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Funktionalanalysis
WiSe 2020–21

Problem Set 3

Due: Thursday, 26.11.2020 (18pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1 $(*)$
For $\mathcal{H}$ a Hilbert space and $X \subset \mathcal{H}$ a linear subspace with closure denoted by $\bar{X}$, prove $(X^{\perp})^{\perp} = \bar{X}$. Does this remain true in general if $\mathcal{H}$ is assumed to be an inner product space but not complete? [4pts]

Problem 2
Assume X and Y are inner product spaces, and $A : X \to Y$ and $A^* : Y \to X$ are linear maps satisfying the adjoint relation

$$\langle y, Ax \rangle = \langle A^* y, x \rangle \quad \text{for all } x \in X,\ y \in Y.$$

Denote the images of these operators by $\operatorname{im} A \subset Y$ and $\operatorname{im} A^* \subset X$.

(a) Prove: $\ker A^* = (\operatorname{im} A)^{\perp}$ and $\ker A = (\operatorname{im} A^*)^{\perp}$.

(b) $(*)$ Assume Y is complete, $A : X \to Y$ is continuous and its image is closed. Show that for a given $y \in Y$, the equation $Ax = y$ has solutions $x \in X$ if and only if $\langle y, z \rangle = 0$ for all $z \in \ker A^*$. [4pts]

Problem 3
For an inner product space $\mathcal{H}$ and subspace $X \subset \mathcal{H}$ such that $\mathcal{H} = X \oplus X^{\perp}$, the *orthogonal projection to* X is the unique linear map $P : \mathcal{H} \to \mathcal{H}$ such that $P|_X$ is the identity map on X and $\ker P = X^{\perp}$. Prove:

(a) P is bounded and self-adjoint,[1] and satisfies $P^2 = P$.

(b) The orthogonal projection to $X^{\perp}$ is given by $\mathbb{1} - P : \mathcal{H} \to \mathcal{H}$.

(c) $(*)$ If $\mathcal{H}$ is complete and $\Pi : \mathcal{H} \to \mathcal{H}$ is a self-adjoint bounded linear operator with $\Pi^2 = \Pi$, then $\operatorname{im} \Pi \subset \mathcal{H}$ is closed and Π is the orthogonal projection onto $\operatorname{im} \Pi$. *Hint: The image of an orthogonal projection is the kernel of another one.* [4pts]

Problem 4
For a Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, associate to each $x \in \mathcal{H}$ the corresponding dual vector $\Lambda_x := \langle x, \cdot \rangle \in \mathcal{H}^*$.[2]

(a) Show that the formula $\langle \Lambda_x, \Lambda_y \rangle := \langle y, x \rangle$ defines an inner product on $\mathcal{H}^*$ such that the operator norm $\| \cdot \|$ satisfies $\|\Lambda\|^2 = \langle \Lambda, \Lambda \rangle$ for all $\Lambda \in \mathcal{H}^*$, thus making $\mathcal{H}^*$ into a Hilbert space over $\mathbb{K}$.

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[1] A linear operator $L : \mathcal{H} \to \mathcal{H}$ on an inner product space is called *self-adjoint* if it satisfies $\langle x, Ly \rangle = \langle Lx, y \rangle$ for all $x, y \in \mathcal{H}$.

[2] Recall that in the case $\mathbb{K} = \mathbb{C}$, our convention is that $\langle\ ,\ \rangle$ is complex-antilinear in its first argument and complex-linear in its second. It follows that the isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \Lambda_x$ is complex-antilinear.

(b) Prove that every Hilbert space is reflexive.

Problem 5
Let ν denote the counting measure on a set I, i.e. every subset $E \subset I$ is ν-measurable and $\nu(E) \in \mathbb{N} \cap \{0, \infty\}$ is the number of points in E. It follows that every function $f : I \to \mathbb{C}$ is ν-measurable, and by a straightforward exercise in measure theory, a ν-integrable function can be nonzero on at most countably many points $\alpha_1, \alpha_2, \alpha_3, \ldots \in I$, so that its integral is given by an absolutely convergent series

$$\int_I f \, d\nu = \sum_{\alpha \in I} f(\alpha) := \sum_{n=1}^{\infty} f(\alpha_n) \in \mathbb{C}.$$

All summations appearing in the following should be understood in this sense. The complex Hilbert space $L^2(I, \nu)$ now consists of all functions $f : I \to \mathbb{C}$ that are nonzero on at most countably many points and satisfy $\|f\|_{L^2}^2 = \sum_{\alpha \in I} |f(\alpha)|^2 < \infty$, with the inner product of two functions in this space given by

$$\langle f, g \rangle_{L^2} = \sum_{\alpha \in I} \overline{f(\alpha)} g(\alpha) \in \mathbb{C}.$$

(a) Show that if the set I is finite or countably infinite, then $L^2(I, \nu)$ is separable.
Hint: Show that every $f \in L^2(I, \nu)$ can be approximated arbitrarily well by functions that have real and imaginary parts in $\mathbb{Q}$ at all points and are nonzero on at most finitely many.

