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Differentialgeometrie I

WiSe 2021–22

Problem Set 8

To be discussed: 15.12.2021

Problem 1

Show that if X is a topological space with open subset $\mathcal{U} \subset X$ and a locally finite collection of continuous functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in I}$ whose supports satisfy $\operatorname{supp}(f_\alpha) \subset \mathcal{U}$ for every $\alpha \in \mathcal{U}$, then $\sum_{\alpha \in I} f_\alpha$ also has support in $\mathcal{U}$.

Problem 2

Without mentioning Riemannian metrics, prove that a smooth n-manifold M admits a volume form $\omega \in \Omega^n(M)$ if and only if M is orientable.
Hint: If you were to take the existence of Riemannian metrics as given, then the existence of the volume form $\omega \in \Omega^n(M)$ would follow because every oriented Riemannian manifold has a canonical volume form. But do not use this. Try instead constructing ω directly, with the aid of a partition of unity.

Problem 3

Prove the following improvement on the theorem from lecture that every manifold M is paracompact: every open cover $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of M admits a locally finite refinement $\{\mathcal{O}_\beta\}_{\beta \in J}$ in which each of the sets $\mathcal{O}_\beta$ is the domain of a chart.
Hint: The proof we worked through in lecture requires only one minor adjustment.

Problem 4

Suppose E is a smooth vector bundle (real of complex) of rank $m \geqslant 0$ over an n-manifold M. We proved in lecture that the total space of E admits a smooth atlas such that the natural bundle projection $\pi : E \to M$ is a smooth map. By a theorem from the second lecture in this course, the atlas on E determines a natural topology, and before we're allowed to call E a "manifold", we must prove that this topology is metrizable. Prove this by constructing a Riemannian metric on E, using only the fact that M (but not necessarily E) is metrizable.
Hint: It would help to know that every open cover of E admits a subordinate partition of unity, but you do not know this. You do know it however for M.

Problem 5

For a smooth vector bundle E over M with local trivialization[1] $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{F}^m$, every section $s : M \to E$ is determined on the subset $\mathcal{U}_\alpha \subset M$ by its so-called *local representation*, which is the unique function $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ such that

$$\Phi_\alpha(s(p)) = (p, s_\alpha(p)) \qquad \text{for all } p \in \mathcal{U}_\alpha.$$

Show that if $(\mathcal{U}_\alpha, \Phi_\alpha)$ and $(\mathcal{U}_\beta, \Phi_\beta)$ are two local trivializations of E and $s : M \to E$ is a section, then the local representations $s_\alpha : \mathcal{U}_\alpha \to \mathbb{F}^m$ and $s_\beta : \mathcal{U}_\beta \to \mathbb{F}^m$ are related to each other on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ in terms of the transition function $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{F})$ by

$$s_\beta(p) = g_{\beta\alpha}(p) s_\alpha(p) \qquad \text{for } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta.$$

[1] Here, as in the lecture, $\mathbb{F}$ denotes a field which is either $\mathbb{R}$ or $\mathbb{C}$, and we are assuming that the fibers of our vector bundle are real or complex accordingly.

Problem 6

In lecture we considered a real line bundle ℓ over S^1, defined as follows: viewing S^1 as the unit circle in $\mathbb{C}$, define the set $\ell \subset S^1 \times \mathbb{R}^2$ as the union of the sets $\{e^{i\theta}\} \times \ell_{e^{i\theta}} \subset S^1 \times \mathbb{R}^2$ for all $\theta \in \mathbb{R}$, where the 1-dimensional subspace $\ell_{e^{i\theta}} \subset \mathbb{R}^2$ is given by

$$\ell_{e^{i\theta}} = \mathbb{R}\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix} \subset \mathbb{R}^2.$$

For any $\theta_0 \in \mathbb{R}$, we can set $p := e^{i\theta_0} \in S^1$ and define a local trivialization for ℓ over $S^1 \backslash \{p\} \subset S^1$ by

$$\Phi : \ell|_{S^1\backslash\{p\}} \to (S^1\backslash\{p\}) \times \mathbb{R} : \left(e^{i\theta}, c\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{pmatrix}\right) \mapsto (e^{i\theta}, c), \tag{1}$$

with θ assumed to vary in the interval $(\theta_0, \theta_0 + 2\pi)$. Prove:

(a) Any two local trivializations defined as in (1) with different choices of $\theta_0 \in \mathbb{R}$ are smoothly compatible.

(b) ℓ is a smooth subbundle of the trivial 2-plane bundle $S^1 \times \mathbb{R}^2$.

