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Funktionalanalysis
WiSe 2020–21

Problem Set 10

Due: Thursday, 11.02.2021 (15pts + 5 for free)[1]

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1
For this problem, we consider functions valued in a fixed finite-dimensional complex inner product space $(V, \langle\ ,\ \rangle)$. Recall that for $s \in \mathbb{R}$, the Hilbert space $H^s(\mathbb{R}^n)$ is defined to consist of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ whose Fourier transforms $\widehat{f} \in \mathscr{S}'(\mathbb{R}^n)$ are represented by functions of the form $\widehat{f}(p) = (1+|p|^2)^{-s/2} g(p)$ for some $g \in L^2(\mathbb{R}^n)$. The inner product on $H^s(\mathbb{R}^n)$ is given by

$$\langle f, g \rangle_{H^s} := \left\langle (1+|p|^2)^{s/2} \widehat{f}, (1+|p|^2)^{s/2} \widehat{g} \right\rangle_{L^2}.$$

If a distribution $f \in H^s(\mathbb{R}^n)$ is representable by a locally integrable function, we generally identify it with this function; note that this is always possible when $s \geqslant 0$, but not when $s < 0$. For an open subset $\Omega \subset \mathbb{R}^n$, the closure in $H^s(\mathbb{R}^n)$ of the space $C_0^\infty(\Omega)$ of smooth functions on $\mathbb{R}^n$ with compact support in Ω defines a closed subspace $\widetilde{H}^s(\Omega) \subset H^s(\mathbb{R}^n)$, and the quotient of $H^s(\mathbb{R}^n)$ by the closed subspace of distributions that vanish on test functions supported in Ω is denoted by $H^s(\Omega)$.

(a) (∗) Given $n \in \mathbb{N}$, for which $s \in \mathbb{R}$ is the Dirac δ-distribution in $H^s(\mathbb{R}^n)$? [3pts]

(b) Prove that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for every $s \in \mathbb{R}$.

(c) Prove that the pairing $\mathscr{S}(\mathbb{R}^n) \times \mathscr{S}(\mathbb{R}^n) \to \mathbb{C} : (\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_{L^2}$ extends to a continuous real-bilinear pairing

$$H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{C} : (f, g) \mapsto \langle f, g \rangle := \langle (1+|p|^2)^{-s/2} \widehat{f}, (1+|p|^2)^{s/2} \widehat{g} \rangle_{L^2},$$

such that the real-linear map $f \mapsto \langle f, \cdot \rangle$ sends $H^{-s}(\mathbb{R}^n)$ isomorphically to the dual space of $H^s(\mathbb{R}^n)$.

(d) Given an open subset Ω with compact closure in $(0,1)^n$, associate to each $f \in C_0^\infty(\Omega)$ the unique function $F \in C^\infty(\mathbb{T}^n)$ such that $f(x) = F(x)$ for $x \in (0,1)^n$. Show that the map $C_0^\infty(\Omega) \to C^\infty(\mathbb{T}^n) : f \mapsto F$ extends to bounded linear injections

$$L^2(\Omega) \cong \widetilde{H}^0(\Omega) \hookrightarrow L^2(\mathbb{T}^n) \qquad \text{and} \qquad \widetilde{H}^1(\Omega) \hookrightarrow H^1(\mathbb{T}^n)$$

whose images are closed.
Hint: Avoid Fourier analysis here by replacing the usual H^1-norm with the equivalent norm $\|u\| := \sum_{|\alpha| \leqslant 1} \|\partial^\alpha u\|_{L^2}$. This works equally well on $\mathbb{R}^n$ or $\mathbb{T}^n$.

[1] This version of the problem set has been revised to correct some errors that invalidated the original version of Problem 1(j) (worth 5 points).

(e) Deduce from the compactness of the inclusion $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ that the map $\widetilde{H}^s(\Omega) \to H^{-s}(\Omega) : f \mapsto [f]$ is compact for every $s \geqslant 1$ and every bounded open set $\Omega \subset \mathbb{R}^n$.