(b) Show that if I is uncountable, then $L^2(I, \nu)$ is not separable.

(c) (*) If $\mathcal{H}$ is a complex[3] Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha \in I}$, show that the map

$$\mathcal{H} \to L^2(I, \nu) : x \mapsto f_x \quad \text{where} \quad f_x(\alpha) := \langle e_\alpha, x \rangle$$

is a unitary isomorphism of Hilbert spaces, i.e. it is an isomorphism and satisfies $\langle f_x, f_y \rangle_{L^2} = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$. Conclude that both this map and its inverse are continuous, and that $\mathcal{H}$ is separable if and only if I is not uncountable. [6pts]

Comment: Almost all infinite-dimensional Hilbert spaces that one encounters in applications (e.g. $L^2(\mathbb{R})$ or $L^2([0,1])$ and the related Sobolev spaces that we will study later) turn out to be separable. Thus all of them are unitarily isomorphic to $\ell^2 := L^2(\mathbb{N}, \nu)$.

Problem 6
For $\mathcal{H}$ a Hilbert space containing an infinite orthonormal set $e_1, e_2, e_3, \ldots \in \mathcal{H}$, prove that the bounded sequence $\{e_n\}_{n=1}^{\infty}$ has no convergent subsequence. In particular, the closed unit ball in $\mathcal{H}$ is not compact.
Comment: A topological space X is called "locally compact" if for every point $x \in X$, every neighborhood of x contains a compact neighborhood of x, e.g. in a Hilbert space, such a neighborhood could be a sufficiently small closed ball about x. Local compactness in a Hilbert space is in fact equivalent to the condition that the closed unit ball is compact, so this problem in combination with a standard result from first-year analysis proves that a Hilbert space is locally compact if and only if it is finite dimensional. We will later prove that the same is true in Banach spaces; in fact, it is true in arbitrary Hausdorff topological vector spaces. If you're curious to see a proof of the latter statement, see
`https://terrytao.wordpress.com/2011/05/24/locally-compact-topological-vector-spaces/`

[3]The analogous statement for a real Hilbert space is obtained by taking functions in $L^2(I, \nu)$ to be real valued and omitting complex conjugation from all formulas.

Problem 1 (*)
For $\mathcal{H}$ a Hilbert space and $X \subset \mathcal{H}$ a linear subspace with closure denoted by $\bar{X}$, prove $(X^\perp)^\perp = \bar{X}$. Does this remain true in general if $\mathcal{H}$ is assumed to be an inner product space but not complete? [4pts]

1. To prove $(X^\perp)^\perp = \bar{X}$.

Suppose $x \in \bar{X}$. $\Rightarrow \exists\ (x_n) \in X$ s.t $\|x_n - x\| \to 0$ as $n \to \infty$. Let $y \in X^\perp \Rightarrow \langle x_n, y \rangle = 0 \quad \forall n$

$\langle \cdot, \cdot \rangle$ inner product is a continuous function

$\Rightarrow \quad \lim_{n\to\infty} \langle x_n, y \rangle = \langle \lim_{n\to\infty} x_n, y \rangle = \langle x, y \rangle = 0$

$\therefore \quad \langle x, y \rangle = 0$. y is arbitrary in $X^\perp$

$\Rightarrow \quad x \in (X^\perp)^\perp. \quad \Rightarrow \quad \bar{X} \subseteq (X^\perp)^\perp.$

Let $x \in (X^\perp)^\perp$ want to prove that $x \in \bar{X}$.

