Hilbert spaces (continued)

$(\mathcal{H}, \langle\ ,\ \rangle)$ an inner product space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

convention in case $\mathbb{K} = \mathbb{C}$: $\langle v, \lambda w\rangle = \lambda\langle v, w\rangle = \langle \bar{\lambda} v, w\rangle$ for $v, w \in \mathcal{H}$, $\lambda \in \mathbb{C}$.

$\langle v, w\rangle = \overline{\langle w, v\rangle}$

"standard" interesting ex: (X, μ) a measure space, $(V, \langle\ ,\ \rangle)$ a fin-dim. inner prod. sp over $\mathbb{K}$

$$\mathcal{H} := L^2(X,\mu) = \{\text{fns. } f : X \to V \text{ of class } L^2\} / \begin{matrix} f \sim g \\ \text{iff } f = g \text{ a.e.}\end{matrix}$$

$$\langle f, g\rangle_{L^2} := \int_X \langle f(x), g(x)\rangle \, d\mu(x)$$

e.g. if $V = \mathbb{C}$ & $\langle v, w\rangle := \bar{v} w$

$$\langle f, g\rangle_{L^2} = \int_X \overline{f(x)}\, g(x)\, d\mu(x).$$

EX: For $(X, \mu) = ([0,1], m := \text{Lebesgue measure})$, the $\mathbb{C}$-val'd fns

$\{e_k(t) := e^{2\pi i k t}\}_{k \in \mathbb{Z}}$ form an orthonormal set in $L^2([0,1])$: $\begin{cases} \langle e_k, e_\ell\rangle_{L^2} = 0 & \text{for } k \neq \ell, \\ = 1 & \text{for } k = \ell. \end{cases}$

thm ("Pythagoras"): Spse $X \subseteq \mathcal{H}$ is a fin-dim. subspace w/ O-N basis $e_1, \dots, e_N$ of X; then $\forall\ x \in \mathcal{H}$,

(1) $x = \sum_{j=1}^{N} \langle e_j, x\rangle e_j + y$ for a ! $y \in X^\perp \subseteq \mathcal{H}$,

(2) $\|x\|^2 = \sum_{j=1}^{N} |\langle e_j, x\rangle|^2 + \|y\|^2$.

pf: (1) $y := x - \sum_{j=1}^{N} \langle e_j, x\rangle e_j$, then $\langle e_j, y\rangle = \langle e_j, x\rangle - \langle e_j, x\rangle = 0$

$\Rightarrow y \in X^\perp$.

(2) follows from

prop: If $x = v + w$ where $v \perp w$ (i.e. $\langle v, w\rangle = 0$), $\|x\|^2 = \|v\|^2 + \|w\|^2$.

pf: $\langle v+w, v+w\rangle = \langle v, v\rangle + \langle w, w\rangle$. ☒

thm (Cauchy-Schwarz ineq.): $\forall\ x, y \in \mathcal{H}, \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$

pf: Assume $x, y \neq 0$ (otherwise trivial). Set $X := \mathrm{Span}\{e_1 := \frac{x}{\|x\|}\}$, then

Pythagoras $\Rightarrow \quad y = \left\langle \frac{x}{\|x\|}, y \right\rangle \frac{x}{\|x\|}$ + something ortho. to X

$$\Rightarrow \|y\|^2 \geq \left\| \left\langle \frac{x}{\|x\|}, y \right\rangle \frac{x}{\|x\|} \right\|^2 = \frac{|\langle x, y\rangle|^2}{\|x\|^2}. \qquad \blacksquare$$

corollaries: (1) $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{K}$ is a contin. fn.

pf: $x_n \longrightarrow x,\ y_n \longrightarrow y \implies |\langle x, y\rangle - \langle x_n, y_n\rangle|$

$$\leq |\langle x - x_n, y\rangle| + |\langle x_n, y - y_n\rangle| \leq \underbrace{\|x - x_n\|}_{\searrow 0} \cdot \|y\| + \underbrace{\|x_n\|}_{\text{bdd}} \cdot \underbrace{\|y - y_n\|}_{\searrow 0} \longrightarrow 0. \qquad \blacksquare$$

(2) For any subspace $X \subseteq \mathcal{H}$, $X^\perp \subseteq \mathcal{H}$ is a closed subspace.

pf: $x_n \in X^\perp \iff \langle x_n, v\rangle = 0 \quad \forall\ v \in X$, then $x_n \longrightarrow x \implies$

$$0 = \langle x_n, v\rangle \longrightarrow \langle x, v\rangle \implies x \in X^\perp. \qquad \blacksquare$$

(3) For any subspace $X \subseteq \mathcal{H}$, $X^\perp = (\overline{X})^\perp$ ($\overline{X}$:= closure of X).

pf: $X \subseteq \overline{X} \implies (\overline{X})^\perp \subseteq X^\perp$. To show: if $v \perp X$, then also $v \perp \overline{X}$. Given $x_n \in X$ with $x_n \longrightarrow x \in \overline{X}$, $\langle v, x_n\rangle = 0 \longrightarrow \langle v, x\rangle \implies \langle v, x\rangle = 0. \qquad \blacksquare$

From now on, assume $\mathcal{H}$ complete.

important corollary: A subspace X in a Hilbert sp. $\mathcal{H}$ is dense iff $X^\perp = \{0\}$.

pf: $\Rightarrow$: $\overline{X} = \mathcal{H} \implies X^\perp = \mathcal{H}^\perp = \{0\}$.

