diagonalization

a spectral representation of $A \in \mathcal{L}(\mathcal{H})$ consists of a (σ-finite) measure space (X,μ), bdd measurable fn. $F : X \to \mathbb{C}$ & unitary iso.

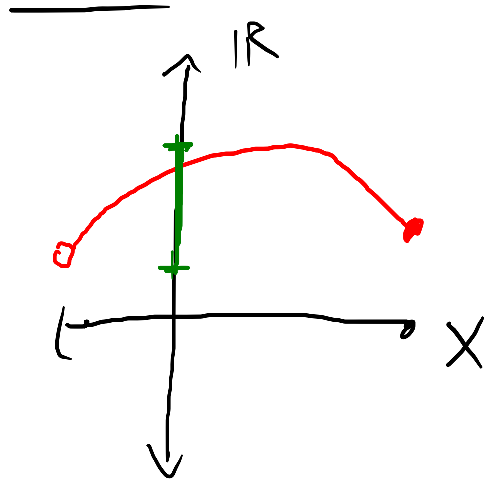
$U : \mathcal{H} \to \underbrace{L^2(X,\mu)}_{\text{cpx-valued fns}}$ s.t. $UAU^{-1} = T_F : L^2(X,\mu) \to L^2(X,\mu)$

$u \mapsto Fu$.

rk: $U(\lambda - A)U^{-1} = \lambda - T_F \Rightarrow \sigma(A) = \sigma(T_F)$.

EX (PSET 11): $\sigma(T_F) =$ the "essential range" of $F : X \to \mathbb{C}$

$:= \{ \lambda \in \mathbb{C} \mid \mu(F^{-1}(B_\varepsilon(\lambda))) > 0 \quad \forall \varepsilon > 0 \}$



rk: $\langle u, T_F v\rangle_{L^2} = \langle u, Fv\rangle_{L^2} = \langle \bar{F} u, v\rangle_{L^2} = \langle T_{\bar{F}} u, v\rangle_{L^2}$

$\Rightarrow T_F^* = T_{\bar{F}}$.

$\Rightarrow$ (i) T_F is self-adj. $\Leftrightarrow F(X) \subseteq \mathbb{R}$.

(ii) T_F is unitary $\Leftrightarrow F(X) \subseteq S^1 := \{ |\lambda| = 1 \} \subseteq \mathbb{C}$.

In general $T_F T_F^* = T_{F\bar{F}} = T_{\bar{F}F} = T_F^* T_F \Rightarrow T_F$ is a normal op.

(i.e. commutes w/ its adjoint). $\Rightarrow$ Only normal ops. can admit a spectral repr.

big thm: If $\mathcal{H}$ is separable, $A \in \mathcal{L}(\mathcal{H})$ is normal $\Leftrightarrow$ admits a spectral repr., & one can always assume $\mu(X) < \infty$.

We'll prove this for A self-adj: then $F(X) \subseteq \mathbb{R}$.

strategy:

(1) $A \in \mathcal{L}(\mathcal{H})$ self-adj. & $x \in \mathcal{H}$ $\rightsquigarrow$ pos. linear fnl $\Lambda : C^0(\sigma(A)) \longrightarrow \mathbb{C}$
$f \longmapsto \langle x, f(A) x \rangle$

(2) Riesz-Markov thm $\Rightarrow$ $\Lambda(f) = \int_{\sigma(A)} f \, d\mu_x$ for some measure μ_x on $\sigma(A)$.

Construct (X, μ) out of $(\sigma(A), \mu_{x_i})$ for some subset $\{x_1, x_2, x_3, \dots\} \subseteq \mathcal{H}$.

step 1: Continuous functional calculus

$\{f : \sigma(A) \xrightarrow{C^0} \mathbb{C}\}$ with $\|\cdot\|_{C^0}$

thm: If $A \in \mathcal{L}(\mathcal{H})$ self adj., then $\exists!$ bdd lin. map $C(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H})$ s.t.
$f \longmapsto f(A)$

"algebraic *-homomorphism"
(i) $(fg)(A) = f(A) g(A)$, $\quad \bar{f}(A) = f(A)^*$
(ii) For $f(x) = \lambda$ const $\in \mathbb{C}$, $\quad f(A) = \lambda \cdot \mathrm{Id} \in \mathcal{L}(\mathcal{H})$
(iii) For $f(x) = x$, $\quad f(A) = A$.

It also has following properties:

(iv) $\sigma(f(A)) = f(\sigma(A))$; in particular $Ax = \lambda x \Rightarrow f(A)x = f(\lambda)x$.

(v) If $f(\sigma(A)) \subseteq [0, \infty)$, then $f(A)$ is pos-semidef., i.e. $\langle x, f(A)x \rangle \geq 0$ $\forall x \in \mathcal{H}$.

(vi) $\|f(A)\| = \|f\|_{C^0}$.

Pf: (i)–(iii) determine $P(A)$ $\forall$ polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ $\quad (a_k \in \mathbb{C})$,
namely $P(A) = \sum_{k=0}^{n} a_k A^k$. $\quad \sigma(A)$ is closed + bdd $\Rightarrow$ cpt,
Weierstrass $\Rightarrow$ these fns. are dense in $C(\sigma(A))$.

