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Differentialgeometrie I

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Problem Set 6: Solution to Problem 4

Problem 4

Using Cartesian coordinates (x, y, z) on $\mathbb{R}^3$, let $\omega := x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \Omega^2(\mathbb{R}^3)$, and let $i : S^2 \hookrightarrow \mathbb{R}^3$ denote the inclusion of the unit sphere.

(a) Show that for an appropriate choice of orientation on S^2, $d\mathrm{vol}_{S^2} := i^*\omega \in \Omega^2(S^2)$ is the Riemannian volume form corresponding to the Riemannian metric on S^2 that is induced by the Euclidean inner product of $\mathbb{R}^3$.
Hint: Pick a good vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ with which to write ω as $\iota_X(dx \wedge dy \wedge dz)$.

We claim first that $\omega = \iota_X(dx \wedge dy \wedge dz)$ for the "radial" vector field $X := x\partial_x + y\partial_y + z\partial_z$. To see this, recall that $dx \wedge dy \wedge dz$ is a sum of permutations of tensor products such as $dx \otimes dy \otimes dz$, where terms like $dx \otimes dz \otimes dy$ for which the permutation is *odd* come with minus signs. Computing the interior product with ∂_x, only permutations that place dx at the beginning will contribute, since $dy(\partial_x) = dz(\partial_x) = 0$, thus

$$\begin{aligned}\iota_{\partial_x}(dx \wedge dy \wedge dz) &= (dx \otimes dy \otimes dz)(\partial_x, \cdot, \cdot) - (dx \otimes dz \otimes dy)(\partial_x, \cdot, \cdot) \\ &= dx(\partial_x)\,dy \otimes dz - dx(\partial_x)\,dz \otimes dy = dy \otimes dz - dz \otimes dy = dy \wedge dz.\end{aligned}$$

Observe next that $dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy$, since both of the last two expressions can be obtained via *even* permutations of the 1-forms dx, dy and dz. The interior products of $dx \wedge dy \wedge dz$ with ∂_y and ∂_z can thus be derived via exactly the same calculation as above, but using the other two expressions for $dx \wedge dy \wedge dz$, which give

$$\iota_y(dx \wedge dy \wedge dz) = dz \wedge dx, \qquad \text{and} \qquad \iota_z(dx \wedge dy \wedge dz) = dx \wedge dy.$$

Since $\iota_X(dx \wedge dy \wedge dz)$ depends linearly on X, we conclude

$$\iota_{x\partial_x + y\partial_y + z\partial_z}(dx \wedge dy \wedge dz) = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy = \omega$$

as claimed.
Now observe that along S^2, X is a unit normal vector field for the sphere, and since $dx \wedge dy \wedge dz$ is the Riemannian volume form for the Riemannian metric on $\mathbb{R}^3$ given by the Euclidean inner product, a result proved in lecture (see Prop. 11.14 in the notes) implies that the restriction of $\iota_X(dx \wedge dy \wedge dz)$ to S^2 is a volume form compatible with its induced Riemannian metric. That restriction is precisely $i^*\omega \in \Omega^2(S^2)$.

(b) Show that in the spherical coordinates (θ, ϕ) of Problem Set 1 #1, $d\mathrm{vol}_{S^2} = \cos\phi\,d\theta \wedge d\phi$.

The Cartesian and spherical coordinates are related to each other by

$$x = r\cos\theta\cos\phi, \qquad y = r\sin\theta\cos\phi, \qquad z = r\sin\phi.$$

These can be understood as equalities between smooth functions that are valid on whichever open subset of $\mathbb{R}^3$ we choose as the domain of the spherical chart; a standard choice would be the complement of the set $\widetilde{E} := \{(x, 0, z) \mid x \geqslant 0\} \subset \mathbb{R}^3$, so that the image of the chart (r, θ, ϕ) is $(0, \infty) \times (0, 2\pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^3$. Restricting to $r = 1$, we obtain a chart (θ, ϕ) on S^2 with domain

$$\mathcal{U} := S^2 \backslash E \subset S^2, \qquad \text{where} \qquad E := \widetilde{E} \cap S^2 \subset S^2,$$

and image $(0, 2\pi) \times (-\pi/2, \pi/2) \subset \mathbb{R}^2$. The coordinates (x, y, z) no longer define a chart when restricted to $\mathcal{U} \subset S^2$, but they are still well-defined smooth functions on $\mathcal{U}$ and are now related to θ and ϕ by

$$x = \cos\theta\cos\phi, \qquad y = \sin\theta\cos\phi, \qquad z = \sin\phi, \qquad \text{on } \mathcal{U} \subset S^2. \tag{1}$$

