<u>Recall</u>: $L^2(\mathbb{R}^n) \supseteq \mathcal{S}(\mathbb{R}^n) \xrightarrow{i\partial_j} L^2(\mathbb{R}^n)$ is essentially self-adj.

<u>Q</u>: What is $\mathcal{D}(\overline{i\partial_j})$?

<u>A</u>: Use $\mathcal{F}: L^2 \to L^2$ as spectral rep: $\mathcal{F}(\mathcal{S}) = \mathcal{S}$

$$\mathcal{F}(i\partial_j)\mathcal{F}^{-1} = T_{2\pi p_j} : L^2 \to L^2 : u \mapsto 2\pi p_j \cdot u$$

This has a self-adj. ext. w/ domain $\mathcal{F}(\mathcal{D}) = \left\{ f \in L^2 \,\middle|\, \begin{array}{l} p \mapsto 2\pi p_j f(p) \\ \text{is in } L^2(\mathbb{R}^n) \end{array} \right\}$

$\Rightarrow \mathcal{D} = \{ f \in L^2 \mid \exists \text{ weak deriv. } \partial_j f \in L^2 \} \supseteq H^1(\mathbb{R}^n)$.

<u>Q</u>: Why, in a nutshell, is the spectral thm. for unbdd ops. true?

<u>A</u>: Self-adj. $\Rightarrow$ symmetric & $\sigma(A) \subseteq \mathbb{R}$ $\Rightarrow$ $\pm i \notin \sigma(A)$ $\Rightarrow$ $T_\pm := R_{\pm i}(A)$ are both bdd normal ops., spectral rep. of $T_+ \rightsquigarrow$ spectral rep. of A.

<u>Q</u>: What if $T_\pm$ are cpct?

<u>defn</u>: $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ has <u>compact resolvent</u> if $\exists\ \lambda \in \mathbb{C} \setminus \sigma(A)$ s.t.

$R_\lambda(A): \mathcal{H} \to \mathcal{H} : x \mapsto (\lambda - A)^{-1} x$ is cpct.

<u>ex</u>: Any $L^2(\mathbb{T}^n) \supseteq H^s(\mathbb{T}^n) \xrightarrow{A} L^2(\mathbb{T}^n)$ with $s > 0$ & $\sigma(A) \neq \mathbb{C}$,

Then for $\lambda \notin \sigma(A)$,
$$\begin{tikzcd} L^2(\mathbb{T}^n) \arrow[r, "(\lambda - A)^{-1}"] \arrow[rr, bend right, "R_\lambda(A)"'] & H^s(\mathbb{T}^n) \arrow[r, hook, "\text{compact}"] & L^2(\mathbb{T}^n) \end{tikzcd}$$

If $A \in \mathcal{L}(H^s(\mathbb{T}^n), L^2(\mathbb{T}^n))$,

then IMT $\Rightarrow (\lambda - A)^{-1} \in \mathcal{L}(L^2, H^s)$, $\Rightarrow R_\lambda(A)$ is compact.

Observe: A self-adj. $\Rightarrow \sigma(A) \subseteq \mathbb{R}$, i.e. $\neq \mathbb{C}$.

$\Rightarrow$ thm: If $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ is self-adj. w/ cpct resolvent, then its spectrum consists of countably many e-vals $\{\lambda_n \in \mathbb{R}\}_{n \in \mathbb{N}}$ s.t. $|\lambda_n| \to \infty$ as $n \to \infty$, each of finite multiplicity, & $\mathcal{H}$ admits an O-N basis of e-vectors of A.

pf: Spse $\mu \in \mathbb{C} \setminus \sigma(A)$, $R_\mu(A) \in \mathcal{L}(\mathcal{H})$ cpct, then since $R_\mu(A)$ is cpct & normal, spectral thm for cpct normal ops $\Rightarrow$ $\exists$ O-N basis of e-vecs of $R_\mu(A)$ with e-val. $\{\alpha_n \in \mathbb{C}\}_{n \in \mathbb{N}}$ s.t. $|\alpha_n| \to 0$. $R_\mu(A)$ inj $\Rightarrow$ 0 not an e-val. of $R_\mu(A)$.

