GROMOV-WITTEN THEORY, WINTERSEMESTER 2022–2023, HU BERLIN

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This is not a set of lecture notes, but merely a brief summary of the contents of each lecture, with reading suggestions and a compendium of exercises. The suggested reading will usually not correspond precisely to what was covered in the lectures, but there will often be a heavy overlap.

1. Week 1

Lecture 1 (18.10.2022): Rough sketch of the Gromov-Witten invariants. This lecture was meant as a rough overview of the subject this course is about, so it contained very few precise definitions and no complete proofs.

- Counting lines through two points in $\mathbb{C}^2$ and $\mathbb{CP}^2$
- $H_2(\mathbb{CP}^n) \cong \mathbb{Z}$ generated by $[L] \in H_2(\mathbb{CP}^n)$ for any line $L \subset \mathbb{CP}^n$
- Definition: a **rational curve of degree** $d \in \mathbb{N}$ **in** $\mathbb{CP}^n$ is an equivalence class $[f]$ of holomorphic maps $f : \mathbb{CP}^1 \to \mathbb{CP}^n$ with $[f] := f_*[\mathbb{CP}^1] = d[L] \in H_2(\mathbb{CP}^n)$, where $f \sim g$ if and only if $f = g \circ \varphi$ for some holomorphic diffeomorphism (i.e. **biholomorphic map**) $\varphi : \mathbb{CP}^1 \to \mathbb{CP}^1$.
- Statement of Kontsevich's recursion formula for the number N_d of rational curves of degree d in $\mathbb{CP}^2$ through $3d - 1$ generic points
- Definition of the moduli space $\mathcal{M}_{g,m}(M, A)$ of holomorphic curves $u : \Sigma \to M$ of genus g in a complex manifold M, homologous to $A \in H_2(M)$, with m marked points $\zeta_1, \dots, \zeta_m \in \Sigma$, quotiented by reparametrizations
- Use of the evaluation map

  $$\mathrm{ev} = (\mathrm{ev}_1, \dots, ev_m) : \mathcal{M}_{g,m}(M, A) \to M^{\times m} : [u, (\zeta_1, \dots, \zeta_m)] \mapsto (u(\zeta_1), \dots, u(\zeta_m))$$

  to define N_d as a count of $\mathrm{ev}^{-1}(p_1, \dots, p_m)$ for $M := \mathbb{CP}^2$, $A := d[L]$, $m := 3d - 1$ and generic tuples $(p_1, \dots, p_m) \in M^{\times m}$.
- The almost complex structures $J : TM \to TM$ on a complex manifold M and $j : T\Sigma \to T\Sigma$ on a Riemann surface Σ (= "complex 1-dimensional manifold")
- $u : \Sigma \to M$ is holomorphic $\Leftrightarrow Tu \circ j = J \circ Tu \Leftrightarrow \partial_s u + J\partial_t u = 0$ in local holomorphic coordinates $s + it$ on regions in Σ (cf. Exercise 1.1)
- Brief sketch (to be discussed in detail later) of the local identification of $\mathcal{M}_{g,m}(M, A)$ with $\bar{\partial}_J^{-1}(0)$ for a smooth nonlinear Fredholm section $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$ of a Banach space bundle $\mathcal{E} \to \mathcal{B}$
- Riemann-Roch formula (to be discussed later) $\Rightarrow$ linearization of $\bar{\partial}_J$ at a zero has complex Fredholm index $(n - 3)(1 - g) + \langle c_1(TM), A\rangle + m \in \mathbb{Z}$
- Convenient fictions:
  1. The linearization of $\bar{\partial}_J$ is always surjective (so implicit function theorem then implies $\mathcal{M}_{g,m}(M, A)$ is a smooth manifold of complex dimension $(n-3)(1-g)+\langle c_1(TM), A\rangle+ m$
  2. $\mathcal{M}_{g,m}(M, A)$ is compact

One can then naively define Gromov-Witten invariants as multilinear maps $\mathrm{GW}_{g,m,A} : H^*(M)^{\times m} \to \mathbb{Q}$ by

$$\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m) := \# \operatorname{ev}^{-1}(\bar{\alpha}_1 \times \dots \times \bar{\alpha}_m) = \int_{\mathcal{M}_{g,m}(M,A)} \operatorname{ev}_1^* \alpha_1 \wedge \dots \operatorname{ev}_m^* \alpha_m. \tag{1.1}$$

Here $\bar{\alpha}_j \subset M$ for each $j = 1, \dots, m$ is any closed oriented submanifold representing the homology class Poincaré dual to α_j,[1] chosen generically so that ev is transverse to $\bar{\alpha}_1 \times \dots \times \bar{\alpha}_m \subset M^{\times m}$, and the differential forms in the integral are arbitrary choices of closed forms representing the cohomology classes $\alpha_j \in H^*(M)$. The two expressions are equivalent due to Poincaré duality, and they make sense if and only if the dimensional condition

$$\sum_{j=1}^m |\alpha_j| = \dim_{\mathbb{R}} \mathcal{M}_{m,g}(M, A) = 2(n-3)(1-g) + 2\langle c_1(TM), A\rangle + 2m$$

is satisfied; if it isn't, we set $\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m) := 0$.

- Interpretation of (1.1): $\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m)$ is the algebraic count of holomorphic curves $u : \Sigma \to M$ of genus g with marked points $\zeta_1, \dots, \zeta_m \in \Sigma$ satisfying the constraints $u(\zeta_j) \in \bar{\alpha}_j$ for $j = 1, \dots, m$. In practice, (1.1) is difficult to define rigorously because $\mathcal{M}_{g,m}(M, A)$ is hardly ever actually a compact smooth manifold of the correct dimension, but mathematical definitions of $\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m)$ typically view it as some generalization of a homological intersection number. According to Witten [Wit88b], on the other hand, the integral in (1.1) is also the result of computing a Feynman path integral via stationary phase approximation in a certain kind of quantum field theory known as a topological nonlinear sigma-model. Maybe someday I'll understand what that means.