(f) Let $\Delta := \sum_{j=1}^n \partial_j^2$ denote the Laplace operator. Show that the linear map

$$\Phi : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n) : u \mapsto u - \frac{1}{4\pi^2}\Delta u$$

has a unique extension to a unitary isomorphism $\Phi : H^1(\mathbb{R}^n) \to H^{-1}(\mathbb{R}^n)$.

(g) Let $I : H^{-1}(\mathbb{R}^n) \to (H^1(\mathbb{R}^n))^*$ denote the real-linear isomorphism from part (c). Show that the map $\widetilde{H}^1(\Omega) \to \left(\widetilde{H}^1(\Omega)\right)^* : u \mapsto I\Phi(u)|_{\widetilde{H}^1(\Omega)}$ is an isometric real-linear isomorphism, and deduce that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Phi(u)]$ is an isomorphism.
Hint: Write down an explicit formula for $I\Phi(u)f$ for $u, f \in \widetilde{H}^1(\Omega)$.

(h) Deduce that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Delta u]$ is a Fredholm operator of index 0.

(i) Show that the equation $\Delta u = 0$ has no nontrivial solutions $u \in C_0^\infty(\mathbb{R}^n)$.
Hint: What does integration by parts tell you about $\int_{\mathbb{R}^n} \langle u, \Delta u\rangle \, dm$?

(j) Prove that $\widetilde{H}^1(\Omega) \to H^{-1}(\Omega) : u \mapsto [\Delta u]$ is an isomorphism.
Hint: Extend the formula for $\int_{\mathbb{R}^n} \langle u, \Delta u\rangle \, dm$ in part (i) to all $u \in \widetilde{H}^1(\Omega)$, and use this to prove injectivity.

Problem 2
Assume X is a complex Banach space and $T \in \mathscr{L}(X)$. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ for all n such that $(\lambda - T)x_n \to 0$. Prove:

(a) Every approximate eigenvalue of T belongs to the spectrum $\sigma(T)$.

(b) $(*)$ If $\lambda \in \sigma(T)$ is neither an eigenvalue nor belongs to the residual spectrum of T, then it is an approximate eigenvalue of T. [4pts]

(c) $(*)$ For the operator $T : \ell^1 \to \ell^1 : (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots)$, 1 is not an eigenvalue but is an approximate eigenvalue. [4pts]

Problem 3
Given a complex Banach space X and $T \in \mathscr{L}(X)$, let $T' \in \mathscr{L}(X^*)$ denote the transpose, also known as the dual operator of T.[2] Prove:

(a) If $\lambda \in \sigma(T)$ is in the residual spectrum of T then it is an eigenvalue of T'.

(b) $(*)$ If $\lambda \in \sigma(T')$ is an eigenvalue of T', then it is either an eigenvalue of T or belongs to the residual spectrum of T. [4pts]

Now suppose X is a complex Hilbert space $\mathcal{H}$, and $T^* : \mathcal{H} \to \mathcal{H}$ denotes the adjoint operator, defined via the condition $\langle x, Ty\rangle = \langle T^*x, y\rangle$ for all $x, y \in \mathcal{H}$. Prove:

(c) $\sigma(T^*) = \left\{\bar{\lambda} \in \mathbb{C} \;\middle|\; \lambda \in \sigma(T')\right\}$

(d) $\sigma(T) = \sigma(T')$

Hint: $T^* : \mathcal{H} \to \mathcal{H}$ and $T' : \mathcal{H}^* \to \mathcal{H}^*$ are closely related via the complex-antilinear isomorphism $\mathcal{H} \to \mathcal{H}^* : x \mapsto \langle x, \cdot\rangle$.

[2] We have sometimes denoted T' in the past by T^*, but will now be reserving the latter notation for the adjoint of an operator on a complex Hilbert space.