Problem Set 10

1) $\pi : E \to M$ smooth v.b ∇ connection

$\Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{F}^m$

$\Phi_\beta : E|_{U_\beta} \to U_\beta \times \mathbb{F}^m$

$g_{\alpha\beta}, g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m, \mathbb{F})$

$A_\alpha \in \Omega^1(U_\alpha, \mathbb{F}^{m\times m})$
$A_\beta \in \Omega^1(U_\beta, \mathbb{F}^{m\times m})$

a) Want to prove

$$A_\alpha(X) = g_{\alpha\beta}(p) A_\beta(X) g_{\beta\alpha}(p) + g_{\alpha\beta}(p) dg_{\beta\alpha}(X)$$

$$\Updownarrow$$

$$A_\alpha = g^{-1} A_\beta g + g^{-1} dg \quad \text{on } U_\alpha \cap U_\beta.$$

let $s_\alpha : U_\alpha \to \mathbb{F}^m$

then $(\nabla_X s)_\alpha(p) = ds_\alpha(X) + A_\alpha(X) s_\alpha(p)$ 1

$s_\beta = g_{\beta\alpha} s_\alpha$, $s_\alpha = g_{\alpha\beta} s_\beta$

We get

$$g_{\beta\alpha}(p)\,(\nabla_X s)_\alpha(p) = d(g_{\beta\alpha} s_\alpha)(X)$$
$$+ A_\beta(X)\, g_{\beta\alpha}(p)\, s_\alpha(p)$$

$$= dg_{\beta\alpha}(X)\, s_\alpha(p) + ds_\alpha(X)\, g_{\beta\alpha}(p)$$
$$+ A_\beta(X)\, g_{\beta\alpha}(p)\, s_\alpha(p)$$

$$= \left(dg_{\beta\alpha}(X) + A_\beta(X)\, g_{\beta\alpha}(p) \right) s_\alpha(p)$$
$$+ ds_\alpha(X)\, g_{\beta\alpha}(p)$$

$$= \left(g_{\alpha\beta}\, dg_{\beta\alpha} + g_{\alpha\beta} A_\beta g_{\beta\alpha} \right) s_\alpha(p)$$

$$+ \underbrace{g_{\alpha\beta}\, ds_\alpha(X)\, g_{\beta\alpha}}_{ds_\alpha}$$

$$= g_{\alpha\beta}\, dg_{\beta\alpha} + g_{\alpha\beta} A_\beta g_{\beta\alpha} + ds_\alpha \qquad (2)$$

comparing eq. (1) and (2) we get

$$\boxed{A_\alpha(X) = g_{\alpha\beta}(p)\, A_\beta(X)\, g_{\beta\alpha}(p) + g_{\alpha\beta}(p)\, dg_{\beta\alpha}(X)}$$

gauge-transformation formula.

$\boxed{\text{Tr}}$

(b) $G \subset GL(m, \mathbb{F})$ Lie subgroup $\mathfrak{g} = T_1 G$.

If $g_{\beta\alpha}(p) \in G$ $\quad \forall \quad p \in U_\alpha \cap U_\beta$

$A_\beta(X) \in \mathfrak{g}$ $\quad X \in T_p M$

If $g_{\beta\alpha}(p) \in G \Rightarrow g_{\alpha\beta}(p) \in G$

(inverse)

① $$g^{-1} A_\beta g = \frac{d}{dt} \Big(\underbrace{g^{-1}}_{\in G} \underbrace{e^{tA_\beta}}_{\in G} \underbrace{g}_{\in G} \Big) \Big|_{t=0}$$

$\in$

$\mathfrak{g}$

② $g^{-1}dg$

Consider a curve $c(t) : (-\epsilon, \epsilon) \to G$ w/ $c(0) = g \in G$

then
$$g^{-1}dg = g^{-1}\frac{d}{dt}c(t)\Big|_{t=0}$$
$$\underbrace{= \frac{d}{dt}\Big|_{t=0}\Big(\underset{\in G}{\underbrace{g^{-1}}}\;\underset{\in G}{\underbrace{c(t)}}\Big) \in G}_{\in \mathfrak{g}}$$