Lecture 2 (19.10.2022): Why symplectic manifolds? This was a continuation of the rough overview from Lecture 1, intended mainly to explain why Gromov-Witten theory gives invariants of *symplectic* rather than complex or almost complex manifolds.

- Convenient fiction 1: $\bar{\partial}_J \pitchfork 0$ (transverse to the zero-section), which is equivalent to the condition on surjectivity of the linearization mentioned last time. This could be arranged if sufficiently generic perturbations of the Fredholm section $\bar{\partial}_J(u) := du + J \circ du \circ j$ were allowed. On complex manifolds this cannot be done, because J is fixed, but sometimes it is good enough to generically perturb J in the space
$$\mathcal{J}(M) := \left\{ J \in \Gamma(\mathrm{End}(TM)) \;\middle|\; J^2 = -\mathbb{1} \right\}$$
of smooth almost complex structures. This makes (M, J) an almost complex manifold, but we can no longer assume it admits a complex atlas (i.e. that J is **integrable**). Holomorphic curves in (M, J) are then called ***J*-holomorphic** or **pseudoholomorphic**.
- Convenient fiction 2: $\mathcal{M}_{g,m}(M, A)$ is compact... typically it is not (see Exercise 1.5), though occasionally it is (Exercise 1.4). We will find that under the right set of assumptions, $\mathcal{M}_{g,m}(M, A)$ always admits a natural compactification that is not too hard to describe.
- Useful ingredient for compactness arguments: there is a notion of **energy** $E(u) \in \mathbb{R}$ for holomorphic curves $u : \Sigma \to M$ such that
  1. $E(u) \geqslant 0$, with equality if and only if u is constant;
  2. There is an upper bound $E(u) \leqslant C$ for all $u \in \mathcal{M}_{g,m}(M, A)$, with C depending only on the homology class $A \in H_2(M)$.

[1] Such submanifolds do not always exist, but by a theorem of Thom [Tho54], they do always exist after multiplying α_j by some natural number; this suffices for our purposes since the invariants we are trying to define have rational values, not necessarily integers. (The real reason for them to have rational values has to do with symmetries that we will talk about later.)

- Definition: $\omega \in \Omega^2(M)$ **tames** $J \in \mathcal{J}(M)$ if $\omega(X, JX) > 0$ for all $X \neq 0 \in TM$. Under this condition,
$$E(u) := \int_\Sigma u^*\omega$$
satisfies property (1) above. If $d\omega = 0$ then it also satisfies property (2). This makes ω a **symplectic form** and (M, ω) a **symplectic manifold**.
- Definition: a **symplectomorphism** $\varphi : (M_0, \omega_0) \to (M_1, \omega_1)$ is a diffeomorphism $\varphi : M_0 \to M_1$ with $\varphi^*\omega_1 = \omega_0$. We say (M_0, ω_0) and (M_1, ω_1) are **symplectically deformation equivalent** if there is a diffeomorphism $\varphi : M_0 \to M_1$ such that ω_0 and $\varphi^*\omega_1$ are homotopic through a smooth 1-parameter family of symplectic forms.
- Main "theorem" of the course (though we will prove a slightly less general version): on any symplectic manifold (M, ω), the maps $\mathrm{GW}_{g,m,A} : H^*(M)^{\times m} \to \mathbb{Q}$ defined by counting J-holomorphic curves for a generic choice of tame $J \in \mathcal{J}(M)$ are independent of this choice, and depend only on the symplectic deformation class of ω.
- Some early history:
  (1) 19th century: origins of enumerative algebraic geometry
  (2) 1982: Witten's paper *Supersymmetry and Morse theory* [Wit82] helped popularize what is now called *Morse homology*, which presents the singular homology of a smooth manifold via a chain complex generated by critical points of a Morse function (this later inspired Floer homology, see [AD14, Sch93]). It also was the first instance of Witten's distinctive paradigm in which topological invariants are interpreted as manifestations of supersymmetric quantum mechanics.
  (3) 1983: Donaldson defined invariants of smooth 4-manifolds (see [DK90]), proving many breakthrough results on the distinction between smooth and continuous topology in dimension four. This was the first example of using the topology of a moduli space of solutions to a nonlinear elliptic PDE (in this case one from gauge theory) to define invariants of the geometric setting in which the PDE lives. The Gromov-Witten invariants follow the same idea, but with a different elliptic PDE in a different setting.
  (4) 1985: Gromov's paper *Pseudoholomorphic curves in symplectic manifolds* [Gro85] demonstrated that for almost complex structures J tamed by a symplectic form, the moduli space of J-holomorphic curves encodes deep invariants of the symplectic structure. This paper proved the famous nonsqueezing theorem and initiated the modern field of symplectic topology.
  (5) 1987–88: Inspired in part by Witten's Morse theory paper, Floer produced two versions of infinite-dimensional Morse homology, now known as *instanton homology* [Flo88b] (a 3-dimensional analogue of Donaldson's gauge-theoretic 4-manifold invariants) and *Hamiltonian Floer homology* [Flo88a] (a symplectic invariant based on a variant of Gromov's J-holomorphic curves)
  (6) 1988: Witten's papers *Topological quantum field theory* [Wit88a] and *Topological sigma models* [Wit88b] did for Floer's homological version of Donaldson's gauge-theoretic invariants and Gromov's holomorphic curve theory respectively what Witten's 1982 paper had done for classical Morse theory, giving them new interpretations as by-products of supersymmetric quantum field theories. The paper on sigma models contained the first sketches of what were later called the Gromov-Witten invariants.
  (7) 1993–94: Mathematically rigorous definitions of the Gromov-Witten invariants appeared in parallel work of McDuff-Salamon [MS94] and Ruan-Tian [RT95, RT97]. These definitions were valid for symplectic manifolds satisfying a technical condition ("semipositivity") that is always satisfied up to dimension six, but not always in higher

dimensions. The effort to remove such conditions and define GW for arbitrary symplectic manifolds was a longer story that saw some progress later in the 90's and beyond, but remains a slightly controversial topic today.

