Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi,
L. Upmeier zu Belzen

Funktionalanalysis
WiSe 2020–21

Problem Set 12

Due: Thursday, 25.02.2021 (24pts total)

Problems marked with (∗) will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without (∗), you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *$\mathcal{H}$ is a complex Hilbert space.*

Problem 1

Prove:

(a) A self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$ is positive ($A \geqslant 0$) if and only if $\sigma(A) \subset [0, \infty)$.

(b) If $\langle x, Ax \rangle > 0$ for all $x \neq 0 \in \mathcal{H}$, it does not follow that $0 \notin \sigma(A)$.

Problem 2

The *spectral measure* μ_x corresponding to a self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$ and $x \in \mathcal{H}$ is by definition the unique finite regular measure on the Borel sets in $\sigma(A) \subset \mathbb{R}$ such that

$$\langle x, f(A)x \rangle = \int_{\sigma(A)} f \, d\mu_x \quad \text{for all } f \in C(\sigma(A)).$$

(a) Describe μ_x explicitly in the case where $x \in \mathcal{H}$ is an eigenvector of A.

(b) Describe μ_x explicitly in the case where A is compact and $x \in \mathcal{H}$ is arbitrary.

(c) Show that if A has any eigenvalues of multiplicity greater than 1, then $\mathcal{H}$ does not contain any cyclic vector for A.

(d) (∗) Show that in the case $\mathcal{H} = \mathbb{C}^n$, the converse of part (c) also holds: if $\sigma(A)$ contains n distinct eigenvalues, then a cyclic vector $v \in \mathcal{H}$ for A exists. Give an explicit example of v in the case where $A : \mathbb{C}^n \to \mathbb{C}^n$ is diagonal. [5pts]
Hint: The proof of the spectral theorem will tell you where to look for an example.

Problem 3

Assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $\mathcal{D} \subset X$ is a subspace, and $X \supset \mathcal{D} \xrightarrow{T} Y$ is a linear operator, possibly unbounded, and not necessarily closed. Prove:

(a) If T is closed, then so is the operator $\mathcal{D} \to Y : x \mapsto Tx + Ax$ for every bounded operator $A \in \mathscr{L}(X, Y)$.

(b) T is closed if and only if the so-called *graph norm* $\|x\|_T := \|x\|_X + \|Tx\|_Y$ on $\mathcal{D}$ is complete.

Now assume $X = Y$ is a complex Banach space.

(c) (∗) Show that for every $\lambda \in \mathbb{C}$ such that $\lambda - T : \mathcal{D} \to X$ is bijective, T is closed if and only if the resolvent operator $R_\lambda(T) : X \to X : x \mapsto (\lambda - T)^{-1}x$ is bounded. [4pts][1]

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[1] This result is the reason why one normally never considers the spectrum of a non-closed operator.

Next, assume additionally that T is closed. We call $\lambda \in \mathbb{C}$ an *approximate eigenvalue* of T if there exists a sequence $x_n \in \mathcal{D}$ such that $\|x_n\|_X = 1$ and $(\lambda - T)x_n \to 0$, and λ belongs to the *residual spectrum* of T of the image of $\lambda - T : \mathcal{D} \to X$ is not dense. Prove:

(d) If $\lambda \in \sigma(T)$ is not in the residual spectrum of T, then it is an approximate eigenvalue.

(e) Every approximate eigenvalue of T is in $\sigma(T)$.

Problem 4
Let $\mathrm{AC}^2([0,1])$ denote the space of absolutely continuous complex-valued functions $f(t)$ on $[0,1]$ whose derivatives (defined almost everywhere) are in $L^2([0,1])$.[2] Given the domains

$$\mathcal{D}_0 := \mathrm{AC}^2([0,1]),$$
$$\mathcal{D}_1 := \left\{ f \in \mathrm{AC}^2([0,1]) \mid f(0) = 0 \right\},$$
$$\mathcal{D}_2 := \left\{ f \in \mathrm{AC}^2([0,1]) \mid f(0) = f(1) = 0 \right\},$$

consider for $j = 0,1,2$ the unbounded operators $L^2([0,1]) \supset \mathcal{D}_j \xrightarrow{T_j} L^2([0,1])$ defined by $T_j := i\partial_t = i\frac{d}{dt}$. Prove:

(a) $(*)$ All three domains are dense in $L^2([0,1])$, and all three operators are closed. [6pts]

(b) Every $\lambda \in \mathbb{C}$ is an eigenvalue of T_0, thus $\sigma(T_0) = \mathbb{C}$.

(c) Every $\lambda \in \mathbb{C}$ is in the resolvent set of T_1, and $(\lambda - T_1)^{-1} : L^2([0,1]) \to \mathcal{D}_1$ sends $g \in L^2([0,1])$ to the function $f(t) := i\int_0^t e^{-i\lambda(t-s)} g(s)\, ds$. In particular, $\sigma(T_1) = \varnothing$.[3]

(d) T_2 is symmetric, but not self-adjoint.

