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Differentialgeometrie I
WiSe 2021–22

Problem Set 7

To be discussed: 7–8.12.2021

Problem 1
Prove: For each $k \geqslant 0$, a k-form $\omega \in \Omega^k(M)$ is closed if and only for every compact oriented $(k+1)$-dimensional submanifold $L \subset M$ with boundary, $\int_{\partial L} \omega = 0$.

Problem 2
Prove: On S^1, a 1-form $\lambda \in \Omega^1(S^1)$ is exact if and only if $\int_{S^1} \lambda = 0$.
Hint: Try to construct a primitive $f : S^1 \to \mathbb{R}$ by integrating λ along paths.

Problem 3
Suppose $\mathcal{O}$ is an open subset of either $\mathbb{H}^n$ or $\mathbb{R}^n$. We call $\mathcal{O}$ a *star-shaped* domain if for every $p \in \mathcal{O}$, it also contains the points $tp \in \mathbb{R}^n$ for all $t \in [0,1]$. It follows that $h(t,p) := tp$ defines a smooth homotopy $h : [0,1] \times \mathcal{O} \to \mathcal{O}$ between the identity and the constant map whose value is the origin, making $\mathcal{O}$ smoothly contractible. Use this homotopy to produce an explicit formula for a linear operator $P : \Omega^k(\mathcal{O}) \to \Omega^{k-1}(\mathcal{O})$ for each $k \geqslant 1$ satisfying

$$\omega = P(d\omega) + d(P\omega)$$

for all $\omega \in \Omega^k(\mathcal{O})$. In particular, whenever ω is a closed k-form, $P\omega$ is a primitive of ω.
Hint: Start with the chain homotopy that we constructed in lecture for proving the homotopy invariance of de Rham cohomology. As a sanity check, the answer to this problem can be found at the end of Lecture 13 in the notes, but try to find it yourself first.

Problem 4
Show that the wedge product descends to an associative and graded-commutative product $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, defined by

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

This is called the *cup product* on de Rham cohomology.
Remark: There is similarly a cup product on singular cohomology, to which this one is isomorphic via de Rham's theorem. But this one is easier to define, and is thus often used in practice as a surrogate for the singular cup product.

Problem 5
For this exercise, identify the n-torus $\mathbb{T}^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$ (recall from Problem Set 2 #1 that there is a natural diffeomorphism). For any sufficiently small open set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, the usual Cartesian coordinates $x^1, \dots, x^n : \widetilde{\mathcal{U}} \to \mathbb{R}$ can be used to define a smooth chart $(\mathcal{U}, x)$ on $\mathbb{T}^n$ where

$$\mathcal{U} := \left\{ [p] \in \mathbb{T}^n \;\middle|\; p \in \widetilde{\mathcal{U}} \right\}, \qquad x([p]) := (x^1(p), \dots, x^n(p)) \text{ for } p \in \widetilde{\mathcal{U}}.$$

(a) Show that the coordinate differentials $dx^1, \dots, dx^n \in \Omega^1(\mathcal{U})$ arising from the chart $(\mathcal{U}, x)$ described above are independent of the choice of the set $\widetilde{\mathcal{U}} \subset \mathbb{R}^n$, i.e. the definitions of the coordinate differentials obtained from two different choices $\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2 \subset \mathbb{R}^n$ coincide on the region $\mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathbb{T}^n$ where they overlap.

(b) As a consequence of part (a), the 1-forms $dx^1, \dots, dx^n \in \Omega^1(\mathbb{T}^n)$ are well-defined on the entire torus, and they are obviously locally exact and therefore closed, but they might not actually be exact since none of the coordinates $x^1, \dots, x^n$ admit smooth definitions globally on $\mathbb{T}^n$. Show in fact that for any constant vector $(a_1, \dots, a_n) \in \mathbb{R}^n \backslash \{0\}$, the 1-form

$$\lambda := a_i \, dx^i \in \Omega^1(\mathbb{T}^n)$$

is closed but not exact.
Hint: You only need to find one smooth map $\gamma : S^1 \to \mathbb{T}^n$ such that $\int_{S^1} \gamma^* \lambda \neq 0$.