$\bar{X}$ is a closed subspace of $\mathcal{H}$

$\Rightarrow \quad \mathcal{H} = \bar{X} \oplus \bar{X}^\perp$

$\Rightarrow \quad x = y + z$ where $y \in \bar{X}$ and $z \in \bar{X}^\perp$

$z \in \bar{X}^\perp \Rightarrow z \in X^\perp$ as $X \subseteq \bar{X}$

$\Rightarrow \bar{X}^\perp \subseteq X^\perp$

$\Rightarrow \quad \langle x, z \rangle = 0$ as $x \in (X^\perp)^\perp$

$y \in \overline{X}$ and $z \in \overline{X}^{\perp} \Rightarrow \langle y, z\rangle = 0$

$\therefore \quad \langle x, z\rangle = \langle y, z\rangle + \langle z, z\rangle \Rightarrow 0 = 0 + \|z\|^2$

$\Rightarrow \quad z = 0 \Rightarrow x = y \ , \ y \in \overline{X} \Rightarrow x \in \overline{X}.$

$\therefore \quad (X^{\perp})^{\perp} \subseteq \overline{X} \Rightarrow (X^{\perp})^{\perp} = \overline{X}.$

No. Prob. 3 in PSET2

$\mathbb{R}^{\infty}$, V closed subspace

$V = \overline{V}$, $V^{\perp} = \{0\} \Rightarrow (V^{\perp})^{\perp} = \mathbb{R}^{\infty}$

$\Rightarrow$ if the above were true $V = \overline{V} = \mathbb{R}^{\infty}$ $\times$

as we proved that V is codim 1 in $\mathbb{R}^{\infty}$.

$\boxtimes$

Problem 2

Assume X and Y are inner product spaces, and $A : X \to Y$ and $A^* : Y \to X$ are linear maps satisfying the adjoint relation

$$\langle y, Ax\rangle = \langle A^*y, x\rangle \quad \text{for all } x \in X, \ y \in Y.$$

Denote the images of these operators by $\operatorname{im} A \subset Y$ and $\operatorname{im} A^* \subset X$.

(a) Prove: $\ker A^* = (\operatorname{im} A)^{\perp}$ and $\ker A = (\operatorname{im} A^*)^{\perp}$.

(b) $(*)$ Assume Y is complete, $A : X \to Y$ is continuous and its image is closed. Show that for a given $y \in Y$, the equation $Ax = y$ has solutions $x \in X$ if and only if $\langle y, z\rangle = 0$ for all $z \in \ker A^*$. [4pts]

2.

a) $y \in \ker A^* \Rightarrow A^*y = 0 \Rightarrow \langle A^*y, x\rangle = 0$

$\forall \; x \in X$

$$0 = \langle A^* y, x\rangle = \langle y, \underbrace{Ax}_{\in \text{im} A}\rangle \Rightarrow y \in (\text{im} A)^\perp$$

$\ker A^* = (\text{im} A)^\perp$ and similarly $\ker A = (\text{im} A^*)^\perp$

b) $\mathcal{Y}$ is complete, im A is closed.

Suppose $Ax = y$ has a solution $\Rightarrow \exists \; x_0 \in X$

s.t. $Ax_0 = y \Rightarrow y \in \text{im} A \Rightarrow \langle y, z\rangle = 0$

$\forall \; z \in (\text{im} A)^\perp$

$\Rightarrow \langle y, z\rangle = 0 \; \forall \; z \in \ker A^*$

(part a).

Suppose $\langle y, z\rangle = 0 \;\; \forall \; z \in \ker A^*$

$\Rightarrow \quad y \in (\ker A^*)^\perp$ ($\mathcal{Y}$ is complete)

$\Rightarrow \quad y \in ((\text{im} A)^\perp)^\perp$ Prob. 1.

$\Rightarrow \quad y \in \overline{\text{im} A}$ But im A is closed

$\Rightarrow \quad y \in \text{im} A \Rightarrow \exists$ a solution to $Ax = y$.

$\blacksquare$

Problem 3

For an inner product space $\mathcal{H}$ and subspace $X \subset \mathcal{H}$ such that $\mathcal{H} = X \oplus X^\perp$, the *orthogonal projection to* X is the unique linear map $P : \mathcal{H} \to \mathcal{H}$ such that $P|_X$ is the identity map on X and $\ker P = X^\perp$. Prove:

(a) P is bounded and self-adjoint,[1] and satisfies $P^2 = P$.

(b) The orthogonal projection to $X^\perp$ is given by $\mathbb{1} - P : \mathcal{H} \to \mathcal{H}$.