Main task: show $\|P(A)\| = \|P\|_{C^0(\sigma(A))} = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$. $\quad (*)$

The density $\Rightarrow$ $\exists!$ extension $C(\sigma(A)) \longrightarrow \mathcal{L}(H) : f \longmapsto f(A)$.

Notice: $P(A)^* = \bar{P}(A) \Rightarrow P(A)P(A)^* = (P\bar{P})(A) = (\bar{P}P)(A) = P(A)^* P(A)$
$\Rightarrow P(A)$ is normal. PSET II #1 $\Rightarrow$ $\|P(A)\| =$ spectral radius $r(P(A))$
$= \sup_{\lambda \in \sigma(P(A))} |\lambda|$

$(*)$ then follows from

Lemma: $\forall$ $A \in \mathcal{L}(\mathcal{H})$ & cpx polyn. P, $\sigma(P(A)) = P(\sigma(A))$.

Pf: Let $\lambda \in \sigma(A)$, write $P(x) - P(\lambda) = (x - \lambda) Q(x)$ for some polyn. Q
$\Rightarrow P(A) - P(\lambda) = (A - \lambda) Q(A) = Q(A)(A - \lambda) \Rightarrow P(A) - P(\lambda)$ not invtbl $\Rightarrow P(\lambda) \in \sigma(P(A))$.

Let $\mu \in \sigma(P(A))$, write $P(x) - \mu = a(x - \lambda_1) \dots (x - \lambda_n)$ for some $a, \lambda_1, \dots, \lambda_n \in \mathbb{C}$,
then $P(A) - \mu = a(A - \lambda_1) \dots (A - \lambda_n)$, so $P(A) - \mu$ not invtbl $\Rightarrow$ at least
one of the $(A - \lambda_i)$ is not invtbl $\Rightarrow \lambda_i \in \sigma(A)$, $P(\lambda_i) = \mu \in P(\sigma(A))$. $\boxtimes$

$\Rightarrow \exists$ extension $C(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H}) : f \longmapsto f(A)$, & it satisfies (i) - (iii) + (iv) + (vi).

<u>pf of (v)</u>: If $f : \sigma(A) \longrightarrow [0,\infty)$, then $f = g^2$ for some real-val'd $g \in C(\sigma(A))$

$\Rightarrow f(A) = g(A)^2$. $g(A)$ is self-adj., $\langle x, f(A)x\rangle = \langle x, g(A)g(A)x\rangle$

$= \langle g(A)x, g(A)x\rangle \geq 0$. $\square$

<u>step 2</u>: Choose $x \in \mathcal{H}$ & consider fnal $\Lambda : C(\sigma(A)) \longrightarrow \mathbb{C}$

$f \longmapsto \langle x, f(A)x\rangle$.

Λ is linear, $|\Lambda(f)| \leq \|x\|^2 \cdot \|f(A)\|$

$= \|x\|^2 \cdot \|f\|_{C^0} \Rightarrow$ bdd

Λ is also <u>positive</u>: $f \geq 0 \Rightarrow \Lambda(f) \geq 0$.

<u>Riesz-Markov thm</u>: For X a cpct Hausdorff top. space &

$\Lambda : C(X) \longrightarrow \mathbb{C}$ a pos. lin. fnal, $\exists!$ regular finite measure μ

on the Borel sets of X s.t. $\Lambda(f) = \int_X f\, d\mu$.

<u>pf sketch</u>: Regular $\Rightarrow \mu$ is uniquely det'd by its values on cpct sets $K \subset X$.

Set $\mu(K) := \inf\{\Lambda(f) \mid f : X \xrightarrow{C^0} [0,\infty) \text{ s.t. } f \geq \chi_K\}$,

check that this defns. a measure on X. $\square$

(<u>ref</u>: Salomon, "Measure theory")

cor (spectral measures): $\forall\ x \in \mathcal{H}$ & $A \in \mathcal{L}(\mathcal{H})$ self-adjoint, $\exists !$ finite regular Borel measure μ_x on $\sigma(A) \subseteq \mathbb{R}$ s.t. $\forall\ f \in C(\sigma(A))$,

$$\langle x, f(A)\, x\rangle = \int_{\sigma(A)} f\, d\mu_x .$$

$\square$

defn: $x \in \mathcal{H}$ is cyclic for $A \in \mathcal{L}(\mathcal{H})$ if the subspace spanned by $\{x, Ax, A^2x, A^3x, \dots\} \subseteq \mathcal{H}$ is dense.

Lemma: If $A \in \mathcal{L}(\mathcal{H})$ is self-adj. & $x \in \mathcal{H}$ is cyclic for A, then A has a spectral rep. on $(X, \mu) := (\sigma(A), \mu_x)$ with $F : \sigma(A) \to \mathbb{R} : \lambda \mapsto \lambda$.