(c) One can similarly produce closed k-forms $\omega \in \Omega^k(\mathbb{T}^n)$ for any $k \leqslant n$ by choosing constants $a_{i_1 \dots i_k} \in \mathbb{R}$ and writing

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{T}^n). \tag{1}$$

Show that for every nontrivial k-form of this type, one can find a cohomology class $[\alpha] \in H^{n-k}_{\mathrm{dR}}(\mathbb{T}^n)$ such that the cup product $[\omega] \cup [\alpha] \in H^n_{\mathrm{dR}}(\mathbb{T}^n)$ defined in Problem 4 is nontrivial, and deduce from this that ω is not exact.
Hint: Can you choose $\alpha \in \Omega^{n-k}(\mathbb{T}^n)$ so that $\omega \wedge \alpha$ is a volume form?

Remark: One can show that all cohomology classes in $H^k_{\mathrm{dR}}(\mathbb{T}^n)$ are representable by k-forms with constant coefficients as in (1), thus $\dim H^k_{\mathrm{dR}}(\mathbb{T}^n) = \binom{n}{k}$.

Problem 6

For V an n-dimensional vector space, the main goal of this exercise is to show that for every $v \in V$, the operator $\iota_v : \Lambda^* V^* \to \Lambda^* V^*$ defined by $\iota_v \omega := \omega(v, \cdot, \dots, \cdot)$ satisfies the graded Leibniz rule

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_v \beta) \tag{2}$$

for all $\alpha \in \Lambda^k V^*$ and $\beta \in \Lambda^\ell V^*$. The statement is trivial if $v = 0$, so assume otherwise, in which case we may as well assume v is the first element e_1 of a basis $e_1, \dots, e_n \in V$, whose dual basis we can denote by $e^1_*, \dots, e^n_* \in V^* = \Lambda^1 V^*$.

(a) Prove that (2) holds whenever α and β are both products of the form $\alpha = e^{i_1}_* \wedge \dots \wedge e^{i_k}_*$ and $\beta = e^{j_1}_* \wedge \dots \wedge e^{j_\ell}_*$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_\ell$.
Hint: Consider separately a short list of cases depending on whether each of i_1 and j_1 are 1 and whether the sets $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_\ell\}$ are disjoint.

(b) Deduce via linearity that (2) holds always.

(c) Using (2), prove that for any manifold M and vector field $X \in \mathfrak{X}(M)$, the operator $P_X := d \circ \iota_X + \iota_X \circ d : \Omega^*(M) \to \Omega^*(M)$ satisfies the Leibniz rule

$$P_X(\alpha \wedge \beta) = P_X \alpha \wedge \beta + \alpha \wedge P_X \beta.$$

$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$

This is one of the main steps in a proof of Cartan's formula $\mathcal{L}_X \omega = P_X \omega$.

Problem 7

Prove that for any closed symplectic manifold (M, ω), $H^2_{\mathrm{dR}}(M)$ is nontrivial.
Hint: What can you say about the n-fold cup product of $[\omega] \in H^2_{\mathrm{dR}}(M)$ with itself?

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Problem 1
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$$d\omega = 0 \iff \forall \; L^{k+1} \subset M \quad \int_{\partial L} \omega = 0$$

$\Rightarrow$

Suppose $d\omega = 0$. want:- $\int_{\partial L} \omega = 0$

$$\int_{\partial L} \omega \overset{\text{Stokes' Theorem.}}{=} \int_L d\omega = 0$$

$\Leftarrow$

If $\int_{\partial L} \omega = 0 \quad \forall$ compact, oriented $L^{k+1} \subset M$

then $d\omega = 0$.

Let $p \in M$ choose $\{X_1, \dots, X_{k+1}\}$ Linearly ind. set of vectors T_pM. Let D be a $(k+1)$-dimensional disc; it passes through p, tangent to the subspace spanned by $\{X_1, \dots, X_{k+1}\}$.

$$\int_{\partial D} \omega = \int_D d\omega = 0$$

approximates the values of $d\omega$ itself.