$\Leftarrow$: $X^\perp = \{0\} \implies \overline{X}$ is a closed subspace s.t. $\overline{X}^\perp = \{0\}$.

Thm from Thursday says: $\mathcal{H} = \overline{X} \oplus \overline{X}^\perp = \overline{X} \implies X$ is dense. $\blacksquare$

<u>dual space</u>: $\mathcal{H}$ Hilbert, $\mathcal{H}^* = \mathcal{L}(\mathcal{H}, \mathbb{K})$

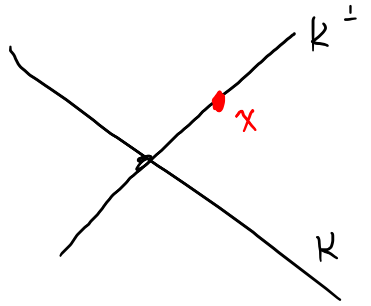
<u>Riesz representation thm</u> (version 1): $\exists$ a real-linear isometric isomorphism

$$\Phi_{\mathcal{H}} : \mathcal{H} \longrightarrow \mathcal{H}^* : x \longmapsto \Lambda_x \quad \text{def'd by} \quad \Lambda_x(y) := \langle x, y \rangle.$$

("isometric" means $\|\Lambda_x\| = \|x\|$. In case $\mathbb{K} = \mathbb{C}$, $\Phi_{\mathcal{H}}$ is cpx-antilinear, i.e. $\Phi_{\mathcal{H}}(\lambda x) = \bar{\lambda}\, \Phi_{\mathcal{H}}(x)$.)

<u>Pf</u>: Cauchy-Schwarz $\Rightarrow |\Lambda_x(y)| = |\langle x, y\rangle| \le \|x\| \cdot \|y\| \Rightarrow \Lambda_x \in \mathcal{H}^*$,
$\|\Lambda_x\| \le \|x\|$. $\quad |\Lambda_x(x)| = \|x\|^2 \Rightarrow \|\Lambda_x\| = \|x\|$.

<u>To prove</u>: $\forall\ \Lambda \in \mathcal{H}^*$, $\exists\ x \in \mathcal{H}$ s.t. $\Lambda(v) = \langle x, v\rangle \ \forall\ v \in \mathcal{H}$.
Assume $\Lambda \neq 0$ (alternative is easy). Then $K := \ker \Lambda \subseteq \mathcal{H}$ is a closed subspace of codim. 1, $\Rightarrow \mathcal{H} = K \oplus K^\perp$.



$K^\perp$ is a 1-dim. subspace of $\mathcal{H}$ $\Rightarrow$
$\exists !\ x \in K^\perp$ s.t. $\langle x, x\rangle = \Lambda(x)$.
(Choose any $y \in K^\perp$, then multiply by a scalar.)
Now $\Lambda_x = \Lambda$ evaluated on x & also on K,
since $K \oplus \underbrace{\operatorname{Span}\{x\}}_{= K^\perp} = \mathcal{H} \Rightarrow \Lambda_x = \Lambda$. $\square$

<u>Cor</u>: For X, Y Hilbert spaces, every $A \in \mathcal{L}(X, Y)$ has a ! <u>adjoint</u>
$A^* \in \mathcal{L}(Y, X)$ s.t. $\langle y, Ax\rangle = \langle A^* y, x\rangle \quad \forall\ x \in X,\ y \in Y$.

<u>Pf</u>: Recall <u>transpose</u> $A^T : Y^* \longrightarrow X^*$, $(A^T \lambda)(x) = \lambda(Ax)$.
$\langle y, Ax\rangle = \Lambda_y(Ax) = (A^T \Lambda_y)(x) = \langle A^* y, x\rangle \iff$
$\Lambda_{A^* y} = A^T \Lambda_y$, i.e.

$$\begin{tikzcd} Y \arrow[r, "\Phi_Y", "\cong"'] \arrow[d, dashed, "A^*"'] \arrow[dr, phantom, "\circlearrowright"] & Y^* \arrow[d, "A^T"] \\ X \arrow[r, "\Phi_X", "\cong"'] & X^* \end{tikzcd}$$

$$\Rightarrow A^* = \Phi_X^{-1} A^T \Phi_Y$$

$$\|A^*\| \le \underbrace{\|\Phi_X^{-1}\|}_{=1} \cdot \|A^T\| \cdot \underbrace{\|\Phi_Y\|}_{=1}$$

$< \infty$. $\square$

defn: Given a nonempty subset $S \subseteq \mathcal{H}$,

$$\mathrm{Span}(S) := \left\{ \sum_{j=1}^{N} \lambda_j e_j \;\middle|\; N \in \mathbb{N},\ e_j \in S,\ \lambda_j \in \mathbb{R},\ j = 1, \dots, N \right\} \subseteq \mathcal{H}.$$

S is orthonormal if $\forall\, e, f \in S$, $\langle e, f \rangle = \begin{cases} 1 & \text{if } e = f \\ 0 & \text{if } e \neq f \end{cases}$.