(8) 1994: Kontsevich and Manin [KM94] wrote down a set of axioms satisfied by the Gromov-Witten invariants (in the spirit of the Eilenberg-Steenrod axioms for homology theories), from which impressive computations such as the Kontsevich recursion formula can be deduced without needing to know how the invariants are constructed.

Gromov-Witten theory remains an active field of research today in both symplectic and algebraic geometry.

- Some basics on symplectic manifolds:
  - Statement of Darboux's theorem ("there are no local symplectic invariants")—we will prove it next week.
  - Hamiltonian vector fields: $\mathrm{Symp}(M,\omega)$ is a very large group
  - Symplectomorphisms are volume preserving; do they also have special properties that volume-preserving maps do not?
- Gromov's nonqueezing theorem, and how it follows from the computation $\mathrm{GW}_{0,1,A}(\alpha) \neq 0$ for $A = [S^2 \times \{\text{const}\}] \in H_2(S^2 \times \mathbb{T}^{2n-2})$ and $\alpha \in H^{2n}(S^2 \times \mathbb{T}^{2n-2})$ Poincaré dual to a point.

Suggested reading. Nothing covered this week was intended to be essential to the remainder of the course, since it was only an overview. An account similar (but not identical) to the proof I sketched in class of Gromov's nonsqueezing theorem can be found in [Wena, §5.1]; it differs in that instead of citing a computation $\mathrm{GW}_{0,1,A}(\alpha) \neq 0$ as a black box, it more directly proves the existence of the required J-holomorphic curve (which is also the main step in that computation). This uses results and methods that we will cover in detail later in the course.

In the mean time, if you'd like to shore up your knowledge of basic symplectic geometry, the first few chapters of [MS17] are helpful, or alternatively, [CdS01]. One particular topic we plan to discuss in the Übung next week is the standard symplectic form on $\mathbb{CP}^n$; you will find a description of it in [Wen18, Example 1.4].

Exercises (for the Übung on 25.10.2022).

Exercise 1.1. Suppose $\mathcal{U} \subset \mathbb{C}$ is an open subset and $f : \mathcal{U} \to \mathbb{C}$ is a smooth function, where the notion of smoothness is defined the same way as in first-year analysis after identifying $\mathbb{C}$ in the obvious way with $\mathbb{R}^2$, so that f becomes a function of two real variables. With this understood, the derivative of f at any point $z \in \mathcal{U}$ gives a *real*-linear map

$$Df(z) : \mathbb{C} \to \mathbb{C},$$

i.e. $Df(z)$ respects addition and multiplication by real scalars, but not necessarily multiplication by imaginary scalars; the latter would make $Df(z)$ a complex-linear map. Show in fact that $Df(z)$ is complex linear at every point $z \in \mathcal{U}$ if and only if f is holomorphic.

Remark. Exercise 1.1 yields the quickest generalization of the notion of a holomorphic map to various other contexts, e.g. for an open subset $\mathcal{U} \subset \mathbb{C}^n$, a smooth map $f : \mathcal{U} \to \mathbb{C}^m$ is called holomorphic if and only if its derivative $Df(z) : \mathbb{C}^n \to \mathbb{C}^m$ is complex linear at every point. (One can show that this is equivalent to the existence at every point of partial derivatives $\frac{\partial f}{\partial z_j}$ with respect to the complex variables $z_1, \ldots, z_n$.)

Exercise 1.2. If you were not previously familiar with complex manifolds and the fact that $\mathbb{CP}^n$ is one, write down an explicit atlas for $\mathbb{CP}^n$ consisting of $\mathbb{C}^n$-valued charts such that all transition maps are holomorphic.

Hint: One such chart comes from the inverse of the embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n : (z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n]$.

Exercise 1.3. Prove that there is a natural equivalence between the notion of rational curves of degree 1 in $\mathbb{CP}^n$ (as defined in lecture via equivalence classes of holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^n$) and the notion of "lines in $\mathbb{CP}^n$". Take the latter to mean the image of any embedding $S^2 \hookrightarrow \mathbb{CP}^n$ that is obtained by extending an embedding of the form $\mathbb{C} \hookrightarrow \mathbb{C}^n : z \mapsto za + b$ (with $b \neq 0 \in \mathbb{C}^n$) over the point $\infty \in S^2 := \mathbb{C} \cup \{\infty\}$ using any of the embeddings $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$ defined by $(z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n]$ or $[z_1 : 1 : z_2 : \ldots : z_n]$ etc.

The next two exercises both pretend that you know what the natural topology on the moduli spaces $\mathcal{M}_{g,m}(M, A)$ is, which is a convenient fiction since we have not yet defined any such topology. We will do so later; for now, don't worry about it too much, and believe me when I tell you that whatever educated guesses you make about the properties this topology should have, you are probably right.

Exercise 1.4. Let $\mathbb{P}(T(\mathbb{CP}^2))$ denote the space of complex lines through the origin in the fibers of $T(\mathbb{CP}^2)$; this is a compact manifold since it is a fiber bundle over $\mathbb{CP}^2$ with fiber $\mathbb{CP}^1$. Prove that the moduli space $\mathcal{M}_{0,0}(\mathbb{CP}^2, [L])$ is compact by showing that it is the image of a continuous surjective map

$$\mathbb{P}(T(\mathbb{CP}^2)) \to \mathcal{M}_{0,0}(\mathbb{CP}^2, [L])$$

that sends each line $\ell \subset T_p(\mathbb{CP}^2)$ at a point $p \in \mathbb{CP}^2$ to the unique line $L \subset \mathbb{CP}^2$ that passes through p and is tangent at that point to L.