$(k+1)$-dim vol D

$p \in M$ was arbitrary.

$$\Rightarrow \quad d\omega(X_1, \ldots, X_{k+1}) = 0$$

$$\Rightarrow \quad d\omega = 0 .$$

$$\left| \int d\omega \right| \leq |d\omega| \, vol^{(k+1)}$$

$\lambda \in \Omega^1(M)$

Problem 2

Prove: On S^1, a 1-form $\lambda \in \Omega^1(S^1)$ is exact if and only if $\int_{S^1} \lambda = 0$.
Hint: Try to construct a primitive $f : S^1 \to \mathbb{R}$ by integrating λ along paths.

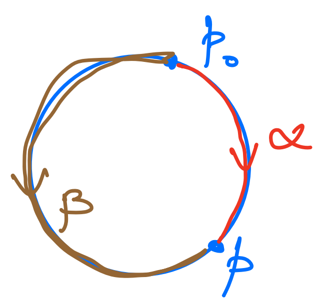
If $\int_{S^1} \lambda = 0$ then λ is exact, i.e., $\lambda = df$ for some $f \in C^\infty(S^1)$.

Let $p_0 \in S^1$ fixed.

Let $p \in S^1$ $\alpha : [0,1] \to S^1$ be a path in S^1 joining p_0 to p.

define $f : S^1 \to \mathbb{R}$

$$f(p) = \int_0^1 \lambda(\alpha'(t))\, dt \quad \text{—} \; 1$$



$$\varphi(e^{2\pi i t}) = \begin{cases} \alpha(t) & , \; 0 \leq t \leq 1/2 \\ \beta(2-2t), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\int_{S^1} \varphi^* \lambda = \int_0^1 \alpha^* \lambda - \int_0^1 \beta^* \lambda = \int_{S^1} \lambda = 0$$

from ①

$$\frac{d}{dt} f(\alpha(t)) = \lambda(\alpha'(t)) \quad \forall\ t \in [0,1]$$

$$\forall\ \alpha : [0,1] \to S^1 \quad w/\ \alpha(0) = p_0$$

$\therefore\ df = \lambda$ which is a primitive

$\Rightarrow$ λ is exact.

$\boxed{\not\!\!/}$

Problem 3

③ Suppose $\mathcal{O}$ is an open subset of either $\mathbb{H}^n$ or $\mathbb{R}^n$. We call $\mathcal{O}$ a *star-shaped* domain if for every $p \in \mathcal{O}$, it also contains the points $tp \in \mathbb{R}^n$ for all $t \in [0,1]$. It follows that $h(t,p) := tp$ defines a smooth homotopy $h : [0,1] \times \mathcal{O} \to \mathcal{O}$ between the identity and the constant map whose value is the origin, making $\mathcal{O}$ smoothly contractible. Use this homotopy to produce an explicit formula for a linear operator $P : \Omega^k(\mathcal{O}) \to \Omega^{k-1}(\mathcal{O})$ for each $k \geqslant 1$ satisfying

$$\omega = P(\cancel{d\omega}) + d(P\omega) \qquad \omega = d(P\omega)$$

for all $\omega \in \Omega^k(\mathcal{O})$. In particular, whenever ω is a closed k-form, $P\omega$ is a primitive of ω.
Hint: Start with the chain homotopy that we constructed in lecture for proving the homotopy invariance of de Rham cohomology. As a sanity check, the answer to this problem can be found at the end of Lecture 13 in the notes, but try to find it yourself first.

$$h : [0,1] \times \mathcal{O} \to \mathcal{O}$$

$h(t,p) = tp$ smooth homotopy b/w

$id(p) = p$, $h(0,p) = 0 \in \mathcal{O}$

$\mathcal{O}$ - contractible.

We want $P : \Omega^k(\mathcal{O}) \to \Omega^{k-1}(\mathcal{O})$

$$\omega = P(d\omega) + d(P\omega).$$

$L^{k-1} \subset \mathcal{O}$ submanifold, compact, oriented. , ω is a $(k-1)$ form

$$h^*(d\omega) \in \Omega^k([0,1] \times \mathcal{O})$$

$$\int_{[0,1]\times L} h^*(d\omega) = \int_{[0,1]\times L} d(h^*\omega) \underset{\text{Stokes'}}{=} \int_{\partial([0,1]\times L)} h^*\omega$$

$$= \int_{\partial[0,1]\times L} h^*\omega - \int_{[0,1]\times\partial L} h^*\omega$$

