{H})$ self-adj. $\Rightarrow$

$\exists$ a finite measure space (X, μ), bdd fn. $F : X \to \mathbb{R}$,

& a unitary iso. $U : \mathcal{H} \to \underbrace{L^2(X,\mu)}_{\text{real-val'd fns}}$ s.t.

$$U T U^{-1} : L^2(X,\mu) \to L^2(X,\mu) : g \mapsto Fg.$$