Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi,
L. Upmeier zu Belzen

Funktionalanalysis
WiSe 2020–21

Problem Set 5

Due: Thursday, 10.12.2020 (18pts total)

Problems marked with $(*)$ will be graded. Solutions may be written up in German or English and should be submitted electronically via the moodle before the Übung on the due date. For problems without $(*)$, you do not need to write up your solutions, but it is highly recommended that you think through them before the next Tuesday lecture. You may also use the results of those problems in your written solutions to the graded problems.

Convention: *You can assume unless stated otherwise that all functions take values in a fixed finite-dimensional inner product space $(V, \langle\ ,\ \rangle)$ over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by m.*

Problem 1

Show that the space of bounded continuous functions on $\mathbb{R}$ is not dense in $L^\infty(\mathbb{R})$.

Problem 2

Fix $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show that if $p > 1$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \langle f, \varphi \rangle \, dm = 0$ for all smooth compactly supported functions $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $f = 0$ almost everywhere.[1]

(b) $(*)$ Assume $1 < p < \infty$, and suppose $T, T^* : C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ are two linear operators satisfying the "adjoint" relation

$$\int_{\mathbb{R}^n} \langle Tf, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, T^* g \rangle \, dm \quad \text{for all} \quad f, g \in C_0^\infty(\mathbb{R}^n).$$

Show that T extends to a bounded linear operator $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if and only if T^* extends to a bounded linear operator $T^* : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$. [6pts]

Hint: Use the isometric identification of L^p with the dual space of L^q. (In part (a), this makes sense only after restricting to a compact subset.) You will also need to use the density of C_0^∞ in L^p.

Problem 3 $(*)$

Show that for any $f, g \in L^1(\mathbb{R}^n)$ and a compactly supported smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} \langle \varphi * f, g \rangle \, dm = \int_{\mathbb{R}^n} \langle f, \varphi^- * g \rangle \, dm,$$

where $\varphi^-(x) := \varphi(-x)$. [4pts]

Hint: Here is a useful fact about integrals of vector-valued functions. If $L : V \to W$ is a linear map between finite-dimensional vector spaces and $f : \mathbb{R}^n \to V$ is Lebesgue integrable, then $Lf : \mathbb{R}^n \to W$ is also Lebesgue integrable and $\int_{\mathbb{R}^n} Lf \, dm = L\left(\int_{\mathbb{R}^n} f \, dm\right)$.

[1] We will see when we study distributions that the result of Problem 2(a) is also true for $p = 1$, but that case is trickier to prove.

Problem 4 ($*$)
For an integer $m \geqslant 0$, let $C^m_b(\mathbb{R}^n)$ denote the Banach space of C^m-functions $\mathbb{R}^n \to V$ whose derivatives up to order m are all bounded, with the usual C^m-norm. Let $C^m(\bar{\mathbb{R}}^n)$ denote the subspace consisting of functions $f \in C^m_b(\mathbb{R}^n)$ whose derivatives of order m are also uniformly continuous.[2] One can show along the lines of Problem Set 1 #3(b) that $C^m(\bar{\mathbb{R}}^m)$ is a closed subspace of $C^m_b(\mathbb{R}^n)$, so it is also a Banach space. Prove that if $f \in C^m(\bar{\mathbb{R}}^n)$ and $\{\rho_j : \mathbb{R}^n \to [0,\infty)\}_{j\in\mathbb{N}}$ is an approximate identity with shrinking support, then

$$\lim_{j\to\infty} \|\rho_j * f - f\|_{C^m} = 0,$$

and conclude that $C^\infty(\mathbb{R}^n) \cap C^m(\bar{\mathbb{R}}^n)$ is dense in $C^m(\bar{\mathbb{R}}^n)$. [8pts]
Hint: A similar (though non-identical) result is proved at the end of §5 *in the lecture notes. We did not cover it in lecture.*

[2] Note that for $f \in C^m(\bar{\mathbb{R}}^n)$, the derivatives of any order $k < m$ are also uniformly continuous, but this is not an extra condition; it follows (via the fundamental theorem of calculus) from the assumption that the derivatives of order $k+1$ are bounded.