Course description and syllabus

General information

Instructors: Prof. Chris Wendl (lectures)
HU Institut für Mathematik (Rudower Chaussee 25), room 1.301
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Office hour: Tuesdays 17:00–18:00

Dr. Shubham Dwivedi (problem sessions)
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PD Dr. habil. Olaf Müller (problem sessions)
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Website: www.mathematik.hu-berlin.de/~wendl/Winter2021/Diffgeo1/

Moodle: moodle.hu-berlin.de/course/view.php?id=106777
enrolment key: Riemann
The moodle will be used mainly for communication: you must join it if you want to receive occasional important announcements about the course via e-mail, and you can also use the moodle forum to discuss or ask questions about homework problems. Essential course materials such as lecture notes will be posted on the course website rather than the moodle.

Lectures: Tuesdays 11:15–12:45 in 1.115 (Rudower Chaussee 25)
Wednesdays 11:15–12:45 in 1.115 (Rudower Chaussee 25)

Problem sessions (Dwivedi): Wednesdays 9:15–10:45 in 3.007 (Rudower Chaussee 25)

Problem sessions (Müller): Mondays or Tuesdays 13.15-14:45 (to be decided)

Language: This course is counted as a *BMS Basic Course*, thus it is offered in English, with the exception of Olaf Müller's problem session, which will be in German.

Prerequisites: The contents of the HU's courses *Analysis I–II* and *Lineare Algebra und Analytische Geometrie I–II*, plus the basic existence/uniqueness theory of ordinary differential equations and integration of functions of n variables (as in *Analysis III*).
The single most important prerequisite is a solid grasp of calculus for functions $\mathbb{R}^n \to \mathbb{R}^m$, including the statements (if not the proofs) of the inverse and implicit function theorems, and the change of variables formula for integration.

Course description

Differential geometry naturally begins with the study of smooth 1-dimensional curves and 2-dimensional surfaces in Euclidean space, but these are only special cases of much more general objects, called smooth n-dimensional *manifolds*. Manifolds arise naturally in many branches of mathematics as well as in physics, e.g. as configuration spaces for constrained dynamical systems, or as the curved spacetime in Einstein's theory of gravitation (General Relativity). This course will study manifolds from a fairly general perspective, without limiting the discussion to curves and surfaces, though for the purposes of visualization, most of the examples we consider will be 2-dimensional. The first task is to understand basic notions such as smoothness of maps between manifolds, the derivatives of such maps, tangent vectors, vector fields and the flows that they generate. Tensors are then introduced as a linear-algebraic means of encoding local geometric information, and as a special case, we consider differential forms, which define notions of volume on manifolds and can thus be integrated. The first portion of the course culminates with the general version of Stokes' theorem for integrals of differential forms on manifolds: this is the natural n-dimensional generalization of the fundamental theorem of calculus, and also implies the standard vector calculus theorems of Gauss, Green and Stokes.

The second half of the course is based on the general notion of a vector bundle, of which several examples (e.g. the set of all tangent vectors on a manifold) will already be familiar from the first half. The need to define derivatives on bundles leads naturally to the notions of parallel transport, connections and covariant differentiation. This raises a natural question as to when covariant derivatives in different directions can be assumed to commute, and the answer requires the introduction of *curvature*, a tensor whose vanishing characterizes the existence of "covariantly constant" vector fields on manifolds. In order to prove this, we introduce smooth distributions on manifolds and prove the Frobenius integrability theorem. The most convincing initial applications of these ideas are in the study of *Riemannian* manifolds: these are manifolds equipped with extra structure so that lengths of paths and angles between them can be defined. We consider geodesics, which define shortest paths between nearby points on Riemannian manifolds, and discuss the geometric meaning of the Riemann curvature tensor in n dimensions, as well as its simpler variant for surfaces, the so-called "Gaussian" curvature. We can then prove one of the most beautiful and fundamental results about 2-dimensional Riemannian manifolds: the Gauss-Bonnet theorem, which relates the sum of the angles in a geodesic triangle to the amount of curvature it encloses, or for the case of compact surfaces without boundary, computes the total curvature in terms of a purely topological invariant, the Euler characteristic.

