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Differentialgeometrie I
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Problem Set 5: Solution to Problem 1

Problem 1

Suppose M is a 3-manifold and $\alpha \in \Omega^1(M)$ is nowhere zero, so for every $p \in M$, there is a well-defined 2-dimensional subspace $\xi_p := \ker \alpha_p \subset T_pM$. The set $\xi := \bigcup_{p\in M} \xi_p \subset TM$ in this situation is called a *smooth 2-plane field* in M. We say that ξ is *integrable* if its defining 1-form α satisfies the condition $\alpha \wedge d\alpha \equiv 0$.

(a) Show that the integrability condition depends only on ξ and not on α, i.e. for any $\beta \in \Omega^1(M)$ that is also nowhere zero and satisfies $\ker \beta_p = \xi_p$ for all $p \in M$, $\alpha \wedge d\alpha \equiv 0$ if and only if $\beta \wedge d\beta \equiv 0$.
Hint: If $\ker \alpha_p = \ker \beta_p$, how are the two cotangent vectors $\alpha_p, \beta_p \in T_p^*M$ related?

Suppose $\alpha, \beta \in \Omega^1(M)$ are both nowhere zero and satisfy $\ker \alpha_p = \ker \beta_p = \xi_p$ for all $p \in M$. Here is a basic fact from linear algebra: if two nontrivial linear functionals $\alpha_p, \beta_p : T_pM \to \mathbb{R}$ have the same kernel, then one is a multiple of the other. It follows that there exists a nowhere-zero funtion $f : M \to \mathbb{R}$ such that $\beta = f\alpha$ everywhere. Since α and β are both smooth, f will also be smooth. Now use the Leibniz rule to compute:

$$\beta \wedge d\beta = f\alpha \wedge d(f\alpha) = f\alpha \wedge (df \wedge \alpha + f\, d\alpha) = f^2\, \alpha \wedge d\alpha,$$

where the first term in parentheses has disappeared because $\alpha \wedge (df \wedge \alpha) = -\alpha \wedge (\alpha \wedge df) = -(\alpha \wedge \alpha) \wedge df = 0$, since the wedge product of a 1-form with itself is always 0. Since $f \neq 0$ everywhere, we now see that $\beta \wedge d\beta$ can vanish if and only if $\alpha \wedge d\alpha$ vanishes.

(b) Prove that the following conditions are each equivalent to integrability:

(i) $(d\alpha)_p|_{\xi_p} \in \Lambda^2 \xi_p^*$ vanishes for every $p \in M$.
Hint: Evaluate $(\alpha \wedge d\alpha)_p$ on a basis of T_pM that includes two vectors in ξ_p.

Since $\dim M = 3$, $\alpha \wedge d\alpha \in \Omega^3(M)$ is a top-dimensional form, thus $(\alpha \wedge d\alpha)_p \in \Lambda^3 T_p^*M$ vanishes at a point $p \in M$ if and only if $(\alpha \wedge d\alpha)(X_1, X_2, X_3) = 0$ for some basis $X_1, X_2, X_3 \in T_pM$. For this we can choose any basis we like, so let us choose one so that X_2, X_3 form a basis of the 2-dimensional subspace $\xi_p \subset T_pM$ and $X_1 \notin \xi_p$, which means $\alpha(X_2) = \alpha(X_3) = 0$ but $\alpha(X_1) \neq 0$. Using Equation (9.4) from the notes,

$$\begin{aligned}(\alpha \wedge d\alpha)(X_1, X_2, X_3) &= \frac{3!}{1!2!} \frac{1}{3!} \sum_{\sigma \in S_3} (-1)^{|\sigma|} (\alpha \otimes d\alpha)(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) \\ &= \frac{1}{2} \sum_{\sigma \in S_3} (-1)^{|\sigma|} \alpha(X_{\sigma(1)}) \cdot d\alpha(X_{\sigma(2)}, X_{\sigma(3)}),\end{aligned}$$

but in this expression, permutations that satisfy $\sigma(1) \neq 1$ will contribute nothing because $\alpha(X_2) = \alpha(X_3) = 0$, so there are only two nontrivial terms in the sum, and since $d\alpha$ is antiysymmetric,

$$\begin{aligned}(\alpha \wedge d\alpha)(X_1, X_2, X_3) &= \frac{1}{2} \left[\alpha(X_1) \cdot d\alpha(X_2, X_3) - \alpha(X_1) \cdot d\alpha(X_3, X_2)\right] \\ &= \alpha(X_1) \cdot d\alpha(X_2, X_3).\end{aligned}$$

We already know $\alpha(X_1) \neq 0$, so this expression vanishes if and only if $d\alpha(X_2, X_3) = 0$. Now recall that X_2, X_3 is a basis of ξ_p, and observe that the restriction of $(d\alpha)_p \in \Lambda^2 T_p^* M$ to a bilinear form on $\xi_p \subset T_p M$ is a top-dimensional alternating form on ξ_p, i.e. an element of $\Lambda^2 \xi_p^*$, which therefore vanishes if and only if it evaluates to zero on the basis X_2, X_3, thus $(\alpha \wedge d\alpha)_p = 0$ is now equivalent to the condition $(d\alpha)_p|_{\xi_p} = 0 \in \Lambda^2 \xi_p^*$.

(ii) For every pair of vector fields $X, Y \in \mathfrak{X}(M)$ with $X(p), Y(p) \in \xi_p$ for all $p \in M$, $[X, Y] \in \mathfrak{X}(M)$ also satisfies $[X, Y](p) \in \xi_p$ for all $p \in M$.
Hint: Use our original definition of the exterior derivative, via C^∞-linearity.