$$= \int_{\{1\}\times L} h^*\omega - \int_{\{0\}\times L} h^*\omega - \int_{[0,1]\times\partial L} h^*\omega$$

$$= \int_L \omega - 0 - \int_{[0,1]\times\partial L} h^*\omega$$

$$\therefore \int_{[0,1]\times L} h^*(d\omega) = \int_L \omega - \int_{[0,1]\times\partial L} h^*\omega \quad \text{—— } (1)$$

$$P : \Omega^k \to \Omega^{k-1} \quad \omega$$

$$\underset{\mathbb{R}^n}{(P\omega)_p}(X_1, \dots, X_{k-1}) = \int_0^1 (h^*\omega)_{(t,p)}(\partial_t, X_1, \dots, X_{k-1})\, dt \qquad h = tp \quad \mathbb{R}^n$$

$$= \int_0^1 ((tp)^*\omega)_{tp}(\partial_t, X_1, \dots, X_{k-1})\, dt$$

$$= \int_0^1 \omega_{tp}((tp)_*\partial_t, (tp)_*X_1, \dots, (tp)_*X_{k-1})\, dt$$

$$= \int_0^1 \omega_{tp}(p, tX_1, tX_2, \dots, tX_{k-1})\, dt$$

$$= \int_0^1 t^{k-1}\omega(p, X_1, \dots, X_{k-1})\, dt \quad \text{—} (2)$$

$(t, p) \mapsto tp$

$\varphi : M^n \to N$

X

$$\varphi_* X = \mathrm{Jac}[\varphi] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\int_{[0,1]\times L} h^*(d\omega) = \int_L P(d\omega) \Rightarrow$$

$\therefore$ in eq (1)

$$\int_L P(d\omega) = \int_L \omega - \int_{\partial L} P\omega$$

$$\Rightarrow \int_L P(d\omega) + \int_L d(P\omega) = \int_L \omega$$

$$\Rightarrow \int_L P(d\omega) + d(P\omega) = \int_L \omega$$

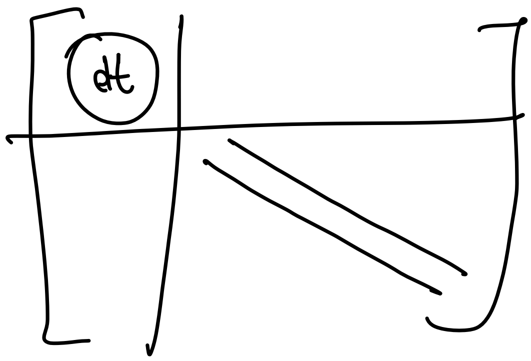
$\because$ L was arbitrary

$\omega = P(d\omega) + d(P\omega)$ w/ P is as given in eq. (2).

$\boxtimes$

Remark:- This was one of the main steps in the proof of homotopy invariance of de Rham coh. groups.

$[0,1] \times \mathbb{R} \to \mathbb{R}$



$\varphi : M^m \to N^n$

$\varphi_* X = \left[\mathrm{Jac}(\varphi) \right]_{n \times m} \left[\quad \right]$

(4)

Problem 4

Show that the wedge product descends to an associative and graded-commutative product $\cup : H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \to H^{k+\ell}_{\mathrm{dR}}(M)$, defined by

$$[\alpha] \cup [\beta] := [\alpha \wedge \beta].$$

Well-defined.

$\tilde{\alpha} \in [\alpha] \Rightarrow \tilde{\alpha} = \alpha + d\eta, \quad \eta \in \Omega^{k-1}(M)$

$\tilde{\beta} \in [\beta] \Rightarrow \tilde{\beta} = \beta + d\gamma, \quad \gamma \in \Omega^{\ell-1}(M)$

$[\tilde{\alpha} \wedge \tilde{\beta}] = \quad [\alpha \wedge \beta]$

$$\tilde{\alpha} \wedge \tilde{\beta} = (\alpha + d\eta) \wedge (\beta + d\gamma)$$

$$= \alpha \wedge \beta + \alpha \wedge d\gamma + d\eta \wedge \beta + d\eta \wedge d\gamma$$

$$[\tilde{\alpha} \wedge \tilde{\beta}] = [\alpha \wedge \beta] + \underbrace{[\alpha \wedge d\gamma]}_{d(-)} + [d\eta \wedge \beta] + [d\eta \wedge d\gamma]$$

$$0 = \qquad\qquad 0 \qquad\qquad 0$$

$$\underbrace{d\alpha}_{=0} \wedge \gamma + (-1)^k \alpha \wedge d\gamma = d(\alpha \wedge \gamma) \qquad d(\eta \wedge \beta) \qquad d(\eta \wedge d\gamma)$$

cup-product is well-defined. $\square$

in practice as a surrogate for the singular cup product.

