Humboldt-Universität zu Berlin
Institut für Mathematik
C. Wendl, S. Dwivedi, O. Müller

Differentialgeometrie I
WiSe 2021–22

Problem Set 13

To be discussed: 16.02.2022

Problem 1

If Σ is a compact oriented (and not necessarily connected) surface with at least two boundary components $\ell_1, \ell_2 \cong S^1$, one can form a new compact oriented surface Σ' by "gluing" ℓ_1 to ℓ_2, i.e. define $\Sigma' := \Sigma/\sim$, where the equivalence relation identifies each point $p \in \ell_1$ with $\varphi(p) \in \ell_2$ for some choice of orientation-reversing diffeomorphism $\varphi : \ell_1 \to \ell_2$.

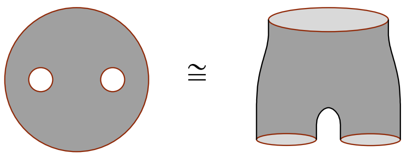
(a) Prove $\chi(\Sigma') = \chi(\Sigma)$.
Hint: You can always subdivide triangulations to arrange that ℓ_1 and ℓ_2 have the same number of vertices.

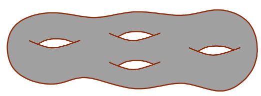
(b) A *pair of pants* is a compact surface P obtained by removing two small disjoint open disks from a closed disk $\mathbb{D}^2$. (The picture below on the left explains the terminology.) Prove $\chi(P) = -1$.

(c) Suppose Σ is a closed oriented surface, and Σ' is a new closed oriented surface obtained by "attaching a handle" to Σ: this means removing an open disk $\mathcal{D}_1 \subset \Sigma$ from Σ and another open disk from $\mathcal{D}_2 \subset \mathbb{T}^2$ from the torus, then gluing together the two boundary components of $(\Sigma \backslash \mathcal{D}_1) \sqcup (\mathbb{T}^2 \backslash \mathcal{D}_2)$. Prove $\chi(\Sigma') = \chi(\Sigma) - 2$.

(d) The *closed oriented surface* Σ_g of *genus* g is obtained by attaching $g \geqslant 0$ handles to S^2. (The figure below on the right shows one possible embedding of Σ_4 into $\mathbb{R}^3$.) Prove $\chi(\Sigma_g) = 2 - 2g$.

(e) Show that for each $g \geqslant 2$, Σ_g can be constructed by gluing together boundary components of a disjoint union of $2g - 2$ pairs of pants. Show also that $\Sigma_0 \cong S^2$ and $\Sigma_1 \cong \mathbb{T}^2$ cannot be decomposed in this way into pairs of pants.





Problem 2

A smooth polygon P in a Riemannian 2-manifold (Σ, g) is called a *geodesic n-gon* (or *geodesic triangle, hexagon* etc.) if it has exactly n edges, all of which are geodesic segments, and none of the angles at its vertices are π. Let H^2 denote the hyperbolic plane.

(a) Show that every geodesic triangle in H^2 has area strictly less than π, but there exist examples with area arbitrarily close to π.

(b) Suppose $P \subset H^2$ is a geodesic n-gon with right angles at all its vertices. What can n be? (Find an inequality that n must satisfy, and try to convince yourself with pictures that suitable n-gons actually exist whenever the inequality is satisfied.)

Remark: One can use right-angled geodesic hexagons in H^2 as puzzle pieces that glue together to form closed oriented surfaces Σ_g of any genus $g \geqslant 2$, i.e. gluing two hexagons together along every other edge produces a pair of pants, and Σ_g can be constructed by gluing together $2g - 2$ pairs of pants. This is one of the popular ways to construct metrics *with everywhere negative Gaussian curvature on Σ_g for $g \geqslant 2$. Note that by Problem Set 12 #3, Σ_g with such a metric can never be embedded isometrically into $\mathbb{R}^3$, and by Gauss-Bonnet, no such metric exists at all if $g \leqslant 1$.*

Problem 3

Suppose $E \to \Sigma$ is a complex line bundle over an oriented surface and $s \in \Gamma(E)$ has an isolated zero at $p \in \Sigma$ with $\text{ind}(s; p) \neq 0$. Show that for every neighborhood $\mathcal{U} \subset \Sigma$ of p, sufficiently small perturbations of s are guaranteed to vanish somewhere in $\mathcal{U}$. In other words, zeroes with nontrivial index cannot be "perturbed away".
Hint: Use the homotopy-invariance of winding numbers.

Problem 4

For a vector bundle $\pi : E \to M$ and section $s \in \Gamma(E)$, the *linearization* of s at a point $p \in s^{-1}(0)$ is the linear map $Ds(p) : T_pM \to E_p : X \mapsto \nabla_X s$ defined via any choice of connection ∇.

(a) Show that $Ds(p)$ is independent of the choice of connection.

(b) Assume $M = \Sigma$ is a closed oriented surface and E is a complex line bundle. Show that if $p \in s^{-1}(0)$ and $Ds(p) : T_p\Sigma \to E_p$ is invertible, then the index $\text{ind}(s; p)$ is ± 1. What determines the sign? (Think in terms of orienations.)

(c) For $\Sigma = \mathbb{R}^2$ and $E = \Sigma \times \mathbb{C}$ the trivial line bundle with sections $s \in \Gamma(E)$ regarded as functions $s : \mathbb{R}^2 \to \mathbb{C}$, consider a section of the form $s(x, y) := (x - iy)^k(x + iy)^\ell$ for integers $k, \ell \geqslant 0$. What is $\text{ind}(s; 0)$? Does the condition $\text{ind}(s; 0) = \pm 1$ imply $Ds(0)$ is invertible?
Hint: Use polar coordinates.

(d) Find an example as in part (c) such that $\text{ind}(s; 0) = 0$ and s admits arbitrarily close perturbations that have no zeroes at all.

Problem 5

(a) Show that for two complex line bundles $E, E' \to \Sigma$ over a closed oriented surface, $\int_\Sigma c_1(E \otimes E') = \int_\Sigma c_1(E) + \int_\Sigma c_1(E')$.

(b) Show that if E is a line bundle, the bundle $E \otimes E^*$ is trivial.
Hint: There is always (also for higher rank bundles) a natural bundle map from $E \otimes E^*$ to the trivial line bundle. Show that it's an isomorphism if $\text{rank}(E) = 1$.

(c) Find an explicit example of a complex line bundle that is not isomorphic to its dual bundle.
Remark: This cannot happen with real bundles, cf. Problem Set 9 #5.

Problem 6

Use the following picture and the Poincaré-Hopf theorem to give a proof independent of Problem 1(d) that $\chi(\Sigma_g) = 2 - 2g$ for $g \geqslant 2$.