To write down $i^*\omega$, we can first use the fact that wedge products and exterior derivatives are respected by pullbacks, giving rise to the slightly pedantic formula

$$i^*\omega = (i^*x)\, d(i^*y) \wedge d(i^*z) + (i^*y)\, d(i^*z) \wedge d(i^*x) + (i^*z)\, d(i^*x) \wedge d(i^*y). \tag{2}$$

I call this "pedantic" because it can be made to look a lot simpler: the function $i^*x = x \circ i$ is actually just the restriction of the coordinate function $x : \mathbb{R}^3 \to \mathbb{R}$ to S^2, and similarly with the other coordinates, which can then be written on $\mathcal{U} \subset S^2$ in terms of θ and ϕ using (1), so we obtain

$$\begin{aligned} i^*\omega = {} & (\cos\theta\cos\phi)\, d(\sin\theta\cos\phi) \wedge d(\sin\phi) + (\sin\theta\cos\phi)\, d(\sin\phi) \wedge d(\cos\theta\cos\phi) \\ & + (\sin\phi)\, d(\cos\theta\cos\phi) \wedge d(\sin\theta\cos\phi). \end{aligned}$$

To simplify this, we use the fact that any function f has differential $df = \frac{\partial f}{\partial x^i}\, dx^i$ on the domain of any chart $(x^1, \ldots, x^n)$, so using (θ, ϕ) as the chart on $\mathcal{U}$, we find

$$\begin{aligned} i^*\omega = {} & (\cos\theta\cos\phi)\,(\cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi) \wedge (\cos\phi\, d\phi) \\ & + (\sin\theta\cos\phi)(\cos\phi\, d\phi) \wedge (-\sin\theta\cos\phi\, d\theta - \cos\theta\sin\phi\, d\phi) \\ & + (\sin\phi)\,(-\sin\theta\cos\phi\, d\theta - \cos\theta\sin\phi\, d\phi) \wedge (\cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi)\,. \end{aligned}$$

The next step is to combine all terms that contain wedge products of $d\theta$ with $d\phi$, use the relation $d\phi \wedge d\theta = -d\theta \wedge d\phi$ to reorder them all into products of smooth functions with $d\theta \wedge d\phi$, and throw out all terms that contain $d\theta \wedge d\theta = d\phi \wedge d\phi = 0$: this gives

$$\begin{aligned} i^*\omega &= \left(\cos^2\theta\cos^3\phi + \sin^2\theta\cos^3\phi + \sin^2\theta\sin^2\phi\cos\phi + \cos^2\theta\sin^2\phi\cos\phi\right)\, d\theta \wedge d\phi \\ &= \left(\cos^3\phi + \sin^2\phi\cos\phi\right)\, d\theta \wedge d\phi = \cos\phi\, d\theta \wedge d\phi. \end{aligned}$$

Note that this is a volume form since the values of ϕ on the domain of our spherical chart lie in $(-\pi/2, \pi/2)$, so that $\cos\phi > 0$. The positivity of $\cos\phi$ also indicates that if we assign to S^2 the orientation for which $i^*\omega$ is a positive volume form, then (θ, ϕ) is an oriented chart. (This is why I chose to write the spherical chart as (θ, ϕ) instead of (ϕ, θ); the latter would not have turned out to be an oriented chart.)

(c) On the open upper hemisphere $\mathcal{U}_+ := \{z > 0\} \subset S^2 \subset \mathbb{R}^3$, one can define a chart $(x, y) : \mathcal{U}_+ \to \mathbb{R}^2$ by restricting to $\mathcal{U}_+$ the usual Cartesian coordinates x and y, which are then related to the z-coordinate on this set by $z = \sqrt{1 - x^2 - y^2}$. Show that $d\mathrm{vol}_{S^2} = \frac{1}{z}\, dx \wedge dy$ on $\mathcal{U}_+$.