Using the $k = 1$ case of Proposition 8.6 in the notes, any 1-form α and vector fields X, Y satisfy

$$d\alpha(X, Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X, Y]). \tag{1}$$

If X and Y both take values in ξ everywhere, then the first two terms on the right hand side vanish, leaving only $\alpha([X, Y])$, which vanishes precisely at the points where $[X, Y]$ has its values in ξ. If that is true everywhere, it follows that $d\alpha(X, Y)$ vanishes everywhere, and if this is assumed to be true for *every* pair of vector fields valued in ξ, then it means $(d\alpha)|_\xi \equiv 0$, since one can always choose X and Y to form a basis of ξ_p at any given point p. This means that the condition of part (b)(i) is satisfied, and ξ is therefore integrable. Conversely, if the condition $d\alpha|_\xi \equiv 0$ is satisfied, then the left hand side of (1) vanishes for all $X, Y \in \mathfrak{X}(M)$ with values in ξ, thus forcing $\alpha([X, Y])$ to vanish, which means $[X, Y]$ takes values in ξ everywhere.

(c) Using Cartesian coordinates (x, y, z) on $M := \mathbb{R}^3$, suppose $\alpha = f(x)\,dy + g(x)\,dz$ for smooth functions $f, g : \mathbb{R} \to \mathbb{R}$. Under what conditions on f and g is ξ integrable? Show that if these conditions hold, then for every point $p \in \mathbb{R}^3$ there exists a 2-dimensional submanifold $\Sigma \subset \mathbb{R}^3$ such that $p \in \Sigma$ and $T_q\Sigma = \xi_q$ for all $q \in \Sigma$.

We can regard f and g as functions on $\mathbb{R}^3$ whose partial derivatives in the y and z directions vanish everywhere, thus $df = f'(x)\,dx$ and $dg = g'(x)\,dx$. We then compute $d\alpha = d(f\,dy + g\,dz) = df \wedge dy + dg \wedge dz = f'\,dx \wedge dy + g'\,dx \wedge dz$, thus

$$\begin{aligned}
\alpha \wedge d\alpha &= (f\,dy + g\,dz) \wedge \left(f'\,dx \wedge dy + g'\,dx \wedge dz\right) \\
&= ff'\,dy \wedge dx \wedge dy + fg'\,dy \wedge dx \wedge dz + gf'\,dz \wedge dx \wedge dy + gg'\,dz \wedge dx \wedge dz \\
&= (-fg' + gf')\,dx \wedge dy \wedge dz,
\end{aligned}$$

where we have eliminated the two terms that contained wedge products of dy or dz with themselves, and used permutations to rewrite $dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$ and $dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$. This shows that $\alpha \wedge d\alpha$ vanishes if and only if the function $f(x)g'(x) - g(x)f'(x)$ vanishes. I like to imagine $x \mapsto (f(x), g(x))$ as a path in $\mathbb{R}^2$ and $fg' - gf'$ as the determinant of a 2-by-2 matrix: its vanishing then tells us that for all x, the vectors $(f(x), g(x))$ and $(f'(x), g'(x))$ in $\mathbb{R}^2$ are linearly dependent, which means that the path $(f(x), g(x))$ is confined to a single line through the origin. It cannot ever touch the origin, since that would cause α to vanish somewhere, so the conclusion is that there exists a constant nonzero vector $(a, b) \in \mathbb{R}^2$ and a nowhere-zero smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $(f(x), g(x)) = \varphi(x) \cdot (a, b) \in \mathbb{R}^2$, and α can thus be written in the form

$$\alpha = \varphi(x)\,(a\,dy + b\,dz)\,.$$

Recalling part (a), observe that the function $\varphi(x)$ does not affect the kernel of α at any point, so if we just want to understand the 2-plane field ξ, we are now free to ignore φ and write

$$\xi_p = \ker(a\,dy + b\,dz) \qquad \text{for all } p \in \mathbb{R}^3.$$

The difference between this situation and the picture below is that since a and b are constant, the 2-plane field we are considering here does not "twist": in fact there are two constant nonzero vector fields

$$V(x,y,z) := b\frac{\partial}{\partial y} - a\frac{\partial}{\partial z} \qquad \text{and} \qquad Z(x,y,z) := \frac{\partial}{\partial x}$$

on $\mathbb{R}^3$ whose span at every point $p = (x,y,z) \in \mathbb{R}^3$ is ξ_p. The flows of these vector fields are easy to compute, and they commute with each other; if you now start at any given point $p \in \mathbb{R}^3$ and follow the flows of both V and Z, you obtain a surface (more specifically a plane) whose tangent space at each point is identical to ξ at that point. In other words, the surface I'm describing is the image of

$$\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 : (s,t) \mapsto \varphi_V^s \circ \varphi_Z^t(p),$$

and more precisely, if $p = (x_0, y_0, z_0) \in \mathbb{R}^3$, this surface is

$$\Sigma = \left\{ (x_0 + s, y_0 + bt, z_0 - at) \;\middle|\; s,t \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Remark: The result of part (c) is a special case of the Frobenius integrability theorem, which we will prove later in this course. In case you're curious, the following picture gives an example of what $\xi \subset T\mathbb{R}^3$ might look like if it is ***not*** *integrable. Can you picture a 2-dimensional submanifold that is everywhere tangent to ξ? (I didn't think so.)*