Exercise 1.5. Show that the evaluation map

$$\mathrm{ev} : \mathcal{M}_{0,3}(\mathbb{CP}^1, [\mathbb{CP}^1]) \to (\mathbb{CP}^1)^{\times 3}$$

is a homeomorphism onto the complement of the so-called **fat diagonal**

$$\Delta := \left\{ (p_1, p_2, p_3) \in (\mathbb{CP}^1)^{\times 3} \;\middle|\; p_1 = p_2,\ p_2 = p_3 \text{ or } p_1 = p_3 \right\} \subset (\mathbb{CP}^1)^{\times 3}.$$

This proves that $\mathcal{M}_{0,3}(\mathbb{CP}^1, [\mathbb{CP}^1])$ is not compact, while at the same time providing you with a pretty good guess as to what its natural compactification might be homeomorphic to.

Exercise 1.6. Let $x_0 = [1 : 0 : 0] \in \mathbb{CP}^2$, and consider the holomorphic map

$$\pi : \mathbb{CP}^2 \backslash \{x_0\} \to \mathbb{CP}^1 : [z_0 : z_1 : z_2] \mapsto [z_1 : z_2].$$

Show that the closure of each level set $\pi^{-1}(\text{const}) \subset \mathbb{CP}^2$ can be parametrized by a holomorphic embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ that passes through x_0, thus it defines a complex submanifold $\Sigma \subset \mathbb{CP}^2$ which is diffeomorphic to S^2.

Additional topic for the Übung: we will discuss the standard symplectic form on $\mathbb{CP}^n$, whose existence implies that all complex submanifolds of $\mathbb{CP}^n$ (so in particular all smooth projective varieties) are also symplectic manifolds. (ADDED LATER: We did not end up discussing the symplectic form on $\mathbb{CP}^n$ in the Übung, but will instead discuss that in Lecture 4.)

2. Week 2

Lecture 3 (25.10.2022): Basics on symplectic manifolds. Following the first week's overview, this lecture can be considered the official beginning of the course, i.e. the part where one can expect to see precise definitions and proofs.

- Quick review of symplectic forms, Hamiltonian vector fields ($\omega(X_H, \cdot) = -dH$)
- The standard symplectic form $\omega_{\mathrm{std}} = \sum_j dp^j \wedge dq^j$ on $\mathbb{R}^{2n} \ni (q^1, \ldots, q^n, p^1, \ldots, p^n)$ and Hamilton's equations in coordinates

- The canonical 1-form $\lambda_{\text{can}} = \sum_j p^j\, dq^j$ on T^*M (with local coordinates $(q^1, \ldots, q^n$ on M giving coordinates for 1-forms $p^1\, dq^1 + \ldots + p^n\, dq^n \in T^*M)$, and canonical symplectic form $\omega_{\text{can}} := d\lambda_{\text{can}}$
- Symplectic linear algebra:
  - The **symplectic orthogonal complement** $W^{\perp\omega}$ of a subspace $W \subset V$ with nondegenerate $\omega \in \Lambda^2 V^*$; why "complement" is a bad name but nonetheless $\dim W + \dim W^{\perp\omega} = \dim V$
  - **Symplectic**, **isotropic**, **coisotropic** and **Lagrangian** subspaces
  - Existence of a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n \in V$ with $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$
- Darboux's theorem: Near every point x in a symplectic $2n$-manifold (M, ω), there exists a chart $(q^1, \ldots, q^n, p^1, \ldots, p^n)$ such that $\omega = \sum_j dp^j \wedge dq^j$; proof by the Moser deformation trick
- Statements of other results provable via the Moser deformation trick:
  - Lagrangian neighborhood theorem: Every Lagrangian submanifold $L \subset (M, \omega)$ has a neighborhood symplectomorphic to a neighborhood of the zero-section in $(T^*L, \omega_{\text{can}})$.
  - Moser stability theorem: for M closed and $\{\omega_t\}_{t\in[0,1]}$ a smooth family of cohomologous symplectic forms, there exists for $t \in [0, 1]$ a smooth family of symplectomorphisms $\psi_t : (M, \omega_0) \to (M, \omega_t)$ with $\psi_0 = \text{Id}$.
- The spaces $\mathcal{J}(E)$ (arbitrary), $\mathcal{J}_\tau(E, \omega)$ (**tame**) and $\mathcal{J}(E, \omega)$ (**compatible**) of complex structures on a symplectic vector bundle $(E, \omega) \to M$, $\omega \in \Gamma(\Lambda^2 E^*)$ fiberwise nondegenerate
- Almost complex structures and integrability: statement (without proof) of the Newlander-Nirenberg theorem
- Corollary (due originally to Gauss, and to be proved later by more direct means): All almost complex structures on a surface are integrable, hence "Riemann surface" = "complex 1-manifold" = "almost complex manifold of real dimension 2"

Lecture 4 (26.10.2022): Almost complex structures.

- The spaces $\mathcal{J}_\tau(V, \omega)$, $\mathcal{J}_\tau(E, \omega)$ and $\mathcal{J}_\tau(M, \omega) := \mathcal{J}_\tau(TM, \omega)$ of tame complex structures on a symplectic vector space (V, ω), symplectic vector bundle (E, ω) or symplectic manifold (M, ω) (for this case add the word "almost"); similar spaces of compatible complex structures (remove the τ)
- If $J \in \mathcal{J}(V, \omega)$ and $\dim V = 2n$, then (V, ω, J) is isomorphic to $(\mathbb{R}^{2n}, \omega_{\text{std}}, i)$; define i on $\mathbb{R}^{2n}$ via the identification with $\mathbb{C}^n$ defined by $(q^1, \ldots, q^n, p^1, \ldots, p^n) \leftrightarrow (p^1 + iq^1, \ldots, p^n + iq^n)$
- Theorem: $\mathcal{J}_\tau(E, \omega)$ and $\mathcal{J}(E, \omega)$ are nonempty and contractible spaces.
- First proof for $\mathcal{J}(E, \omega)$, using convexity of the space of bundle metrics
- Proposition: If $\dim_{\mathbb{R}} V = 2n$, then $\mathcal{J}(V) \subset \text{End}(V)$ is a smooth submanifold of dimension $2n^2$ with tangent spaces $T_J\mathcal{J}(V) = \overline{\text{End}}_{\mathbb{C}}(V, J) := \{Y \in \text{End}(V) \mid YJ = -JY\}$; proof by identification with the homogeneous space $\text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$
- Corollary: The map

$$Y \mapsto \Psi(Y) = J_Y := \left(\mathbb{1} + \frac{1}{2}JY\right) J \left(\mathbb{1} + \frac{1}{2}JY\right)^{-1}$$

  embeds a neighborhood of 0 in $\overline{\text{End}}_{\mathbb{C}}(V, J)$ smoothly onto a neighborhood of J in $\mathcal{J}(V)$ such that the derivative at 0 is the identity.