We can start from (2), but write x instead of i^*x and so forth since the latter is just the restriction of $x : \mathbb{R}^3 \to \mathbb{R}$ to S^2. Incorporating also the relation $z = \sqrt{1 - x^2 - y^2}$, we have $z^2 = 1 - x^2 - y^2$ and thus

$$d(z^2) = 2z\, dz = d(1 - x^2 - y^2) = -2x\, dx - 2y\, dy,$$

implying

$$dz = -\frac{x}{z}\,dx - \frac{y}{z}\,dy \qquad \text{on } \mathcal{U}_+ \subset S^2.$$

Combining this with (2) and the fact that $x^2 + y^2 + z^2 = 1$ on S^2 gives

$$\begin{aligned} i^*\omega &= x\,dy \wedge \left(-\frac{x}{z}\,dx - \frac{y}{z}\,dy\right) + y\left(-\frac{x}{z}\,dx - \frac{y}{z}\,dy\right) \wedge dx + z\,dx \wedge dy \\ &= -\frac{x^2}{z}\,dy \wedge dx - \frac{y^2}{z}\,dy \wedge dx + z\,dx \wedge dy = \frac{z^2 + x^2 + y^2}{z}\,dx \wedge dy = \frac{1}{z}\,dx \wedge dy. \end{aligned}$$

Note that since $z > 0$ on $\mathcal{U}_+$, this computation proves that (x, y) is also an oriented chart for the orientation on S^2 such that $i^*\omega > 0$.

(d) Compute the surface area of $S^2 \subset \mathbb{R}^3$ in two ways: once using the formula for $d\mathrm{vol}_{S^2}$ in part (b), and once using part (c) instead. In both cases, you should be able to express the answer in terms of a single Lebesgue integral[1] over a region in $\mathbb{R}^2$, and there will be no need for any partition of unity.

Here's a computation using the formula $d\mathrm{vol}_{S^2} = i^*\omega = \cos\phi\, d\theta \wedge d\phi$ from part (b). The domain on which that formula is valid is the complement $\mathcal{U} = S^2 \backslash E$ of the set $E = \{(x, 0, z) \in S^2 \mid x \geqslant 0\}$, which is a semicircle connecting the north and south poles $(0, 0, \pm 1) \in S^2$. It should be easy to convince yourself that E has measure zero, i.e. its intersection with the domain of any chart looks like a set of measure zero in the corresponding coordinates. (I will skip this detail.) Now, Exercise 11.2 in the notes implies

$$\int_{S^2} d\mathrm{vol}_{S^2} = \int_{\mathcal{U}} d\mathrm{vol}_{S^2} + \int_E d\mathrm{vol}_{S^2} = \int_{\mathcal{U}} d\mathrm{vol}_{S^2}.$$

Since the domain of integration in the last expression is contained in the domain of a single chart (θ, ϕ), and we saw above that this chart has the correct orientation, Proposition 11.3 from the notes allows us to use *only* that chart for the computation and avoid choosing a partition of unity. The image of $(\theta, \phi) : \mathcal{U} \to \mathbb{R}^2$ is $(0, 2\pi) \times (-\pi/2, \pi/2)$, so we find

$$\int_{\mathcal{U}} d\mathrm{vol}_{S^2} = \int_{\mathcal{U}} \cos\phi\, d\theta \wedge d\phi = \int_{(0,2\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} \cos\phi\, d\theta\, d\phi = 2\pi \int_{-\pi/2}^{\pi/2} \cos\phi = 4\pi.$$