5.

Problem 5

For this exercise, identify the n-torus $\mathbb{T}^n$ with the quotient $\mathbb{R}^n/\mathbb{Z}^n$ (recall from Problem Set 2 #1 that there is a natural diffeomorphism). For any sufficiently small open set $\tilde{\mathcal{U}} \subset \mathbb{R}^n$, the usual Cartesian coordinates $x^1, \dots, x^n : \tilde{\mathcal{U}} \to \mathbb{R}$ can be used to define a smooth chart $(\mathcal{U}, x)$ on $\mathbb{T}^n$ where

$$\mathcal{U} := \left\{ [p] \in \mathbb{T}^n \;\middle|\; p \in \tilde{\mathcal{U}} \right\}, \qquad x([p]) := (x^1(p), \dots, x^n(p)) \text{ for } p \in \tilde{\mathcal{U}}.$$

$x \mapsto x$

$x \mapsto x-1$

(a) Show that the coordinate differentials $dx^1, \dots, dx^n \in \Omega^1(\mathcal{U})$ arising from the chart $(\mathcal{U}, x)$ described above are independent of the choice of the set $\tilde{\mathcal{U}} \subset \mathbb{R}^n$, i.e. the definitions of the coordinate differentials obtained from two different choices $\tilde{\mathcal{U}}_1, \tilde{\mathcal{U}}_2 \subset \mathbb{R}^n$ coincide on the region $\mathcal{U}_1 \cap \mathcal{U}_2 \subset \mathbb{T}^n$ where they overlap.

$y_1, \dots, y_n$ on $\tilde{\mathcal{U}} \subset \mathbb{R}^n$

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

(b) As a consequence of part (a), the 1-forms $dx^1, \dots, dx^n \in \Omega^1(\mathbb{T}^n)$ are well-defined on the entire torus, and they are obviously locally exact and therefore closed, but they might not actually be exact since none of the coordinates $x^1, \dots, x^n$ admit smooth definitions globally on $\mathbb{T}^n$. Show in fact that for any constant vector $(a_1, \dots, a_n) \in \mathbb{R}^n \backslash \{0\}$, the 1-form

$$\lambda := a_i \, dx^i \in \Omega^1(\mathbb{T}^n)$$

is closed but not exact.
Hint: You only need to find one smooth map $\gamma : S^1 \to \mathbb{T}^n$ such that $\int_{S^1} \gamma^* \lambda \neq 0$.

(c) One can similarly produce closed k-forms $\omega \in \Omega^k(\mathbb{T}^n)$ for any $k \leqslant n$ by choosing constants $a_{i_1 \dots i_k} \in \mathbb{R}$ and writing

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{T}^n). \tag{1}$$

Show that for every nontrivial k-form of this type, one can find a cohomology class $[\alpha] \in H^{n-k}_{\mathrm{dR}}(\mathbb{T}^n)$ such that the cup product $[\omega] \cup [\alpha] \in H^n_{\mathrm{dR}}(\mathbb{T}^n)$ defined in Problem 4 is nontrivial, and deduce from this that ω is not exact.
Hint: Can you choose $\alpha \in \Omega^{n-k}(\mathbb{T}^n)$ so that $\omega \wedge \alpha$ is a volume form?

$a = (a_1, \dots, a_n)$ constant vector in $\mathbb{R}^n \setminus \{0\}$.

$\lambda = a_i dx^i \in \Omega^1(T^n)$ is closed but not exact.

$$d\lambda = d(a_i) \wedge dx^i + a_i \, d(dx^i)$$
$$= 0 + 0 \qquad = 0.$$

$\exists \ \gamma : S^1 \to \mathbb{T}^n$ s.t. $\int_{S^1} \gamma^* \lambda \neq 0$.