- Sketch of second proof of contractibility of $\mathcal{J}(E,\omega)$ (and also $\mathcal{J}_\tau(E,\omega)$): Given $J \in \mathcal{J}_\tau(E,\omega)$, the map $Y \mapsto J_Y$ identifies $\mathcal{J}_\tau(E,\omega)$ with the space of smooth sections of $\overline{\mathrm{End}}_{\mathbb{C}}(E,J)$ taking values in a fiberwise convex subset.
- Corollary: Topological invariants of complex vector bundles are also invariants of symplectic vector bundles.
- Axiomatic characterization of the **first Chern class** $c_1(E) \in H^2(M)$ for a complex or symplectic vector bundle $E \to M$:
  1. $c_1(E \oplus F) = c_1(E) + c_1(F)$
  2. For $f : N \to M$, $c_1(f^*E) = f^*c_1(E) \in H^2(N)$
  3. For a complex line bundle $E \to \Sigma$ over a closed Riemann surface, the **first Chern number** $c_1(E) := \langle c_1(E), [\Sigma] \rangle \in \mathbb{Z}$ is the algebraic count of zeroes of any section $\eta \in \Gamma(E)$ with only finitely many:
  $$\#\eta^{-1}(0) := \sum_{z \in \eta^{-1}(0)} \mathrm{ord}(\eta; z) = c_1(E),$$
  where $\mathrm{ord}(\eta; z) \in \mathbb{Z}$ is the winding number of the loop $S^1 \to \mathbb{C}\backslash\{0\} : e^{i\theta} \mapsto \eta(z + \epsilon e^{i\theta})$ for $\epsilon > 0$ small after choosing a local holomorphic coordinate on Σ and trivialization of E near z
- Corollary of the Poincaré-Hopf theorem: $c_1(T\Sigma) = \chi(\Sigma)$ for a closed Riemann surface
- The Fubini-Study symplectic form ω_{FS} on $\mathbb{CP}^n$ and its characterization via
$$\mathrm{pr}^* \omega_{\mathrm{FS}} = \omega_{\mathrm{std}}|_{TS^{2n+1}}$$
using the quotient projection $\mathrm{pr} : S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{CP}^n$
- Corollary: Every complex submanifold of $\mathbb{CP}^n$ (e.g. all smooth complex projective varieties) inherits a canonical symplectic form compatible with its complex structure.

Suggested reading. Almost everything mentioned this week can be found in the early chapters of [MS17], and probably also in [CdS01]. For full details of the proof that $\mathcal{J}_\tau(E,\omega)$ is contractible, see [Wena, Prop. 2.2.17]. If you like homotopy theory, you might prefer the more abstract alternative proof (originating in Gromov's paper [Gro85]) that is given one page earlier in my notes, though our more direct proof (based on an idea of Sévennec) has some practical advantages that we'll occasionally make use of. Alternative presentations of the first Chern class can be found in many places; the treatment in [MS17] is different from ours but also geared toward symplectic geometry, and is phrased in terms of the *Maslov index* (a very useful concept if one wants to study Lagrangian manifolds, though we don't plan on it in this course).

Exercises (for the Übung on 01.11.2022).

Exercise 2.1. Here is a coordinate-invariant way to express the canonical 1-form $\lambda_{\mathrm{can}} \in \Omega^1(T^*M)$ on the cotangent bundle of a smooth n-manifold M. Using the derivative $T\pi : T(T^*M) \to TM$ of the bundle projection $\pi : T^*M \to M$, define $\lambda_{\mathrm{can}} : T(T^*M) \to \mathbb{R}$ on a vector $\xi \in T_\alpha(T^*M)$ at $\alpha \in T^*_x M$ for $x \in M$ by
$$\lambda_{\mathrm{can}}(\xi) := \alpha(T\pi(\xi)).$$
You could be forgiven for finding this concise definition too abstract to be revealing, but here is another useful way to think about it. For bookkeeping purposes, let's write elements of T^*M as pairs (q,p) where $q \in M$ and $p \in T^*_q M$. Any choice of connection on the vector bundle $T^*M \to M$ determines at each point $(q,p) \in T^*M$ a splitting of $T_{(q,p)}(T^*M)$ into horizontal and vertical subspaces
$$T_{(q,p)}(T^*M) = H_{(q,p)}(T^*M) \oplus V_{(q,p)}(T^*M),$$

where $T\pi$ gives a natural isomorphism of $H_{(q,p)}(T^*M)$ to T_qM and $V_{(q,p)}(T^*M) := T_p(T_q^*M)$ is canonically isomorphic to T_q^*M. In this way, we obtain an isomorphism

$$T_{(q,p)}(T^*M) \cong T_qM \oplus T_q^*M,$$

and can use it to write tangent vectors $\xi \in T_{(q,p)}(T^*M)$ as pairs (X, α), the two components being interpreted as horizontal part $X \in T_qM \cong H_{(q,p)}(T^*M)$ and vertical part $\alpha \in T_q^*M \cong V_{(q,p)}(T^*M)$. For example, if we use this notation to write a smooth path $\gamma(t) \in T^*M$ in the form $\gamma(t) = (q(t), p(t))$, then its derivative becomes

$$\dot{\gamma}(t) = (\dot{q}(t), \nabla_t p(t))$$

since the covariant derivative $\nabla_t p(t)$ of a section $p(t) \in T^*_{q(t)}M$ of T^*M along a path $q(t) \in M$ is the vertical part of the derivative of the corresponding path in the total space. With this notation understood, we can now write λ_{can} in the form

$$\lambda_{\mathrm{can}}(X, \beta) = p(X) \qquad \text{for } (X, \beta) \in T_{(q,p)}(T^*M) = T_qM \oplus T_q^*M.$$

If we had defined λ_{can} this way in the first place, we would now have to worry about whether it depends on the choice of connection (since the horizontal-vertical splitting does), but our original definition shows that this is not so.