(c) ($*$) If $\mathcal{H}$ is complete and $\Pi : \mathcal{H} \to \mathcal{H}$ is a self-adjoint bounded linear operator with $\Pi^2 = \Pi$, then $\operatorname{im}\Pi \subset \mathcal{H}$ is closed and Π is the orthogonal projection onto $\operatorname{im}\Pi$. *Hint: The image of an orthogonal projection is the kernel of another one.* [4pts]

a) P is bounded. Let $h \in \mathcal{H} \Rightarrow h = x + y$ where $x \in X$, $y \in X^\perp$

$$\Rightarrow \quad Ph = P(x+y) = Px + Py = x + 0 = x$$

$$\Rightarrow \quad \|Ph\| \le \|h\| \quad \Rightarrow \quad P \text{ is bounded}$$

for SA, we want $\langle Ph, h' \rangle = \langle h, Ph' \rangle$

Let $h' = x' + y'$, $x' \in X$, $y' \in X^\perp$

$$\langle Ph, h' \rangle = \langle P(x+y), x'+y' \rangle = \langle x, x'+y' \rangle$$
$$= \langle x, x' \rangle$$
$$= \langle x, Ph' \rangle$$
$$= \langle x+y, Ph' \rangle$$
$$= \langle h, Ph' \rangle$$

$\langle x', y \rangle = 0$ as $x' \in X$, $y \in X^\perp$

$\Rightarrow$ P is self-adjoint.

$$P^2(h) = P(P(x+y)) = P(x) = x = P(h)$$
$$\Rightarrow \quad P^2 = P.$$

(c) $\mathcal{H}$ is complete

$$\Pi : \mathcal{H} \to \mathcal{H}$$

$\Rightarrow \quad \langle \Pi h, h' \rangle = \langle h, \Pi h' \rangle \quad$ SA

$\Pi^2 = \Pi$

h_n in im Π $\Rightarrow h_n = \Pi(x_n)$, $x_n \in \mathcal{H}$

$h_n \to h$, $h = \Pi(x)$

Want :- 1) im Π is closed in $\mathcal{H}$.

2) Π is orthogonal projection onto im Π.

<u>Claim</u> $\quad$ im $\Pi = \underbrace{\ker(Id - \Pi)}_{\text{closed subspace}}$

Let $h \in \ker(Id - \Pi)$

$\Rightarrow (Id - \Pi)(h) = 0 = Id(h) - \Pi(h) \Rightarrow h - \Pi(h)$

$\Rightarrow \quad \Pi(h) = h \quad \Rightarrow \quad h \in \text{im}\,\Pi.$

$\Rightarrow \quad \ker(Id - \Pi) \subseteq \text{im}\,\Pi$

Conversely, $x \in \text{Im}\,\Pi \Rightarrow x = \Pi(h)$ for some $h \in \mathcal{H}$

$$\begin{aligned} \Rightarrow \quad (Id - \Pi)x &= (Id - \Pi)(\Pi(h)) \\ &= Id(\Pi(h)) - \Pi^2(h) \\ &= \Pi(h) - h \end{aligned}$$

$\Rightarrow \quad x \in \ker(Id - \pi)$

$\Rightarrow \quad \operatorname{Im}\pi \subseteq \ker(Id - \pi)$

$\Rightarrow \quad \ker(Id - \pi) = \operatorname{Im}\pi \longrightarrow$ closed subspace.

$\because \operatorname{im}\pi$ is closed $\Rightarrow \mathcal{H} = \operatorname{im}\pi \oplus (\operatorname{im}\pi)^\perp$

If $x \in \ker \pi \Leftrightarrow \langle \pi x, h \rangle = 0 \quad \forall h \in \mathcal{H}$

$\Leftrightarrow \langle x, \pi^* h \rangle = 0 = \langle x, \pi h \rangle = 0$

$\Leftrightarrow x \in (\operatorname{im}\pi)^\perp$.

$\Rightarrow \pi$ is an orthogonal projection.