(e) Every $\lambda \in \mathbb{C}$ is in the residual spectrum of T_2, hence $\sigma(T_2) = \mathbb{C}$.

Problem 5 $(*)$
Fix an L^2-function $P : [0,1] \to \mathbb{R}$ and define $\mathcal{D}$ to be the vector space of C^1-functions $x : [0,1] \to \mathbb{C}$ such that $x(0) = x(1) = 0$ and the derivative $\dot{x}$ belongs to the space $\mathrm{AC}^2([0,1])$ from Problem 4, so every $x \in \mathcal{D}$ has an almost everywhere defined second derivative $\ddot{x} \in L^2([0,1])$.[4] Setting $Tx := \ddot{x} + Px$, show that $L^2([0,1]) \supset \mathcal{D} \xrightarrow{T} L^2([0,1])$ is an unbounded self-adjoint operator. [5pts]
Hint: Interpret the condition defining the domain of T^* in terms of weak derivatives.

Problem 6 $(*)$
A closed (but not necessarily bounded) self-adjoint operator $\mathcal{H} \supset \mathcal{D} \xrightarrow{A} \mathcal{H}$ is called *positive* if $\langle x, Ax \rangle \geqslant 0$ for all $x \in \mathcal{D}$. Prove (without citing the spectral theorem) that under this assumption, $\sigma(A)$ contains no negative real numbers. [4pts]

[2] Observe that $\mathrm{AC}^2([0,1])$ is equivalent to the Sobolev space $W^{1,2}((0,1))$. This follows mostly from Problem Set 9 #3–4: Problem 3 gives an inclusion $\mathrm{AC}^2([0,1]) \hookrightarrow W^{1,2}((0,1))$, and also shows that every equivalence class in $W^{1,2}((0,1))$ has a unique representative that is absolutely continuous on compact subsets, and whose derivatives almost everywhere match their weak derivatives. Problem 4 (the Sobolev embedding theorem) implies in turn that these continuous functions are also in $C^{0,\frac{1}{2}}((0,1))$, thus they are uniformly continuous on $(0,1)$, and therefore admit continuous extensions over $[0,1]$. One can deduce from the fundamental theorem of calculus that the extensions are also absolutely continuous.

[3] The invertibility of $\lambda - T_1$ can also be deduced from general principles without writing down an explicit formula. The essential question is: given $g \in L^2([0,1])$ and $\lambda \in \mathbb{C}$, how many absolutely continuous functions $f : [0,1] \to \mathbb{C}$ satisfy the initial value problem $f'(t) = H(t, f(t)) := -i[\lambda f(t) - g(t)]$ with $f(0) = 0$? Intuitively, the Picard-Lindelöf theorem suggests that the answer must be exactly one, though strictly speaking, the theorem does not apply here since g cannot be assumed continuous. But since $H(t,x) = -i(\lambda x - g(t))$ is Lipschitz continuous with respect to x, the usual proof can be adapted for this case.

[4] Similarly to the situation in Problem 4, $\mathcal{D}$ in Problem 5 is equivalent to the Sobolev space $W^{2,2}((0,1))$.

Supplement

The following (unstarred) problem serves two purposes: (1) it fills in some gaps in the lectures' coverage of the spectral theorem for bounded normal operators, and (2) it provides a structured review (with mild generalizations) of the proof of the spectral theorem for bounded self-adjoint operators. From that perspective, working through it informally should serve as valuable preparation for the final exam.

Problem 7
The spectral theorem for bounded normal operators (proved in lecture) provides for any normal operator $A \in \mathscr{L}(\mathcal{H})$ a σ-finite measure space (X, μ), unitary isomorphism $U : \mathcal{H} \to L^2(X, \mu)$ and bounded measurable function $F : X \to \mathbb{C}$ such that $UAU^{-1} = T_F : L^2(X, \mu) \to L^2(X, \mu) : u \mapsto Fu$. One easy corollary is that the Borel functional calculus extends to normal operators, i.e. there is a natural linear map

$$\mathscr{B}(\sigma(A)) \to \mathscr{L}(\mathcal{H}) : f \mapsto f(A) := U^{-1} T_{f \circ F} U,$$

where $\mathscr{B}(\sigma(A))$ denotes the algebra of bounded Borel-measurable functions $f : \sigma(A) \to \mathbb{C}$. Show that this map has the following properties:

(a) $(fg)(A) = f(A)g(A)$, $\bar{f}(A) = f(A)^*$, $f(A) = \lambda \mathbb{1}$ for each constant function $f(z) = \lambda$, and $f(A) = A$ for the identity function $f(z) = z$.

(b) For any pointwise convergent sequence $f_n \to f \in \mathscr{B}(\sigma(A))$ satisfying a uniform bound $\sup_{z \in \sigma(A)} |f_n(z)| \leqslant C$ for all n, $f_n(A)x \to f(A)x$ for every $x \in \mathcal{H}$.