$\therefore$ both the terms on the RHS of part (a) $\in \mathfrak{g}$. $\square$

c) If the group G in part (b) is abelian, then the formula in part (a)
$$A_\alpha(x) = A_\beta(x) + g_{\alpha,\beta}(x)\, d g_{\beta\alpha}(x).$$

$$g^{-1}A_\beta g = \frac{d}{dt}\Big|_{t=0}\left(g^{-1}e^{tA_\beta}g\right)$$

$$= \frac{d}{dt}\Big|_{t=0}\left(e^{tA_\beta}\right) = A_\beta$$

$\therefore$ the transformation formula

$$A_\alpha = A_\beta + g^{-1}dg.$$

d) $G = U(1)$

$$\mathfrak{u}(1) \cong i\mathbb{R} \subset \mathbb{C}$$

$U(1)$ abelian , part (c) $\Rightarrow$

$$A_\alpha(x) = A_\beta(x) + g_{\alpha\beta}(x)\, dg_{\beta\alpha}(x)$$

$$dA_\alpha = dA_\beta + d\big(\underbrace{g^{-1}dg}\big)$$

$$d\left(\frac{d}{dt}\Big|_{t=0}\big(\underbrace{\qquad}_{\hat{G}}\big)\right)$$

$$d(\text{constant}) = 0$$

$\therefore$ $dA_\alpha = dA_\beta$ $\boxed{\checkmark}$

2 $\quad G \subset GL(m, \mathbb{F})$ Lie subgroup

$A \in \mathfrak{g} = T_{\mathbb{1}} G$

$\dot{\Phi}(t) = A \Phi(t) \quad , \quad \Phi(0) = \mathbb{1}$

$\Phi(t) \in G \quad \forall t.$

$$X(B) = AB \in \underset{\underset{T_B GL(m, \mathbb{F})}{\shortparallel}}{\mathbb{F}^{m \times m}} \quad , \quad A \in \mathfrak{g}$$

X defines a smooth v.f. on G. if $B \in G$.

* consider the curve $\gamma : [0,1] \to G$

s.t. $\gamma(t) = e^{tA} B$

$\dot{\gamma}(0) = ? \in T_B G$

X v.f on G , smooth - ?

(3) $\nabla_X Y$

$\rightsquigarrow \nabla_X(fY) - f\nabla_X Y + df$

difference of any two connections on M is always a tensor.

$TM \to M$

a) $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

defines a $(1,2)$ tensor field on M.

Torsion tensor.

∇ is symmetric if $T \equiv 0$. $(\nabla g = 0)$

To prove:-

T is a tensor field.

$T(fX, Y) = f\,T(X,Y)$, $f \in C^\infty(M)$

$T(X, fY) = f\,T(X,Y)$, $f \in C^\infty(M)$

$$T(fX,Y) = \nabla_{fX} Y - \nabla_Y (fX) - [fX,Y]$$

just a calculation:-

(b) $$T(\partial_j, \partial_k) = \nabla_{\partial_j} \partial_k - \nabla_{\partial_k} \partial_j - \underbrace{[\partial_j, \partial_k]}_{=0}$$

$$= \left(\Gamma^i_{jk} - \Gamma^i_{kj}\right) \partial_i$$

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

$\Rightarrow$ ∇ is symmetric $\Rightarrow$ $T(\partial_j, \partial_k) = 0$ $\forall j, k.$

$\Rightarrow$ $\Gamma^i_{jk} = \Gamma^i_{kj}$

$\Leftarrow$ $T \equiv 0$ $\Rightarrow$ ∇ is symmetric.

For the Levi-Civita connection ∇^g on (M,g)

$\Gamma^i_{jk} = \Gamma^i_{kj}$. for the LC-connection.

☺

(4) $\pi : E \to M$, N smooth manifold.

$f_0, f_1 : N \to M$ are two smoothly homotopic maps, f_0^*E and f_1^*E are isomorphic.

$$\begin{array}{ccc} f_0^*E & & f_1^*E \\ \downarrow & & \downarrow \\ N & & N \end{array}$$

Suppose h is the smooth hom. b/w f_0 and f_1

$$h : [0,1] \times N \to M$$
$$h(0,p) = f_0(p) \qquad \forall\ p \in N$$
$$h(1,p) = f_1(p)$$

$$\begin{array}{c} h^*E \\ \downarrow \\ [0,1]\times N \end{array}$$

Let ∇ be a connection on $\begin{array}{c} h^*E \\ \downarrow \\ [0,1]\times N. \end{array}$

Let $\gamma(s)$ is a path in $[0,1]\times N$

s.t. $\gamma(0) = (0,p)$, $\gamma(1) = (1,p)$

Let $P_{\gamma(s)}$ is the parallel transport of $\gamma(s)$

$$\Rightarrow \quad P_\gamma^t : h^*E_{(0,p)} \to h^*E_{(t,p)}$$

$\therefore \ \forall t, \quad h^*E_{(0,p)} \cong h^*E_{(t,p)}$

$\Rightarrow \ t=1$

$$f_0^*E_p = h^*E_{(0,p)} \cong h^*E_{(1,p)} \cong f_1^*E_p$$

$\therefore$ $\forall p \in N$ the fibres $f_0^*E_p \cong f_1^*E_p$ as vector space

and $\therefore$

$$\begin{array}{ccc} f_0^*E & \cong & f_1^*E \\ \downarrow & & \downarrow \\ N & & N. \end{array}$$

(b) M is smoothly contractible the $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ is a trivial bundle.

