Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi,
L. Upmeier zu Belzen

Funktionalanalysis
WiSe 2020–21

Problem Set 2

Due: Thursday, 19.11.2020 (20pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Problem 1

Assume X is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $V \subset X$ is a subspace. One says that V has *codimension* k if the quotient vector space X/V has dimension k. (Note that k may be finite even if X and V are both infinite dimensional.)

(a) Show that the following conditions are equivalent:

  (i) $\operatorname{codim} V = 1$;
  (ii) There exists a vector $w \in X \backslash V$ such that every $x \in X$ can be written as $x = v + \lambda w$ for unique elements $v \in V$ and $\lambda \in \mathbb{K}$;
  (iii) $V = \ker \Lambda$ for some nontrivial linear map $\Lambda : X \to \mathbb{K}$.[1]

(b) Show that if $V = \ker \Lambda = \ker \Lambda'$ for two linear functionals $\Lambda, \Lambda' : X \to \mathbb{K}$, then $\Lambda' = c\Lambda$ for some nonzero scalar $c \in \mathbb{K}$.

(c) $(*)$ Assuming X is a normed vector space and $V = \ker \Lambda$ for a nontrivial linear functional $\Lambda : X \to \mathbb{K}$, show that the following conditions are equivalent:

  (a) $V \subset X$ is closed;
  (b) $V \subset X$ is not dense;
  (c) $\Lambda : X \to \mathbb{K}$ is continuous.

Hint: Show that the closure of any subspace is also a subspace. If $\Lambda : X \to \mathbb{K}$ is not bounded, there exists a bounded sequence $x_n = v_n + \lambda_n w \in X$ with $v_n \in V$, $w \in X \backslash V$ and $\lambda_n \in \mathbb{K}$ such that $|\Lambda(x_n)| \to \infty$. What can you say about $\frac{v_n}{\lambda_n}$? [8pts]

Problem 2

For vectors $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, consider the norms

$$|x|_p := \left(\sum_{j=1}^n x_j^p \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty, \qquad |x|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

(a) Show (by drawing pictures of the unit ball) that the normed vector spaces $(\mathbb{R}^n, |\cdot|_1)$ and $(\mathbb{R}^n, |\cdot|_\infty)$ are not strictly convex.

(b) $(*)$ Show that the spaces of real-valued functions of class L^1 or L^∞ on $[0, 1]$ are not strictly convex. [6pts]

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[1] Linear maps $X \to \mathbb{K}$ are also called *linear functionals*, and subspaces $V \subset X$ of codimension 1 are also called *hyperplanes*.

Problem 3
Let $\mathbb{R}^\infty$ denote the vector space of infinite tuples $x = (x_1, x_2, \ldots)$ of real numbers such that at most finitely many of the coordinates $x_n \in \mathbb{R}$ are nonzero. This becomes an inner product space if we define on $\mathbb{R}^\infty$ the obvious generalization of the Euclidean inner product,

$$\langle x, y \rangle := \sum_{n=1}^{\infty} x_n y_n \in \mathbb{R}, \tag{1}$$

where the sum always converges since only finitely many of its terms are nonzero. Define a subspace $V \subset \mathbb{R}^\infty$ by $V := \left\{ x \in \mathbb{R}^\infty \;\middle|\; \sum_{n=1}^{\infty} \frac{x_n}{n} = 0 \right\}$.

(a) Prove that $V \subset \mathbb{R}^\infty$ is a closed subspace of codimension 1.

(b) (*) Prove that the orthogonal complement $V^\perp = \left\{ x \in \mathbb{R}^\infty \;\middle|\; \langle x, v \rangle = 0 \text{ for all } v \in V \right\}$ is the trivial subspace of $\mathbb{R}^\infty$. [6pts]

(c) In lecture we proved that for any closed subspace V in a Hilbert space $\mathcal{H}$, $\mathcal{H} = V \oplus V^\perp$. Where does the proof of this theorem go wrong if you try to carry it out with the Hilbert space $\mathcal{H}$ replaced by the *incomplete* inner product space $\mathbb{R}^\infty$?

Hint: $\mathbb{R}^\infty$ is a dense subspace of the Hilbert space ℓ^2 consisting of tuples $x = (x_1, x_2, \ldots)$ that are allowed to have infinitely many nonzero coordinates but must also satisfy $\sum_{n=1}^\infty x_n^2 < \infty$. Equivalently, ℓ^2 is $L^2(\mathbb{N}, \nu)$, the space of square-integrable functions $\mathbb{N} \to \mathbb{R} : n \mapsto x_n$ with the counting measure ν. Notice that $V = \mathbb{R}^\infty \cap z^\perp$ for an element $z \in \ell^2 \backslash \mathbb{R}^\infty$.

Problem 4
Here is an example of a topological vector space that is not locally convex and has a trivial dual space. Define $L^p([0,1])$ as usual to be the space of equivalence classes (up to equality almost everywhere) of measurable functions $f : [0,1] \to \mathbb{R}$ with $\|f\|_{L^p} := \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} < \infty$, but instead of $p \geqslant 1$, assume $0 < p < 1$. In this case, Minkowski's inequality does not hold, so $\|\cdot\|_{L^p}$ is not a norm, but we shall regard $L^p([0,1])$ as a metric space with the metric defined by $d(f,g) := \|f - g\|_{L^p}^p$.

(a) Show that d is a metric on $L^p([0,1])$.
Hint: Show first that $(x+y)^p \leqslant x^p + y^p$ holds for all $x, y \geqslant 0$. The latter can be deduced from the relation $a^q + b^q \leqslant (a+b)^q$ for $a, b \geqslant 0$ and $q := 1/p > 1$, which follows in turn from $(1+x)^q \geqslant 1 + x^q$ for $x \geqslant 0$, which you can prove by differentiating with respect to x.

(b) Prove that $L^p([0,1])$ with the topology defined via d is a topological vector space.

(c) Prove that the space of bounded measurable real-valued functions is dense in $L^p([0,1])$.
Hint: Given $f \in L^p([0,1])$, define functions f_n that match f wherever $|f| \leqslant n$.

(d) Prove by induction on $N \in \mathbb{N}$ that if K is any convex subset of a vector space, then for any finite collections $x_1, \ldots, x_N \in K$ and $\tau_1, \ldots, \tau_N \in [0,1]$ with $\sum_{i=1}^N \tau_i = 1$, $\sum_{i=1}^N \tau_i x_i \in K$. (This is known as a *convex combination* of $x_1, \ldots, x_N$.)

(e) Prove that for any given $\epsilon > 0$, every bounded measurable function $f : [0,1] \to \mathbb{R}$ can be written as $f = \frac{1}{N} \sum_{i=1}^N f_i$ for some finite collection of functions $f_i \in L^p([0,1])$ satisfying $d(f_i, 0) < \epsilon$ for all $i = 1, \ldots, N$. Conclude that the only closed convex subset of $L^p([0,1])$ containing a neighborhood of 0 is $L^p([0,1])$ itself.
Hint: Define each f_i to have support in an interval of length $1/N$, then make N large.

(f) Prove that all continuous linear functionals $\Lambda : L^p([0,1]) \to \mathbb{R}$ are trivial.
Hint: What kind of subset is $\{f \in L^p([0,1]) \mid |\Lambda(f)| \leqslant 1\}$?