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Differentialgeometrie I
WiSe 2021–22

Problem Set 8

To be discussed: 15.12.2021

Problem 1
Show that if X is a topological space with open subset $\mathcal{U} \subset X$ and a locally finite collection of continuous functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in I}$ whose supports satisfy $\operatorname{supp}(f_\alpha) \subset \mathcal{U}$ for every $\alpha \in \mathcal{U}$, then $\sum_{\alpha \in I} f_\alpha$ also has support in $\mathcal{U}$.

Problem 2
Without mentioning Riemannian metrics, prove that a smooth n-manifold M admits a volume form $\omega \in \Omega^n(M)$ if and only if M is orientable.
Hint: If you were to take the existence of Riemannian metrics as given, then the existence of the volume form $\omega \in \Omega^n(M)$ would follow because every oriented Riemannian manifold has a canonical volume form. But do not use this. Try instead constructing ω directly, with the aid of a partition of unity.

Problem 3
Prove the following improvement on the theorem from lecture that every manifold M is paracompact: every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of M admits a locally finite refinement $\{\mathcal{O}_\beta\}_{\beta \in J}$ in which each of the sets $\mathcal{O}_\beta$ is the domain of a chart.
Hint: The proof we worked through in lecture requires only one minor adjustment.

Problem 4
Suppose E is a smooth vector bundle (real of complex) of rank $m \geqslant 0$ over an n-manifold M. We proved in lecture that the total space of E admits a smooth atlas such that the natural bundle projection $\pi : E \to M$ is a smooth map. By a theorem from the second lecture in this course, the atlas on E determines a natural topology, and before we're allowed to call E a "manifold", we must prove that this topology is metrizable. Prove this by constructing a Riemannian metric on E, using only the fact that M (but not necessarily E) is metrizable.
Hint: It would help to know that every open cover of E admits a subordinate partition of unity, but you do not know this. You do know it however for M.

Problem 5
For a smooth vector bundle E over M with local trivialization[1] $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$, every section $s : M \to E$ is determined on the subset $\mathcal{U}_\alpha \subset M$ by its so-called *local representation*, which is the unique function $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ such that

$$\Phi_\alpha(s(p)) = (p, s_\alpha(p)) \qquad \text{for all } p \in \mathcal{U}_\alpha.$$

Show that if $(\mathcal{U}_\alpha, \Phi_\alpha)$ and $(\mathcal{U}_\beta, \Phi_\beta)$ are two local trivializations of E and $s : M \to E$ is a section, then the local representations $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ and $s_\beta : \mathcal{U}_\beta \to \mathbb{F}^m$ are related to each other on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ in terms of the transition function $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ by

$$s_\beta(p) = g_{\beta\alpha}(p) s_\alpha(p) \qquad \text{for } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

[1] Here, as in the lecture, $\mathbb{F}$ denotes a field which is either $\mathbb{R}$ or $\mathbb{C}$, and we are assuming that the fibers of our vector bundle are real or complex accordingly.

Problem 6

In lecture we considered a real line bundle ℓ over S^1, defined as follows: viewing S^1 as the unit circle in $\mathbb{C}$, define the set $\ell \subset S^1 \times \mathbb{R}^2$ as the union of the sets $\{e^{i\theta}\} \times \ell_{e^{i\theta}} \subset S^1 \times \mathbb{R}^2$ for all $\theta \in \mathbb{R}$, where the 1-dimensional subspace $\ell_{e^{i\theta}} \subset \mathbb{R}^2$ is given by

$$\ell_{e^{i\theta}} = \mathbb{R}\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \subset \mathbb{R}^2.$$

For any $\theta_0 \in \mathbb{R}$, we can set $p := e^{i\theta_0} \in S^1$ and define a local trivialization for ℓ over $S^1 \backslash \{p\} \subset S^1$ by

$$\Phi : \ell|_{S^1\backslash\{p\}} \to (S^1\backslash\{p\}) \times \mathbb{R} : \left(e^{i\theta}, c\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}\right) \mapsto (e^{i\theta}, c), \tag{1}$$

with θ assumed to vary in the interval $(\theta_0, \theta_0 + 2\pi)$. Prove:

(a) Any two local trivializations defined as in (1) with different choices of $\theta_0 \in \mathbb{R}$ are smoothly compatible.

(b) ℓ is a smooth subbundle of the trivial 2-plane bundle $S^1 \times \mathbb{R}^2$.

(c) There exists no continuous section of ℓ that is nowhere zero.

(d) ℓ is not globally trivial.