(a) Verify that for any choice of local coordinates $q^1, \dots, q^n$ on a region $\mathcal{U} \subset M$, taking the induced chart $(q^1, \dots, q^n, p^1, \dots, p^n)$ on $T^*M|_{\mathcal{U}}$ as explained in the lecture, $\lambda_{\mathrm{can}} = \sum_{j=1}^n p^j \, dq^j$ on $T^*M|_{\mathcal{U}}$. In particular, the expression $\sum_{j=1}^n p^j \, dq^j$ for a 1-form on T^*M is therefore independent of the choice of local coordinates $q^1, \dots, q^n$ (though it does crucially depend on how the other coordinates $p^1, \dots, p^n$ are determined by these).

(b) Show that if the connection ∇ on $T^*M \to M$ used for the splitting of $T(T^*M)$ is induced via duality from a *symmetric* connection on M, then the canonical symplectic form $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}} \in \Omega^2(T^*M)$ is given by the formula

$$\omega_{\mathrm{can}}((X, \alpha), (Y, \beta)) = \alpha(Y) - \beta(X).$$

Hint: It suffices to consider cases where each of (X, α) and (Y, β) is either purely vertical or purely horizontal. If you get stuck, see [Wen21, Lemma 23.14].

(c) Now suppose additionally that the symmetric connection on M is the Levi-Cività connection for some Riemannian (or pseudo-Riemannian) metric $\langle\ ,\ \rangle$, and using the bundle metric induced on $T^*M \to M$ via duality, consider the Hamiltonian

$$H : T^*M \to \mathbb{R}, \qquad H(q, p) := \frac{1}{2}\langle p, p\rangle.$$

Show that the resulting Hamiltonian vector field $X_H \in \mathfrak{X}(T^*M)$ is given by

$$X_H(q, p) = (p^\sharp, 0) \in T_qM \oplus T_q^*M = T_{(q,p)}(T^*M),$$

where $T_q^*M \to T_qM : \alpha \mapsto \alpha^\sharp$ denotes the inverse of the "musical" isomorphism $T_qM \to T_q^*M : X \mapsto X_\flat := \langle X, \cdot\rangle$.

(d) Explain precisely what is meant by the statement, "The Hamiltonian system $(T^*M, \omega_{\mathrm{can}}, H)$ is equivalent to the geodesic equation on $(M, \langle\ ,\ \rangle)$."

Exercise 2.2. Use the Moser deformation trick to prove the *Moser stability theorem*: if M is closed and $\{\omega_t\}_{t\in[0,1]}$ is a smooth family of symplectic forms that all represent the same de Rham cohomology class, then for $t \in [0, 1]$ there exists a smooth family of symplectomorphisms $\varphi_t : (M, \omega_0) \to (M, \omega_t)$ with $\varphi_0 = \mathrm{Id}$.
Hint: You will need to know that $\omega_t = \omega_0 + d\lambda_t$ for a smooth family of 1-forms λ_t. This follows

from the assumption that $[\omega_t] \in H^2_{\mathrm{dR}}(M)$ *is independent of* t*, but is not so obvious; some hints on how to prove it are given in* [MS17, proof of Theorem 3.2.4].

Exercise 2.3. On a symplectic vector space (V, ω), show that $J \in \mathcal{J}(V, \omega)$ if and only if $J \in \mathcal{J}_\tau(V, \omega)$ and $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in V$.

Exercise 2.4. In this exercise, we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via the correspondence

$$\mathbb{R}^{2n} \ni (q^1, \dots, q^n, p^1, \dots, p^n) \leftrightarrow (p^1 + iq^1, \dots, p^n + iq^n) \in \mathbb{C}^n,$$

so that multiplication by i is regarded as a real-linear transformation $i : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and $\mathrm{GL}(n, \mathbb{C})$ becomes the subgroup $\{A \in \mathrm{GL}(2n, \mathbb{R}) \mid Ai = iA\}$.

(a) Check that the standard symplectic form $\omega_{\mathrm{std}} = \sum_j dp^j \wedge dq^j$ on $\mathbb{R}^{2n}$ can be written in terms of the standard Hermitian inner product $\langle u, v\rangle = \sum_j \bar{u}^j v^j$ on $\mathbb{C}^n$, namely as

$$\omega_{\mathrm{std}}(X, Y) = \mathrm{Im}\langle X, Y\rangle.$$

(b) Prove $\mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n)$.

Exercise 2.5. According to the Newlander-Nirenberg theorem, an almost complex structure J on a manifold M is integrable if and only if the induced **Nijenhuis tensor** $N_J : TM \oplus TM \to TM$, given by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

vanishes. Prove:

(a) The expression given above for N_J really does define a tensor field on M, i.e. it is C^∞-linear in both X and Y.
(b) N_J vanishes whenever J is integrable (i.e. the "easy" direction of the Newlander-Nirenberg theorem).
(c) N_J always vanishes if $\dim_{\mathbb{R}} M = 2$.
(d) As mentioned in lecture, the vanishing of N_J when $\dim M = 2$ implies a classical result due to Gauss that almost complex structures on surfaces are always integrable. Actually, what Gauss proved was stated a bit differently: the theorem is that every Riemannian metric g on a surface Σ is **conformally flat**, meaning that every point $p \in \Sigma$ has a neighborhood $\mathcal{U} \subset \Sigma$ such that $(\mathcal{U}, g)$ is isometric to $(\mathcal{V}, f g_E)$ for some open subset $\mathcal{V} \subset \mathbb{R}^2$, smooth function $f : \mathcal{V} \to (0, \infty)$ and the Euclidean metric g_E. Prove that that statement is equivalent to the one mentioned above about integrable almost complex structures.