WLOG, let $a_1 \neq 0$, $S^1 \cong \mathbb{R}/\mathbb{Z}$, $\mathbb{T}^n \cong (\mathbb{R}/\mathbb{Z})^n$

$\gamma : S^1 \to \mathbb{T}^n$

$\mathbb{R}/\mathbb{Z} \ni x \longmapsto [x, 0, \dots, 0] \in (\mathbb{R}/\mathbb{Z})^n$

$$\gamma^* \lambda = \lambda \circ \gamma = a_1 \, dx^1$$

$$\Rightarrow \int_{S^1} \gamma^* \lambda = \int_0^1 a_1 \, dx^1 = a_1 \neq 0$$

$\therefore$ by the characterization of exact forms, λ is not exact.

c) $\omega = \sum' a_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(\mathbb{T}^n)$

$\alpha = \sum' * (dx^{i_1} \wedge \dots \wedge dx^{i_k})$

$* : \Omega^k(M) \to \Omega^{n-k}(M)$, M n-dim, g, dVol.

$\beta \in \Omega^k(M)$, $\quad * \beta \in \Omega^{n-k}(M)$ s.t.

$\beta \wedge * \beta = d\mathrm{vol}_M$.

on $\mathbb{R}^3$ $\quad \{x_1, x_2, x_3\}$, $\quad dx^1 \wedge dx^2 \wedge dx^3$

$*(dx^1) = dx^2 \wedge dx^3$

$[\omega] \cup [\alpha] = [\omega \wedge \alpha] = [d\mathrm{vol}]_{\mathbb{T}^n}$

if ω is exact. then $\omega = d\beta$

$$\int_{\mathbb{T}^n} \omega \wedge \alpha = \int_{\mathbb{T}^n} d\beta \wedge \alpha = \underbrace{\int_{\mathbb{T}^n} d(\beta \wedge \alpha)}_{=0} + \underbrace{(-1)^{k-1} \int_{\mathbb{T}} \beta \wedge d\alpha}_{=0}$$

$\|$ by the choice of α $\qquad$ Stokes' Thm

$$\int_{\mathbb{T}^n} d\mathrm{vol} = \mathrm{vol}(\mathbb{T}^n) \neq 0$$

$\therefore$ ω cannot be exact. $\square$

Problem 7

7

Prove that for any closed symplectic manifold (M, ω), $H^2_{\mathrm{dR}}(M)$ is nontrivial.
Hint: What can you say about the n-fold cup product of $[\omega] \in H^2_{\mathrm{dR}}(M)$ with itself?

ω symplectic if $\forall p \in M$ $\exists \{e^1, \dots, e^n, f^1, \dots, f^n\}$

$$\omega = \sum_{i=1}^{n} de^i \wedge df^i \quad \text{(Darboux coordinates)}$$

Guess:- $[\omega] \in H^2_{dR}(M)$, $[\omega] \neq 0$.

<u>Remark</u>:- If A manifold M^{2n} is symplectic then necc. $H^2_{dR}(M) \neq 0$.

M^{2n}

$d\omega = 0$.

$$\underbrace{[\omega] \cup [\omega] \cup \cdots [\omega]}_{n\text{-times}} \Big|_{\mathcal{U}}$$

$$\| $$

$$[de^1 \wedge df^1 \wedge de^2 \wedge df^2 \wedge \cdots \wedge de^n \wedge df^n] = [dvol] \neq 0$$

$$[\omega]^n = [\omega^n] \neq 0$$

If $[\omega^n] = 0$ then $\omega^n = d\Omega$

$$\int_M \omega^n = \int_M d\Omega = \int_{\partial M} \Omega = 0$$

M-closed compact, w/o boundary

$\Rightarrow [\omega] \neq 0$

$\Rightarrow [\omega] \in H^2_{dR}(M)$ is the non-trivial homology class $\Rightarrow$ $H^2_{dR}(M) \neq 0$. ∎.

$g \to (0,2)$ $\quad \alpha \in \Omega^k(M)$, $\quad H^k_{dR}(M) \neq 0$.