$R_\mu(A) v = \alpha v \iff v = \alpha(\mu - A) v \iff \frac{1}{\alpha} v = \mu v - A v$

$\iff Av = \left(\mu - \frac{1}{\alpha}\right) v$, so v is an e-vec. of A w/ e-val. $\lambda := \mu - \frac{1}{\alpha}$. $\lambda_n := \mu - \frac{1}{\alpha_n}$, then $|\lambda_n| \to \infty$. ☐

ex 1: $L^2(\mathbb{T}^n) \supseteq H^2(\mathbb{T}^n) \xrightarrow{-\Delta} L^2(\mathbb{T}^n)$ ⇝ O-N basis $\{e^{2\pi i k \cdot x}\}_{k \in \mathbb{Z}^n}$,

$-\Delta e^{2\pi i k\cdot x} = 4\pi^2 |k|^2 e^{2\pi i k \cdot x}$.

ex 2: On PSET 1, we studied bndry val. problem $\begin{cases} \ddot{x} + Px = f & \text{on } [0,1] \\ x(0) = x(1) = 0 \end{cases}$

for given $P, f \in C^0([0,1])$. If $\|P\|_{C^0}$ suff. small, $\exists!$ sol. $x \in C^2([0,1])$ $\forall f \in C^0([0,1])$.

PSET 12 #5: $\mathcal{D} := \{x \in C^1([0,1]) \mid \dot{x} \text{ abs contin. } \& \ \ddot{x} \in L^2([0,1])\}$

$L^2([0,1]) \supseteq \mathcal{D} \xrightarrow{Tx := \ddot{x} + Px} L^2([0,1])$ is self-adj. $\forall$ real-val'd $P \in L^2([0,1])$.

claim: T has cpct resolvent.

pf: Pick any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, so $\lambda \notin \sigma(T)$, $\lambda - T : \mathcal{D} \to L^2$ bijective.

Define norm on $\mathcal{D}$ by $\|x\|_{\mathcal{D}} := \|x\|_{C^1} + \|\ddot{x}\|_{L^2}$.

EASY EX: $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$ is a Banach sp. & $\lambda - T$ is a bdd op. $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) \to L^2$.

Inclusion $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}) \hookrightarrow L^2$ is cpct: $x_k \in \mathcal{D}$ w/ $\|x_k\|_{\mathcal{D}} \le C$

$\Rightarrow \|x_k\|_{C^1}$ bdd $\overset{(A.-A.)}{\Longrightarrow}$ $\exists$ C^0-conv. subseq $\Rightarrow$ L^2-conv. ☐

cor: $L^2([0,1])$ has O-N basis of fns $x_\lambda \in \mathcal{D}$ satisfying

$\begin{cases} \ddot{x}_\lambda + (P - \lambda) x_\lambda = 0 \\ x_\lambda(0) = x_\lambda(1) = 0 \end{cases}$ for a countable discrete set e-vals. $\lambda \in \mathbb{R}$.

For any λ not an e-val., the problem

$\begin{cases} \ddot{x} + (P - \lambda) x = f \\ x(0) = x(1) = 0 \end{cases}$ has a ! sol. $x \in \mathcal{D}$ $\forall f \in L^2$.

Notice: if $P, f \in C^0$, $\Rightarrow x \in C^2$. ☐

functional calculus

$\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ self-adj. ⇝ spectral rep. $UAU^{-1} = T_F$ for some $F : X \to \mathbb{R}$,

WLOG $F(X) \subseteq \sigma(A)$ ⇒ can defn. $*$-alg. hom.

$\mathcal{B}(\sigma(A)) \longrightarrow \mathcal{L}(\mathcal{H}) : f \longmapsto f(A) := U^{-1} T_{f \circ F} U$ s.t.

$\mathcal{B}(\sigma(A)) \ni f_n \xrightarrow{p.w.} f$ & $|f_n|$ unif bdd ⇒ $f_n(A)x \longrightarrow f(A)x \ \forall x \in \mathcal{H}$.