$\boxed{}$

④ (a) $\langle \Lambda_x, \Lambda_y \rangle := \underline{\langle y, x \rangle}$

Defines an IP on $\mathcal{H}^*$

$\Lambda_x = \langle x, \cdot \rangle$

Suppose $a \in \mathbb{K}$

$\langle \Lambda_x, a\Lambda_y \rangle = \langle \Lambda_x, \Lambda_{\bar{a}y} \rangle = \langle \bar{a}y, x \rangle$

$= a \langle y, x \rangle$

(following the convec.)

$$= a\langle \Lambda_x, \Lambda_y \rangle$$

$$\langle \Lambda_x + \Lambda_y, \Lambda_z \rangle = \langle z, x+y \rangle$$

operator norm of $\Lambda = \Lambda_x$ for some $x \in \mathcal{H}$

$$\|\Lambda\| = \sup_{\|y\|=1} \left(\|\Lambda y\| = \|\Lambda_x y\| = \|\langle x, y\rangle\| \right)$$

$\|$ Want

$$\langle \Lambda, \Lambda \rangle = \langle \Lambda_x, \Lambda_x \rangle = \|x\|^2$$

$$\|x\| = \sup \left(\|\langle x, y\rangle\| \;\middle|\; \|y\|=1 \right). \text{ — } (1)$$

assume $x \neq 0$.

$$\|x\|^2 = \langle x, x\rangle = \|x\| \cdot \left\langle x, \frac{x}{\|x\|} \right\rangle$$

$$\left\| \frac{x}{\|x\|} \right\| = 1$$

$$\|x\|^2 = \|x\| \, \|\langle x, y\rangle\| ,$$

$$\leq \sup \|x\| \left(\|\langle x, y\rangle\| \;\middle|\; \|y\|=1 \right)$$

$$\Rightarrow \quad \|x\| \leq \sup \left(\|\langle x, y\rangle\| \;\middle|\; \|y\|=1 \right)$$

$$\|\langle x, y\rangle\| \underset{C\text{-}S}{\leq} \|x\| \, \|y\| , \qquad \|y\|=1$$

$$\leq \|x\|$$

$$\Rightarrow \quad \sup \left(\|\langle x, y\rangle\| \;\middle|\; \|y\|=1 \right) \leq \|x\|$$

$\Rightarrow$ (1) is proved $\Rightarrow$ $\|\Lambda\|^2 = \langle \Lambda, \Lambda \rangle$

$\Rightarrow$ $\mathcal{H}^*$ is a Hilbert space.

(b) H is reflexive.

$(\mathcal{H}^*)^* = \mathcal{H}$.

apply Riesz representation thm twice.

$$\mathcal{H} \underset{RPT}{\xrightarrow{\cong}} \mathcal{H}^* \underset{RPT}{\xrightarrow{\cong}} (\mathcal{H}^*)^*$$

$\mathcal{H}$ is reflexive.

5) $L^2(\mathcal{I}, \nu) = \{ f : I \to \mathbb{C} \mid \|f\|^2_{L^2} = \sum_{\alpha \in \mathcal{I}} |f(\alpha)|^2 < \infty \}$

$$\langle f, g \rangle_{L^2} = \sum_{\alpha \in \mathcal{I}} \overline{f(\alpha)} g(\alpha) \in \mathbb{C}$$

a) Want:- $L^2(I, \nu)$ has a countable dense set.

$$S = \left\{ \sum_{i=1}^{n} \lambda_i \chi_{\{\alpha_i\}} \;\middle|\; n \in \mathbb{N},\ \alpha_i \in \mathcal{I},\ \lambda_i \in \mathbb{Q} \right\}$$

Characteristic function.

$\sum \lambda_i \chi_{[a_i, b_i]}$

$\lambda_i, a_i, b_i \in \mathbb{Q}$

S is countable.

S is dense in $L^2(I,\nu)$.

Suppose $f \in L^2(I,\nu) \Rightarrow \sum_{\alpha_i \in I} |f(\alpha_i)|^2 < \infty$

$$\Rightarrow \exists\ N \in \mathbb{N} \text{ s.t. } \sum_{\substack{\alpha_i \\ i > N \\ \alpha_i \in I}} |f(\alpha_i)|^2 < \frac{\epsilon}{2}$$

$f(\alpha_1), f(\alpha_2) \dots, f(\alpha_N)$

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Choose λ_i to be rational number s.t

$|f(\alpha_i) - \lambda_i| < \epsilon \quad , i = 1, \dots, N.$

b) prove the contrapositive if

$L^2(I,\nu)$ is separable $\Rightarrow$ I is countable.

$\Downarrow$

$\exists$ a countable dense subset J

every element in J must be non-zero on at most countable elements in I.

$$I' = \{\alpha \in I \mid f \in J \text{ is non-zero}\}$$

is countable.

If I were uncountable then $I \setminus I'$ must be uncountable.