" an " basis if it is O-N & $\overline{\mathrm{Span}(S)} = \mathcal{H}$.

thm: Every maximal O-N set in a Hilbert space $\mathcal{H}$ is an O-N basis (i.e. an O-N set is maximal $\iff$ it is a basis).

pf: $\Rightarrow$: If $\overline{\mathrm{Span}(S)} \neq \mathcal{H}$, then $\mathrm{Span}(S)^\perp \neq \{0\}$, so $\exists$ a unit vector $v \in \mathcal{H}$ orthogonal to everything in S $\Rightarrow$ $S \cup \{v\}$ is also O-N $\Rightarrow$ S is not maximal.

$\Leftarrow$: Basis means $\overline{\mathrm{Span}(S)} = \mathcal{H}$ $\Rightarrow$ $\mathrm{Span}(S)^\perp = \{0\}$ $\Rightarrow$ $\nexists$ nonzero vectors $\perp$ all $e \in S$ $\Rightarrow$ S is maximal. ☒

thm: If $\mathcal{H}$ has O-N basis $\{e_\alpha\}_{\alpha \in I}$ & $\dim \mathcal{H} = \infty$. Then $\forall\, x \in \mathcal{H}$, the set $I_x := \{\alpha \in I \mid \langle e_\alpha, x \rangle \neq 0\}$ is at most countable, & $x = \sum_{\alpha \in I_x} \langle e_\alpha, x \rangle e_\alpha$ (summation def'd via any choice of ordering for I_x).

pf: $\overline{\{\text{fin. lin. combin. of } e_\alpha\text{'s}\}} = \mathcal{H}$ $\Rightarrow$ $x = \lim_{n \to \infty} x_n$ for $x_n :=$ fin. lin. combin of e_α's $\Rightarrow$ I_{x_n} is finite $\forall n$. $\Rightarrow$ $I_x \subseteq \bigcup_{n=1}^{\infty} I_{x_n}$ is countable.

Choose ordering $I_x = \{\alpha_1, \alpha_2, \alpha_3, \dots\}$ & rewrite $e_j := e_{\alpha_j}$, $j \in \mathbb{N}$.

Let $x_N := \sum_{j=1}^{N} \langle e_j, x \rangle e_j$, want to show $x_N \to x$ as $N \to \infty$.

Pythagoras $\Rightarrow$ $(x - x_N) \perp \mathrm{Span}\{e_1, \dots, e_N\}$ &

$$\|x\|^2 = \sum_{j=1}^{N} |\langle e_j, x \rangle|^2 + \|x - x_N\|^2 \Rightarrow \sum_{j=1}^{\infty} |\langle e_j, x \rangle|^2 < \infty.$$

For $N \geq M$, $\|x_N - x_M\|^2 = \left\| \sum_{j=M+1}^{N} \langle e_j, x \rangle e_j \right\|^2 = \sum_{j=M+1}^{N} |\langle e_j, x \rangle|^2$ arbitrarily small for $M \gg 0$ $\Rightarrow$ x_N is Cauchy.

Let $x' := \lim_{N \to \infty} x_N$. $\forall\, M \leq N$, $\langle e_M, x_N \rangle = \langle e_M, x \rangle$

$\Rightarrow$ $\langle e_M, x' \rangle = \langle e_M, x \rangle$ $\forall\, M \in \mathbb{N}$ $\Rightarrow$ $(x' - x) \perp \underbrace{\overline{\mathrm{Span}\{e_\alpha \mid \alpha \in I\}}}_{= \mathcal{H}}$

$\Rightarrow$ $x' - x = 0$. ☒

defn. A metric space X is separable if it contains a countable dense subset.

thm: Every Hilbert space $\mathcal{H}$ admits an O-N basis, which is at most countable iff $\mathcal{H}$ is separable.

pf: If $\mathcal{H}$ separable, $\exists$ dense seq $\{x_1, x_2, x_3, \dots\} \subseteq \mathcal{H}$, some subseq. is lin. indep. but has same span, then Gram-Schmidt $\Rightarrow$ countable O-N set spanning $\mathcal{H}$.

If $\mathcal{H}$ non-separable: Zorn's lemma.

PSET 3: $\mathcal{H}$ is unitarily isomorphic to $L^2(I, \nu)$ where $\exists$ O-N basis $\{e_\alpha\}_{\alpha \in I}$, $\nu :=$ counting measure.

Separable $\iff$ $\mathcal{H} \cong \ell^2 := L^2(\mathbb{N}, \nu)$.