<u>Caution</u>: $\sigma(A) \subseteq \mathbb{R}$ unbdd ⇒ $\mathrm{Id} \notin \mathcal{B}(\sigma(A))$; similarly $A \notin \mathcal{L}(\mathcal{H})$, but can prove via dominated conv: if $\mathcal{B}(\sigma(A)) \ni f_n$ s.t. $f_n(\lambda) \longrightarrow \lambda$ & $|f_n(\lambda)| \le |\lambda| \ \forall \lambda, n$, then $f_n(A)x \longrightarrow Ax \ \forall x \in \mathcal{D}$.

<u>ex</u>: $f_t(\lambda) := e^{it\lambda}$ for $t \in \mathbb{R}$, then $f_t \in \mathcal{B}(\mathbb{R})$, $f_t(\mathbb{R}) \subseteq S^1 \ \forall t$, f_t is p.w. contin. wrt. t, $f_s f_t = f_{s+t}$, $f_0 \equiv 1$

⇒ $U(t) := f_t(A) = e^{itA} \in \mathcal{L}(\mathcal{H})$ defines a "strongly continuous 1-parameter unitary group":

$U(t) \in \mathcal{U}(\mathcal{H}) := \{\text{unitary ops. } \mathcal{H} \to \mathcal{H}\} \ \forall t$, $U(s)U(t) = U(s+t)$,

$U(0) = \mathrm{Id}$, $U(t)x \longrightarrow U(t_0)x$ as $t \to t_0 \ \forall x \in \mathcal{H}$.

Recall PSET 12 #7(j): $U \in \mathcal{L}(\mathcal{H})$ is unitary ⇔ $U = e^{iA}$ for some $A \in \mathcal{L}(\mathcal{H})$ self-adj.

<u>Can show</u>: $\mathcal{U}(\mathcal{H})$ is a smooth "Banach submanifold" of $\mathcal{L}(\mathcal{H})$ with tangent space $T_{\mathrm{Id}}\,\mathcal{U}(\mathcal{H}) = \{ iA \in \mathcal{L}(\mathcal{H}) \mid A \text{ self-adj.} \}$, & the map

$$\{A \in \mathcal{L}(\mathcal{H}) \mid A \text{ self-adj.}\} \longrightarrow \mathcal{U}(\mathcal{H}), \quad A \longmapsto e^{iA}$$

defines a diffeomorphism from a nbhd of 0 to a nbhd of Id.

Then any "norm continuous" 1-parameter unitary group (i.e. contin. map $U : \mathbb{R} \longrightarrow \mathcal{U}(\mathcal{H})$ s.t. is a group hom.) is of the form $U(t) = e^{itA}$ for a ! self-adj. $A \in \mathcal{L}(\mathcal{H})$. (<u>main tool</u>: implicit fn. thm in Banach spaces)

<u>Trouble</u>: $\exists$ natural exs for which $t \longmapsto U(t)$ is <u>not</u> contin. in operator norm.

<u>ex</u>: $U_j(t) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$, $(U_j(t)f)(x) := f(x + te_j)$ for standard basis $e_1, \dots, e_n \in \mathbb{R}^n$. Here $\forall f \in L^2$, $\|U_j(t)f - U_j(t_0)f\|_{L^2} \longrightarrow 0$ as $t \to t_0$, but

$$\sup_{f \neq 0} \frac{\|U_j(t)f - U_j(t_0)f\|_{L^2}}{\|f\|_{L^2}} \not\longrightarrow 0 \text{ as } t \to t_0.$$

<u>Spectral rep</u>:

$$\widehat{U_j(t)f}(p) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x + te_j)\,dx = e^{2\pi i p \cdot te_j} \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x} f(x)\,dx$$

$$= e^{2\pi i p_j t} \hat{f}(p); \quad \text{compare} \quad \widehat{i\partial_j f}(p) = -2\pi p_j \hat{f}(p)$$

⇒ $\boxed{U_j(t) = e^{-it(i\partial_j)} = e^{t\partial_j}}$ "$-i\partial_j$" is the infinitesimal generator of translations in direction e_j.

<u>Stone's thm</u> (1932): Every strongly contin. 1-parameter unitary group is of the form $U(t) = e^{itA}$ for a self-adj. op $\mathcal{H} \supseteq \mathcal{D} \xrightarrow{A} \mathcal{H}$ with dense domain $\mathcal{D} := \{ x \in \mathcal{H} \mid \frac{d}{dt} U(t)x \big|_{t=0} \text{ exists} \}$.