Exercise 2.6. Suppose (M, ω) is a symplectic manifold with a compatible almost complex structure $J \in \mathcal{J}(M, \omega)$, $g := \omega(\cdot, J\cdot)$ is the induced Riemannian metric and ∇ denotes its Levi-Cività connection. Show that for the induced connections on the vector bundles $\mathrm{End}(TM)$ and $\Lambda^2 T^*M$, $\nabla J = 0$ if and only if $\nabla\omega = 0$, and if either is true, then J is integrable.[2] You may use the Newlander-Nirenberg theorem as a black box.
Hint: Use the symmetry of the connection to write a new formula for the Nijenhuis tensor that involves covariant derivatives instead of brackets.
Comment: The converse is also true: if J *is integrable, then the Levi-Cività connection satisfies* $\nabla J = 0$ *and* $\nabla\omega = 0$*, hence parallel transport respects both* J *and* ω*. For details, see* [MS17, §4.2].

Exercise 2.7. In lecture we have been using the term **symplectic vector bundle** to mean any smooth real vector bundle $E \to M$ that is equipped with a smooth section $\omega \in \Gamma(\Lambda^2 E^*)$ which is nondegenerate on every fiber. In order to justify this terminology, one should prove the following:

[2] In this situation, g is called a **Kähler metric** and (M, J, g) is a **Kähler manifold**. Kähler geometry is essentially the intersection of complex, Riemannian and symplectic geometry.

every symplectic vector bundle (E, ω) of rank $2n$ admits a system of local trivializations covering M such that all transition functions take values in the **linear symplectic group**

$$\mathrm{Sp}(2n) := \left\{ A \in \mathrm{GL}(2n, \mathbb{R}) \;\middle|\; \omega_{\mathrm{std}}(Au, Av) = \omega_{\mathrm{std}}(u, v) \text{ for all } u, v \in \mathbb{R}^{2n} \right\}.$$

Prove this.
Hint: What should the term "symplectic local frame" on (E, ω) mean, and can you show that they always exist?
Comment: If (M, ω) is a symplectic manifold, then (TM, ω) is a symplectic vector bundle over M in an obvious way, but (TM, ω) would also be a symplectic vector bundle if $\omega \in \Omega^2(M)$ is nondegenerate but not closed. The result of this exercise proves that even in that case, one can cover M with neighborhoods that admit symplectic frames. Whenever such a frame consists of coordinate vector fields, the corresponding chart will be a so-called "Darboux chart", i.e. one in which ω matches the standard symplectic form of $\mathbb{R}^{2n}$. Darboux's theorem shows that the latter is possible if and only if ω is closed.

Exercise 2.8. Assume (M, ω) is a symplectic manifold.

(a) Prove the following result stated in lecture: Given subsets $A \subset \mathcal{U} \subset M$ that are closed and open respectively, and given any $J_0 \in \mathcal{J}_\tau(\mathcal{U}, \omega)$, the space $\left\{ J \in \mathcal{J}(M, \omega) \;\middle|\; J = J_0 \text{ on } A \right\}$ is nonempty and contractible.
(b) Show if $\Sigma \subset M$ is a **symplectic submanifold** (meaning $\omega|_{T\Sigma}$ defines a symplectic form on Σ), then there exists $J \in \mathcal{J}(M, \omega)$ such that the action of J on $TM|_\Sigma$ preserves $T\Sigma$.
Comment: This fact is the first step in the proof of various powerful results in [Gro85, McD90] *concerning symplectic 4-manifolds that contain symplectically embedded 2-spheres. The point is: any 2-dimensional symplectic submanifold can in this way be regarded as the image of an embedded J-holomorphic curve, and the properties of the moduli space of such curves can have nontrivial consequenes.*

Exercise 2.9. In lecture we defined the **Fubini-Study** symplectic form ω_{FS} on $\mathbb{CP}^n$ according to the relation

$$\mathrm{pr}^* \omega_{\mathrm{FS}} = \omega_{\mathrm{std}}|_{TS^{2n+1}},$$

where S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ and $\mathrm{pr} : S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{CP}^n$ denotes the quotient projection.

(a) Compute the **symplectic area** of a line in $\mathbb{CP}^n$, i.e. the integral $\int_L \omega_{\mathrm{FS}}$ over any line $L \subset \mathbb{CP}^n$, with its natural orientation as a complex submanifold. Equivalently, this is the evaluation of the cohomology class $[\omega_{\mathrm{FS}}] \in H^2_{\mathrm{dR}}(M)$ on the generator $[L] \in H_2(M)$, which is why the answer will not depend on which line you choose.
Hint: L is the image of a map $\mathbb{C} \cup \{\infty\} \to \mathbb{CP}^n$, and the integral will not change if you ignore the point at infinity and integrate only over $\mathbb{C}$.

Solution:
I'm writing up a solution here because every four or five years I find myself needing to do this exercise again, but sometimes I forget the trick.