Pf: Defn. $T : C(\sigma(A)) \to \mathcal{H} : f \mapsto f(A)\, x$. Then $\operatorname{im} T$ contains $A^n x\ \forall\ n \geq 0 \Rightarrow \operatorname{im} T \subseteq \mathcal{H}$ is dense. For $f \in C(\sigma(A))$,

$$\|Tf\|^2 = \langle f(A)\,x, f(A)\,x\rangle = \langle x, \bar{f}(A) f(A)\, x\rangle = \langle x, |f|^2(A)\, x\rangle$$

$$= \int_{\sigma(A)} |f|^2\, d\mu_x = \|f\|_{L^2}^2 . \qquad C(\sigma(A)) \subseteq L^2(\sigma(A), \mu_x)$$

is dense $\Rightarrow T$ has ! extension to an isometry $T : L^2(\sigma(A), \mu_x) \to \mathcal{H}$, extension has closed image $\Rightarrow T$ surjective, i.e. T is unitary.

Defn $U := T^{-1} : \mathcal{H} \to L^2(\sigma(A), \mu_x)$. Remaining to show: for $F(\lambda) = \lambda$ on $\sigma(A)$, $\quad UAU^{-1}f = Ff \quad \forall\ f \in L^2(\sigma(A), \mu_x)$.

Density $\Rightarrow$ suff. to prove $\forall\ f \in C(\sigma(A))$.

claim: $\forall\ f \in C(\sigma(A))$, $ATf = T(Ff)$.

Pf: $T(Ff) = (Ff)(A)\, x = F(A) f(A)\, x = A f(A)\, x = ATf$. $\square$

<u>Lemma</u>: $\mathcal{H}$ separable & $A \in \mathcal{L}(\mathcal{H})$ self-adj. $\Longrightarrow$ $\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$ $\left(\begin{array}{l}\text{for some} \\ N \in \mathbb{N} \cup \{\infty\}\end{array}\right)$

for closed subspaces $\mathcal{H}_n \subseteq \mathcal{H}$ s.t.

(i) $\mathcal{H}_n \perp \mathcal{H}_m \quad \forall\, n \neq m$

(ii) $A(\mathcal{H}_n) \subseteq \mathcal{H}_n \quad \forall\, n$

(iii) $\forall\, n$, $\exists\, x_n \in \mathcal{H}_n$ cyclic for $A|_{\mathcal{H}_n}$.

Pf: Span $\{y_1, y_2, \dots\} \subseteq \mathcal{H}$ dense. Set $x_1 := y_1$, $\mathcal{H}_1 := \overline{\mathrm{Span}}\{x_1, Ax_1, A^2x_1, \dots\}$,

if $\mathcal{H}_1 \neq \mathcal{H}$, set $x_2' := y_j$ for $j := \min\{n \in \mathbb{N} \mid y_n \notin \mathcal{H}_1\}$,

then $x_2' \in x_2 + \mathcal{H}_1$ for a ! $x_2 \in \mathcal{H}_1^{\perp}$, set $\mathcal{H}_2 := \overline{\mathrm{Span}}\{x_2, Ax_2, A^2x_2, \dots\}$,

note that $A(\mathcal{H}_1) \subseteq \mathcal{H}_1 \Longrightarrow$ (since A self-adj.), $A(\mathcal{H}_1^{\perp}) \subseteq \mathcal{H}_1^{\perp}$, $\Longrightarrow$

$\mathcal{H}_2 \perp \mathcal{H}_1$. If $\mathcal{H}_1 \oplus \mathcal{H}_2 \neq \mathcal{H}$, set $x_3' := y_j$ for $j = \min\{n \in \mathbb{N} \mid y_n \notin \mathcal{H}_1 \oplus \mathcal{H}_2\}$

continue ...

<u>Spectral repr. for $A : \mathcal{H} \to \mathcal{H}$ self-adj. if $\mathcal{H}$ separable:</u>

Write $\mathcal{H} = \bigoplus_{n=1}^{N} \mathcal{H}_n$, $x_n \in \mathcal{H}_n$ cyclic for $A|_{\mathcal{H}_n}$. Rescale x_n s.t. WLOG,

$\|x_n\| = \frac{1}{2^n}$. Then $\mu_{x_n}(\sigma(A)) = \int_{\sigma(A)} 1 \, d\mu_{x_n} = \langle x_n, 1(A) x_n \rangle = \langle x_n, x_n \rangle = \frac{1}{2^{n+1}}$

$\Longrightarrow \sum_n \mu_{x_n}(\sigma(A)) < \infty$.

Identify $\mathcal{H}_n$ with $L^2(\sigma(A), \mu_{x_n}) \Longrightarrow \mathcal{H} = \bigoplus_n \mathcal{H}_n \mathrel{\hat{=}} \bigoplus_n L^2(\sigma(A), \mu_{x_n})$

$= L^2\big((\sigma(A), \mu_{x_1}) \amalg (\sigma(A), \mu_{x_2}) \amalg \dots \big)$,

$F : \sigma(A) \amalg \sigma(A) \amalg \dots \longrightarrow \mathbb{R}$ def'd on each copy as $\lambda \longmapsto \lambda$. $\square$