If we want to use the formula $d\mathrm{vol}_{S^2} = \frac{1}{z}\,dx \wedge dy$ from part (c) instead, then it is useful to observe that the hemisphere $\mathcal{U}_+ \subset S^2$ on which this formula is valid has the same area as its reflection $\mathcal{U}_- := \{z < 0\} \subset S^2$, and the complement of these two sets in S^2 is the circle $\{z = 0\} \subset S^2$, which is a set of measure zero. Exercise 11.2 in the notes thus implies

$$\int_{S^2} d\mathrm{vol}_{S^2} = 2\int_{\mathcal{U}_+} d\mathrm{vol}_{S^2} = 2\int_{\mathcal{U}_+} \frac{1}{z}\,dx \wedge dy,$$

where the latter integral can be computed entirely in the oriented chart $(x, y) : \mathcal{U}_+ \to \mathbb{R}^2$ due to Proposition 11.3 in the notes. The image of this chart is the unit ball $B^2(1) := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 1\right\}$, and since $z = \sqrt{1 - x^2 - y^2}$ on $\mathcal{U}_+$, we have

$$\int_{S^2} d\mathrm{vol}_{S^2} = 2\int_{B^2(1)} \frac{1}{\sqrt{1 - x^2 - y^2}}\,dx\,dy.$$

[1] You may assume that the upper and lower hemispheres have the same area.

This integral on $B^2(1) \subset \mathbb{R}^2$ is unfortunately not as easy to compute as the one we obtained in spherical coordinates, but it becomes computable if we switch from (x, y) to polar coordinates: write $x = \rho\cos\psi$ and $y = \rho\sin\psi$, then the classical change of variables formula gives

$$2\int_{B^2(1)} \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = 2\int_{(0,1)\times(0,2\pi)} \frac{1}{\sqrt{1-\rho^2}}\,\rho\,d\rho\,d\psi = 4\pi\int_0^1 \frac{\rho\,d\rho}{\sqrt{1-\rho^2}} = 4\pi.$$

I'm assuming you don't need any tips on computing $\int_0^1 \frac{\rho\,d\rho}{\sqrt{1-\rho^2}}$.

Comment: what actually happened in this last step was that we replaced (x, y) with yet another chart on S^2 for which the integral turns out to be more easily computable. Strictly speaking, if we want to regard (ρ, ψ) as a chart, then it cannot be defined on all of $\mathcal{U}_+$, but is well defined as soon as we exclude a suitable subset such as $E \cap \mathcal{U}_+ \subset \mathcal{U}_+$; we can denote the complement of this set by $\mathcal{U}'_+ \subset \mathcal{U}_+$ and assume the chart $(\rho, \psi) : \mathcal{U}'_+ \to \mathbb{R}^2$ has image $(0, 1) \times (0, 2\pi)$. Using the relations $x = \rho\cos\psi$, $y = \rho\sin\psi$ and $z = \sqrt{1-x^2-y^2} = \sqrt{1-\rho^2}$ on $\mathcal{U}'_+$, we find

$$\begin{aligned} d\mathrm{vol}_{S^2} &= \frac{1}{z}\,dx \wedge dy = \frac{1}{\sqrt{1-\rho^2}}\,d(\rho\cos\psi) \wedge d(\rho\sin\psi) \\ &= \frac{1}{\sqrt{1-\rho^2}}\,(\cos\psi\,d\rho - \rho\sin\psi\,d\psi) \wedge (\sin\psi\,d\rho + \rho\cos\psi\,d\psi) \\ &= \frac{1}{\sqrt{1-\rho^2}}\,\left(\rho\cos^2\psi + \rho\sin^2\psi\right)\,d\rho \wedge d\psi = \frac{\rho}{\sqrt{1-\rho^2}}\,d\rho \wedge d\psi. \end{aligned}$$

Since the function $\frac{\rho}{\sqrt{1-\rho^2}}$ is positive, this shows that (ρ, ψ) is also an oriented chart on its domain, and since the set $\mathcal{U}_+ \cap E$ we had to exclude in order to define it has measure zero, we can now reframe the computation above as

$$\begin{aligned} 2\int_{\mathcal{U}_+} d\mathrm{vol}_{S^2} &= 2\int_{\mathcal{U}'_+} d\mathrm{vol}_{S^2} = 2\int_{\mathcal{U}'_+} \frac{\rho}{\sqrt{1-\rho^2}}\,d\rho \wedge d\psi = 2\int_{(0,1)\times(0,2\pi)} \frac{\rho}{\sqrt{1-\rho^2}}\,d\rho\,d\psi \\ &= 4\pi\int_0^1 \frac{\rho\,d\rho}{\sqrt{1-\rho^2}} = 4\pi. \end{aligned}$$