(c) There exists no continuous section of ℓ that is nowhere zero.

(d) ℓ is not globally trivial.

Problem Set 8

① X top space $\underset{\text{open}}{U \subset X}$ locally finite collection of cont. functions $\{f_\alpha : X \to \mathbb{R}\}_{\alpha \in I}$, $\text{supp}(f_\alpha) \subset U$ $\forall \alpha \in I$.

Want :- $\text{supp}(\Sigma f_\alpha) \subset U$.

We proceed by contradiction. Let $x \in \text{supp}(\Sigma f_\alpha)$ but $x \notin U$.

$\Rightarrow$ $f_\alpha(x) = 0$ $\forall \alpha \in I$.

$\Rightarrow$ $\exists$ a subsequence $\alpha_k \in I$ and $x_k \to x$ s.t. $f_{\alpha_k}(x_k) \neq 0$.

$\{f_\alpha\}$ is locally finite $\Rightarrow$ $\exists$ a nbd $V \ni x$ s.t. only finitely many f_{α_k}'s are non-zero.

$\Rightarrow$ $\exists$ atleast one $f_{\alpha_{k_0}}$ w/ $f_{\alpha_{k_0}}(x_k) \neq 0$.

and $x_k \to x$ $\Rightarrow$ $x \in \text{supp}(f_{\alpha_{k_0}})$ $\Rightarrow\Leftarrow$

$\therefore$ $x \in U$. ∎

(2) M^n volume form $vol \in \Omega^n(M) \Longrightarrow M$ is orientable.

note:- $\Longleftarrow$

$$\Omega = \{ \omega \in \Lambda^n(V^*) \mid \omega(v_1, \dots, v_n) > 0 \text{ for some pos. orien. basis of } V \}$$

Ω is a convexset.

$\omega_1, \omega_2 \in \Omega$, $t\omega_1 + (1-t)\omega_2 > 0 \quad \forall\, t \in [0,1]$.

$\tau_i \in [0,1]$, $i = 1, \dots, k$, $\sum \tau_i = 1$ then

$$\sum_{i=1}^{k} \tau_i \omega_i > 0.$$

Let $\{(U_\alpha, x_\alpha)\}_{\alpha \in I}$ is an open cover for M s.t U_α is the domain of a chart;, vol_α is an orientation on U_α

$\{\varphi_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$

$vol = \sum \varphi_\alpha vol_\alpha$

$\varphi_\alpha vol_\alpha \in \Gamma(\Lambda^n T^* U_\alpha)$ as a top degree form on M which vanishes outside U_α.

vol is smooth, makes sense.

$\forall\, p \in M$, $vol_p \in \Lambda^n T_p^* M$ is a convex combination of volume forms $\Longrightarrow$ again a vol. form.

$\Rightarrow$ we get $vol \in \Omega^n(M)$ which is the required vol. form.

$\Rightarrow$ vol. form $\Rightarrow$ M is orientable.

$\boxtimes$

③ M is paracompact. Thm. 15.13 in Lec. notes

Let $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M and $M = \bigcup_{j=1}^{\infty} K_j$

w/ K_i compact.

Choose an open nbd V_1 of K_1 s.t $\overline{V_1}$ is compact

$\Rightarrow$ $\overline{V_1} \cup K_2$ is compact.

choose a nbd V_2 of $\overline{V_1} \cup K_2$ s.t. $\overline{V_2}$ is compact

$\Rightarrow$ $\overline{V_2} \cup K_3$ is compact.

$\vdots$

$$\phi = V_0 \subset V_1 \subset \overline{V_1} \subset V_2 \subset \overline{V_2} \cdots \subset \bigcup_{j=1}^{\infty} V_j = M$$

We consider the annular region

$$A_j = \overline{V_j} \setminus V_{j-1} \subset M \quad , j = 1, 2, 3, \ldots$$

$$A_1 = \overline{V_1} \setminus \{V_0 = \phi\} = \overline{V_1}$$

$A_2 = \bar{\mathcal{V}}_2 \setminus \mathcal{V}_1 \quad \cdots$

annular regions are compact and $\bigcup A_j = M$

pick an open covering $\{O^j_\beta \subset M\}_{\beta \in I_j}$ of A_j s.t.

O^j_β is diffeo. to an open ball in $\mathbb{R}^n$ and $O^j_\beta \subset U_\alpha$ for some α.

$O^j_\beta \subset \mathcal{V}_{j+1} \setminus \mathcal{V}_{j-2}$

Union of all these $\{O^j_\beta\}$ is again $M \Rightarrow$ they form an open cover of M

$\downarrow$

refinement of $\{U_\alpha\}$

$\Rightarrow$ M is paracompact and the open sets in the refinement can be assumed to be coordinate charts.