We can construct functions $g: I \to \mathbb{C}$ which is zero at all elements of I' but non-zero at countable elements of $I \setminus I'$ s.t $\sum_{r_i \in I \setminus I'} |g(r_i)|^2 < \infty$

This contradicts the density of J.

$\therefore$ I must be countable.

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(6) $\mathcal{H}$ $\{e_1, e_2, \ldots\}$ o.n. set.

- $\|e_i\|^2 = 1$ as $\uparrow$ o.n. bounded sequence.

For $n, m \in \mathbb{N}$

$$\|e_n - e_m\|^2 = \|e_n\|^2 + \|e_m\|^2 = 2$$

$\Rightarrow \quad \|e_n - e_m\| = \sqrt{2}$

$\Rightarrow$ no subsequence of $\{e_n\}$ can be Cauchy.

$\Rightarrow$ it has no convergent subsequence.

$\overline{B_0(1)}$ in $\mathcal{H}$ is not compact.

we'll find a sequence (x_n) in $\overline{B_0(1)}$ s.t

$\|x_m - x_n\| \geq \frac{1}{2} \quad \forall \; m \neq n, \; \|x_i\| = 1.$

$\forall \; i \in \mathbb{N}.$

Suppose $x_1 \in \overline{B(0,1)} \Rightarrow$ span(x_1) is a proper subspace of $\mathcal{H}$

<u>Riesz's lemma</u> If $\mathcal{H}$, X proper closed subspace of $\mathcal{H}$ then $\forall \; 0 < \epsilon < 1 \;\; \exists \; h_0 \in \mathcal{H}$ w/

$\|h_0\| = 1$ s.t $\|h_0 - x\| \geq 1 - \epsilon$

$\forall \; x \in X.$

By Riesz's lemma, $\epsilon = 1/2 \;\; \exists \; x_2 \in \mathcal{H}$ s.t

$\|x_2\| = 1$ and $\|x_2 - x_1\| \geq 1/2$

$\Rightarrow x_2 \in \overline{B_0(1)}.$

Repeat this procedure with $\mathrm{span}(x_1, x_2)$.

$\neq \mathcal{H}$ as $\mathcal{H}$ is infinite-dimensional.

x_3, $\|x_3\| = 1$, $\|x_3 - x_1\| \geq 1/2$

$\|x_3 - x_2\| \geq 1/2$

$\vdots$ repeating the procedure)

(x_n), $\|x_n\| = 1$, $\|x_n - x_m\| \geq 1/2$.

$\therefore$ $\overline{B_0(p)}$ is not compact.

$\boxtimes$

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5) c). $\mathcal{H} \xrightarrow{T} L^2(I, \nu)$

$x \longmapsto f_x$, $f_x(\alpha) = \langle e_\alpha, x\rangle$

$\{e_\alpha\}_{\alpha \in I}$ o.n.b. of $\mathcal{H}$.

$f_x \in L^2(I, \nu)$

$$\|f_x\|_2^2 = \|\langle e_\alpha, x\rangle\|^2 = \sum_{n=1}^{\infty} \langle e_\alpha, x\rangle^2 \leq \|x\|^2 < \infty$$

$f_x \in L^2(I, \nu)$

$$\|x\|^2 = \left\langle \sum \langle e_\alpha, x\rangle e_\alpha , \sum \langle e_\alpha, x\rangle e_\alpha \right\rangle$$

$$= \sum \langle e_\alpha, x\rangle^2 \underbrace{\langle e_\alpha, e_\alpha\rangle}_{1} = \sum \langle e_\alpha, x\rangle^2$$

$$= \|T(x)\|_2^2$$

$T : \mathcal{H} \to L^2(I, \nu)$ is isometric $\Rightarrow$ injection.

Choose any $f \in L^2(I, \nu)$

Define $h = \sum_{\alpha \in I} f(\alpha) e_\alpha \in \mathcal{H}$

$T(h) = f_n$ simply by the definition.

$\|h\|^2 = \sum_{\alpha \in I} |f(\alpha)|^2 < \infty$ as $f \in L^2(I, \nu)$

$\therefore$ $h \in \mathcal{H}$ as $T(h) = f_n$.

$\therefore$ T is surjective.

T^{-1} is also continuous

$\Rightarrow$ T is an isometric isomorphism $\mathcal{H}$ and $L^2(I, \nu)$.

I is countable $\Rightarrow$ $L^2(I, \nu)$ is separable

$\Rightarrow$ $\mathcal{H}$ is separable. $\blacksquare$