$f_1 : M \to M$ is the iden. map

$f_0 : M \to M$ is the const. map, $p \mapsto p_0$

$$\begin{array}{ccc} f_0^*E & \cong & f_1^*E = E \\ \downarrow & \text{(part (a))} & \\ \text{trivial bundle} & & \end{array}$$

$\forall p \in M$, $f_0^*E_p = E_{f_0(p)} = E_{p_0}$

$\therefore \quad f_0^* E \cong M \times E_{p_0}$

$\therefore \quad E \cong M \times \mathbb{F}^n$ $\quad \square$

5 $\quad \nabla$ connection on $\pi : E \to M$

flat if $\forall$ $p \in M$, $v \in E_p$ $\exists$ a nbd $p \in U \subset M$ and a section $s \in \Gamma(E|_U)$ w/ $\nabla s = 0$ and $s(p) = v$

(a) For any finite subgroup $G \subset GL(m, \mathbb{F})$ a G-structure on $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ determines a flat connection.

If $g_{ij} : U_i \cap U_j \to G$, g_{ij} constant functions

smooth function

$U_i \times G$, we define the horizontal distribution

$$H_{(x,g)}(U_i \times G) = T_{(x,g)}(U_i \times \{g\})$$

$x \in U_i$, $g \in G$.

unique for G-finite.

connection is flat:-

constant section $s: U_i \to U_i \times G$

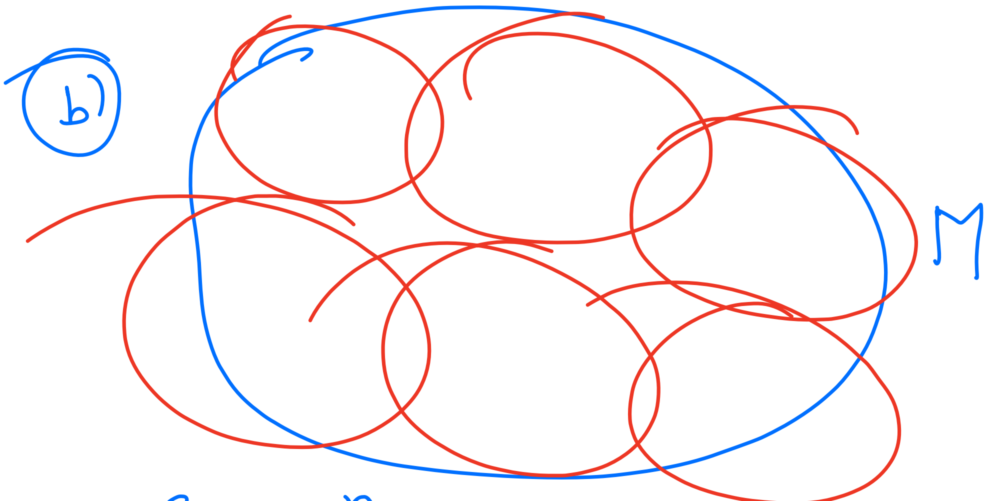
$y \mapsto (y, g)$

$$\mathrm{Vert}_{\pi(x,e)}(\nabla_x s) = 0$$

$\therefore$ $\nabla s \equiv 0$

$\boxtimes$

(b)



$\{U_\alpha\}_{\alpha=1}^{n}$, ∇ flat connection on $E \to M$

we choose $\{U_\alpha\}$ s.t. $\forall\ \alpha = 1, \dots, n$

$\exists\ \{s_a^\alpha\} \in \Gamma(E|_{U_\alpha})$ s.t.

$\nabla s_a^\alpha = 0 \quad \forall\ a = 1, \dots, m$

$(\nabla_x s)_a = (\nabla_x s)_b = 0$ on $U_a \cap U_b$

$(\nabla_x s)_b(p) = g_{ba}(p)(\nabla_x s)_a(p)$
$+ dg_{ba}(x)\, s_a(p)$

$\therefore \quad dg_{ba} = 0 \Rightarrow g_{ba}$ are constant functions on $U_a \cap U_b$

$\therefore$

∇ comes from the G-structure

$G = \{ g_{ba},\ a = 1 \dots m,\ b = 1 \dots m \}$
$\cap$
$GL(m, \mathbb{F})$

c) Show that a parallel section exists on a nbd of every point w/ arbitrary value at that point.

$$F_\nabla \equiv 0$$

$$\cap$$

$$\Omega^2(M, \mathfrak{g})$$