④ $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ v.b. rank k then $\exists$ a bundle metric on E.

$p \in M$, E_p, $\langle , \rangle_p$.

Let $\{(U_\alpha, \varphi_\alpha)\}$ is a trivializing open cover for M

$$\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{R}^k, \quad U_\alpha \subset M$$

$$\downarrow \{x\} \times \mathbb{R}^k$$

Riem. metric $g_M|_{U_\alpha} + g_{eucl}$.

If $\{\rho_\alpha\}$ is a partition of unity on M subordinate to $\{U_\alpha\}$ then

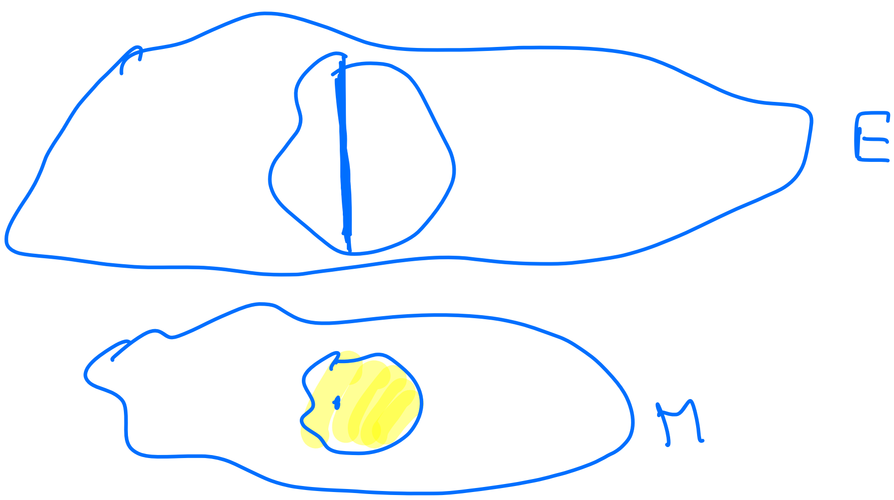
$$g_E = \sum \rho_\alpha \varphi_\alpha^* \left(g_M|_{U_\alpha} + g_{eucl} \right)$$

is a Riemannian metric on E.

$$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \quad \text{bundle}$$

If s and t are sections of E

then $\langle s, t \rangle \in C^\infty(M)$.



(5) $\begin{array}{c} E \\ \downarrow \text{rank } m \\ M \end{array}$ $\quad \Phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{F}^m$

$s : M \to E$ section

$s_\alpha : U_\alpha \to \mathbb{F}^m$ s.t.

$\Phi_\alpha(s(p)) = (p, s_\alpha(p)) \quad \forall p \in U_\alpha.$

<u>Want</u>:- If (U_α, Φ_α) and (U_β, Φ_β) are two local trivializations of E and $s : M \to E$ is a sec.

then $s_\beta(p) = g_{\beta\alpha}(p) \cdot s_\alpha(p) \quad \forall p \in U_\alpha \cap U_\beta$

$g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m, \mathbb{F})$.

$\because$ φ_α and φ_β are local triv. $\Rightarrow$ on $U_\alpha \cap U_\beta$,

$\exists$ $g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(m, \mathbb{F})$

$\forall$ $(p, v) \in U_\alpha \cap U_\beta \times \mathbb{F}^m$

$$\varphi_\beta(p, v) = \varphi_\alpha(p, g_{\beta\alpha}(p) v)$$

$$(p, s_\beta(p)) = \varphi_\beta(s(p)) = g_{\beta\alpha}(p)\, \varphi_\alpha(s(p))$$
$$= (p, g_{\beta\alpha}(p)\, s_\alpha(p))$$

$\forall$ $p \in U_\alpha \cap U_\beta$.