There are a number of additional topics that we may briefly touch upon if time permits, most of them related to physics: these include Hamiltonian dynamics and symplectic geometry, pseudo-Riemannian manifolds and general relativity, Lie groups and Lie algebras, holomorphic vector bundles, principal fiber bundles and gauge theory. Most of these topics will be discussed in detail in the sequel to this course, *Differential Geometry II*.

Syllabus

The following week-by-week plan for the lectures is tentative and subject to change.

1. Differentiable manifolds, implicit function theorem, examples, tangent vectors.
2. Tangent maps, vector fields, Lie bracket and commuting flows.
3. Orientability, tensors, index notation, differential forms.
4. Poincaré lemma, Lie derivative, Cartan's formula.
5. Partitions of unity, existence of volume forms and Riemannian metrics.
6. Integration, Stokes' theorem, low-dimensional examples.
7. Vector bundles and sections, bundle metrics, orientation.
8. Structure groups, fiber bundles, parallel transport.
9. Linear connections, covariant derivatives, compatibility.
10. Connections on tangent bundles, torsion and symmetry, geodesics, Riemann normal coordinates.
11. Levi-Cività connection, geodesics, Riemannian manifolds.

12. Integrability and the Frobenius theorem, curvature.
13. Locally flat manifolds, hypersurfaces, second fundamental form and Gaussian curvature.
14. Euler characteristic and the Gauss-Bonnet theorem for surfaces.
15. Sectional curvature, second variation formula, length-minimizing geodesics.
16. (further topics to be decided if time permits)

Literature

Lecture notes to accompany this course will be posted regularly on the course website. For alternative treatments of the same or similar material, the following sources are recommended:

- John M. Lee, *Introduction to Smooth Manifolds*, second edition, Springer GTM 2012
  (online access available via the HU library)
- Michael Spivak, *A Comprehensive Introduction to Differential Geometry, Volume I*, 3rd edition with corrections, Publish or Perish 2005
  (available in the HU library, Freihandbestand)
- Frank W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer GTM 1983
  (online access available via the HU library)
- Ilka Agricola and Thomas Friedrich, *Globale Analysis: Differentialformen in Analysis, Geometrie und Physik*, Vieweg 2001 (or the English translation, AMS 2002)
  (online access available via the HU library)
- Helga Baum, *Eine Einführung in die Differentialgeometrie*, lecture notes available at
  `https://www.mathematik.hu-berlin.de/~baum/Skript/diffgeo1.pdf`

Note: Some sections of the books by Lee and Warner noticeably assume more knowledge of topology than we will assume in this course, but students who have not yet taken *Topologie I* can safely skip those sections and still understand the sections that are relevant to the course.

Students with an interest in physics may also appreciate the following recommendation:

- Sean Carroll, *Lecture Notes on General Relativity*, available at
  `https://www.preposterousuniverse.com/grnotes/`

Chapters 2 and 3 in these notes present an especially readable introduction to manifolds and curvature that is suitable for both mathematicians and physicists.

Exam and problem sets

Grades in this course will be determined by a 3-hour **written exam** soon after the end of the semester (with a resit option shortly before the beginning of the following semester). Books and notes may be consulted during the exam. The exam problems will be conceived so as to be solvable within 2 hours, so that time pressure should not be the decisive factor.

Problem sets will be distributed and posted on the course website every Wednesday, and solutions discussed in the problem session on the following Wednesday. The problem sets will not be graded, but it is **strongly recommended** that you at least think through every problem before the problem session each week, since this is the single best way to ensure that you are keeping up with the material in the course.

There will also be a special homework assignment midway through the semester, the so-called **take-home midterm**, which you will have two weeks to work on and can submit for a grade. The midterm is voluntary, but your score can be used to boost your final exam grade according to the following rule:

- Midterm 60%–79% $\quad\Rightarrow\quad$ $2,0 \rightsquigarrow 1,7$ or $1,7 \rightsquigarrow 1,3$ etc.
- Midterm 80%–100% $\quad\Rightarrow\quad$ $2,0 \rightsquigarrow 1,3$ or $1,7 \rightsquigarrow 1,0$ etc.