(c) $f \mapsto f(A)$ is the only linear map $\mathscr{B}(\sigma(A)) \to \mathscr{L}(\mathcal{H})$ satisfying both of the properties in parts (a) and (b).
Hint: Since $\sigma(A) \subset \mathbb{C}$ is compact, Weierstrass implies that the polynomial functions in z and $\bar{z}$ are dense in the space of continuous functions $C(\sigma(A)) \subset \mathscr{B}(\sigma(A))$ with the sup-norm. Similarly, $\mathscr{B}(\sigma(A))$ is the smallest class of functions that contains $C(\sigma(A))$ and is closed under the notion of convergence in part (b).

(d) $f(A)$ is normal for every $f \in \mathscr{B}(\sigma(A))$, and it is self-adjoint whenever $f(\sigma(A)) \subset \mathbb{R}$, positive whenever $f(\sigma(A)) \subset [0, \infty)$, and unitary whenever $f(\sigma(A)) \subset S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$.

(e) $\sigma(f(A))$ is contained in the closure of $f(\sigma(A)) \subset \mathbb{C}$ for all $f \in \mathscr{B}(\sigma(A))$.
Hint: If $\mu \notin \overline{f(\sigma(A))}$, then $g(z) := \frac{1}{f(z) - \mu}$ belongs to $\mathscr{B}(\sigma(A))$. What is $g(A)$?

(f) $\sigma(f(A)) = f(\sigma(A))$ for all $f \in C(\sigma(A))$.
Hint: By part (e), you only need to show $f(\lambda) \in \sigma(f(A))$ for every $\lambda \in \sigma(A)$. Compare the essential ranges of F and $f \circ F$ (cf. Problem Set 11 #2(a)).

(g) $\|f(A)\| = \|f\|_{C^0}$ for every $f \in C(\sigma(A))$.

Here are some applications. Prove:

(h) For $A \in \mathscr{L}(\mathcal{H})$ normal and $\lambda \in \mathbb{C} \backslash \sigma(A)$, the resolvent $R_\lambda(A) := (\lambda - A)^{-1}$ satisfies $\frac{1}{\|R_\lambda(A)\|} = \operatorname{dist}(\lambda, \sigma(A))$.

(i) If $A \in \mathscr{L}(\mathcal{H})$ is normal and $f : \mathcal{D} \to \mathbb{C}$ is holomorphic on some disk $\mathcal{D} = \{|z - z_0| < r\} \subset \mathbb{C}$ containing $\sigma(A)$, with Taylor series $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$, then $\sum_{n=0}^\infty a_n (A - z_0 \mathbb{1})^n$ converges absolutely to $f(A)$ in $\mathscr{L}(\mathcal{H})$.

(j) An operator $U \in \mathscr{L}(\mathcal{H})$ is unitary if and only if $U = e^{iA}$ for some bounded self-adjoint operator $A \in \mathscr{L}(\mathcal{H})$.

(k) If $T, A \in \mathscr{L}(\mathcal{H})$ commute and A is normal, then T also commutes with A^*.
Hint: Deduce from $AT = TA$ that $e^{\bar{\lambda}A} T e^{-\bar{\lambda}A} = T$ for all $\lambda \in \mathbb{C}$. Then show that $e^{-\lambda A^*} e^{\bar{\lambda}A}$ is unitary and use this to compute a bound on the norm of $g(\lambda) := e^{-\lambda A^*} T e^{\lambda A^*}$ for all $\lambda \in \mathbb{C}$, concluding that $g : \mathbb{C} \to \mathscr{L}(\mathcal{H})$ is a globally bounded holomorphic function, and thus constant. Finally, compute $\frac{d}{dt} e^{-tA^*} T e^{tA^*}\Big|_{t=0}$.

(l) In the setting of part (k), T also commutes with $f(A)$ for every $f \in \mathscr{B}(\sigma(A))$.

Finally, we can now establish some improvements to the spectral theorem:

(m) Under what conditions does a normal operator $A \in \mathscr{L}(\mathcal{H})$ admit a spectral representation of the form $U : \mathcal{H} \to L^2(\sigma(A), \mu)$ with $UAU^{-1} = T_F$ for $F(\lambda) = \lambda$, where μ is a finite measure on $\sigma(A) \subset \mathbb{C}$ such that $C(\sigma(A))$ is dense in $L^2(\sigma(A), \mu)$?

(n) Show that for every normal operator $A \in \mathscr{L}(\mathcal{H})$, $\mathcal{H}$ splits into a direct sum of (perhaps infinitely many) mutually orthogonal A-invariant subspaces $\mathcal{H}_n \subset \mathcal{H}$ on which $A|_{\mathcal{H}_n}$ admits a spectral representation as described in part (m).

(o) When does a finite collection of normal operators $A_1, \ldots, A_N \in \mathscr{L}(\mathcal{H})$ admit a simultaneous spectral representation?