$\therefore \quad s_\beta(p) = g_{\beta\alpha}(p) \cdot s_\alpha(p).$

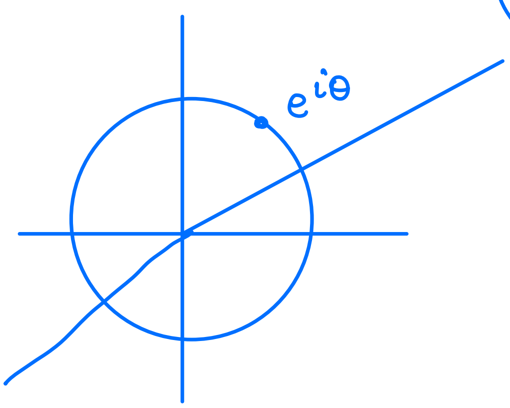
$\boxtimes$

6 $\quad \begin{array}{c} l \\ \downarrow \\ S^1 \end{array}$ $\quad S^1 \subset \mathbb{C}$

$l \subset S^1 \times \mathbb{R}^2$ as the union of the sets $\{e^{i\theta}\} \times l_{e^{i\theta}} \subset S^1 \times \mathbb{R}^2$

$\theta \in \mathbb{R}$, $\quad l_{e^{i\theta}}$ is the line

$$l_{e^{i\theta}} = \mathbb{R}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \subset \mathbb{R}^2$$



$\theta_0 \in \mathbb{R}$, $\quad p = e^{i\theta_0} \in S^1$ define a local triv. for l over $S^1 \setminus \{p\} \subset S^1$ by

$$\Phi : l|_{S^1\setminus\{p\}} \longrightarrow S^1\setminus\{p\} \times \mathbb{R}$$

$$\left(e^{i\theta}, c\begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}\right) \mapsto (e^{i\theta}, c)$$

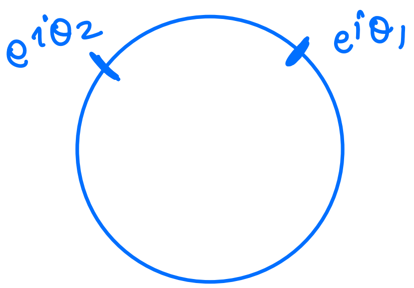
$$\theta \in (\theta_0, \theta_0 + 2\pi)$$

a) Any two local trivia. are smoothly compatible.

$\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 \neq \theta_2$

Then $\Phi_1 : \ell|_{S^1 \setminus \{e^{i\theta_1}\}} \to S^1 \setminus \{e^{i\theta_1}\} \times \mathbb{R}$

$\Phi_1 \circ \Phi_2^{-1} : S^1 \setminus \{e^{i\theta_1}, e^{i\theta_2}\} \times \mathbb{R} \to S^1 \setminus \{e^{i\theta_1}, e^{i\theta_2}\} \times \mathbb{R}$



$\Phi_1 \circ \Phi_2^{-1}(e^{i\theta}, c) \mapsto \Phi_1\left(e^{i\theta}, c\begin{pmatrix}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}\right)$

$$\begin{cases} (e^{i\theta}, c) & \theta \in (\theta_1, \theta_2) \\ (e^{i\theta}, -c) & \theta \in (\theta_2, \theta_1 + 2\pi) \end{cases}$$

The transition function is smooth.

local trivializations are smoothly compatible.

(b) ℓ is a smooth subbundle of $S^1 \times \mathbb{R}^2$.

(c) Suppose $\exists$ nowhere vanishing section s of ℓ

we can define a smooth map

$F : [0, 2\pi] \to \mathbb{R}$ s.t.

$$s(e^{i\theta}) = (e^{i\theta}, F(\theta))$$

$\because$ s is nowhere zero $\Rightarrow$ $\exists\ \theta \in \mathbb{R}$ w/ $F(\theta) \neq 0$.

from the transition functions in part (a)

$$F(\theta) = -F(\theta + 2\pi)$$

$\therefore$ F must have value zero

$\Rightarrow$ s is indeed vanishing

$\therefore$ The line bundle $\ell \to S^1$ cannot have any cont. section which is nowhere zero.

(d) If ℓ were globally trivial

$$\ell \overset{\varphi}{\cong} S^1 \times \mathbb{R} \qquad (M \times \mathbb{R}^k)$$

The constant section $(e^{i\theta}, 1)$ is a nowhere vanishing global section on $S^1 \times \mathbb{R}$ $\Rightarrow$ $\varphi^*(e^{i\theta}, 1)$ should be a nowhere section on $\begin{matrix} \ell \\ \downarrow \\ S^1 \end{matrix}$,

which is impossible from (c)

$\therefore$ $\begin{matrix} \ell \\ \downarrow \\ S^1 \end{matrix}$ is not a trivial bundle.

$$\begin{array}{c} E \\ \big\downarrow \text{rank } n \\ M \end{array}$$ is trivial $\Longleftrightarrow$ there are n global sections that form a basis on each fiber.

$s_1, \ldots, s_n$

$p \in M$, $\{s_1(p), \ldots, s_n(p)\}$ basis for E_p.

$TS^1 \cong S^1 \times \mathbb{R}$ $\quad$ TS^2