We can first show that the answer does not depend on the choice of n. This follows from the observation that for any $1 \leqslant k < n$ and an embedding of the form

$$i : \mathbb{CP}^k \hookrightarrow \mathbb{CP}^n : [z_0 : \ldots : z_k] \mapsto [z_0 : \ldots : z_k : 0 : \ldots : 0],$$

the pullback via i of the Fubini-Study form of $\mathbb{CP}^n$ is the Fubini-Study form of $\mathbb{CP}^k$. This follows via the relation $\mathrm{pr}^* \omega_{\mathrm{FS}} = \omega_{\mathrm{std}}|_{TS^{2n+1}}$ from a similar statement about the pullback

ω_{std} under the embedding

$$\tilde{\imath} : \mathbb{C}^{k+1} \hookrightarrow \mathbb{C}^{n+1} : (z_0, \dots, z_k) \mapsto (z_0, \dots, z_k, 0, \dots, 0),$$

after restricting the latter to an embedding of unit spheres $S^{2k+1} \hookrightarrow S^{2n+1}$. Since the image of the embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ is a line, $\int_L \omega_{\mathrm{FS}}$ will therefore be the same as $\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}}$, where in the latter integral we simply regard ω_{FS} as a 2-form on $\mathbb{CP}^1$, characterized by the relation $\mathrm{pr}^* \omega_{\mathrm{FS}} = \omega_{\mathrm{std}}|_{TS^3}$, where

$$\mathrm{pr} : S^3 \to S^3/S^1 = \mathbb{CP}^1$$

is the quotient projection.

Next, let $B^2 \subset \mathbb{R}^2$ denote the open unit ball, and suppose we can find an embedding $\varphi : B^2 \hookrightarrow S^3 \subset \mathbb{C}^2$ such that $\mathrm{pr} \circ \varphi : B^2 \to \mathbb{CP}^1$ is a diffeomorphism onto the complement of one point $p \in \mathbb{CP}^1$. (We'll discuss in a moment how to find such an embedding.) If we could pretend for a moment that ω_{FS} has compact support in $\mathbb{CP}^1 \backslash \{p\}$ and $\varphi^* \omega_{\mathrm{std}}$ is compactly supported on B^2, we would then have

$$\int_{B^2} \varphi^* \omega_{\mathrm{std}} = \int_{B^2} \varphi^* \mathrm{pr}^* \omega_{\mathrm{FS}} = \int_{B^2} (\mathrm{pr} \circ \varphi)^* \omega_{\mathrm{FS}} = \int_{\mathbb{CP}^1 \backslash \{p\}} \omega_{\mathrm{FS}} = \int_{\mathbb{CP}^1} \omega_{\mathrm{FS}}.$$

To justify this result without any fictional compact support assumption, choose a smooth function $\beta : \mathbb{CP}^1 \to [0,1]$ that equals 1 except in an arbitrarily small neighborhood of p and has compact support in $\mathbb{CP}^1 \backslash \{p\}$; we can clearly arrange the integral $\int_{\mathbb{CP}^1 \backslash \{p\}} \beta \omega_{\mathrm{FS}}$ to be as close to $\int_{CP^1} \omega_{\mathrm{FS}}$ as we want. We then have $\mathrm{pr}^*(\beta \omega_{\mathrm{FS}}) = (\beta \circ \mathrm{pr}) \, \mathrm{pr}^* \omega_{\mathrm{FS}} = (\beta \circ \mathrm{pr}) \omega_{\mathrm{std}}|_{TS^3}$, so that a repeat of the calculation above gives

$$\int_{B^2} (\beta \circ \mathrm{pr} \circ \varphi) \varphi^* \omega_{\mathrm{std}} = \int_{B^2} (\beta \circ \mathrm{pr} \circ \varphi) \varphi^* \mathrm{pr}^* \omega_{\mathrm{FS}} = \int_{B^2} (\mathrm{pr} \circ \varphi)^* (\beta \omega_{\mathrm{FS}}) = \int_{\mathbb{CP}^1 \backslash \{p\}} \beta \omega_{\mathrm{FS}}.$$

Since $\beta \circ \mathrm{pr} \circ \varphi : B^2 \to [0,1]$ has compact support and equals 1 in an arbitrarily large compact subset of B^2, the first integral can be assumed as close as we like to $\int_{B^2} \varphi^* \omega_{\mathrm{std}}$, completing the justification.

Finally, here is a concrete choice of φ that will do the job:

$$\varphi : B^2 \hookrightarrow S^3 : (p, q) \mapsto \left(p + iq, \sqrt{1 - p^2 - q^2} \right) \in \mathbb{C}^2.$$

This traces out an embedded disk $\mathcal{D} \subset S^3 \subset \mathbb{C}^2$, and the integral we are now looking for is $\int_{B^2} \varphi^* \omega_{\mathrm{std}} = \int_{\mathcal{D}} \omega_{\mathrm{std}}$. In coordinates $(p_1 + iq_1, p_2 + iq_2)$ on $\mathbb{C}^2$, we have $\omega_{\mathrm{std}} = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, and since φ never moves in the q_2-direction, the second term in ω_{std} never contributes to $\int_{\mathcal{D}} \omega_{\mathrm{std}}$. What we are left with is $\int_{\mathcal{D}} dp_1 \wedge dq_1$, which depends only on the motion of φ in the p_1 and q_1 directions, thus it will not change if we replace φ with the embedding $\varphi_1 : B^2 \hookrightarrow \mathbb{C}^2 : (p, q) \mapsto (p + iq, 0)$, tracing out a completely "flat" 2-disk in $\mathbb{C}^2$. The answer is then the area of this disk, giving

$$\int_L \omega_{\mathrm{FS}} = \pi.$$

(b) Prove that $\langle c_1(T(\mathbb{CP}^2)), [L] \rangle = 3$.
Hint: Split $T(\mathbb{CP}^2)|_L$ into a direct sum of the tangent and normal bundles of L.

3. Week 3

Lecture 5 (01.11.2022): The nonlinear Cauchy-Riemann equation and its linearization.

- Pseudoholomorphic maps $f : (\Sigma, j) \to (M, J)$ between almost complex manifolds, and why we restrict to the case of **curves** ($\dim_{\mathbb{C}} \Sigma = 1$)

- Statement of the local existence theorem for J-holomorphic curves: On any almost complex manifold (M, J), for any $p \in M$ and $X \in T_pM$, there exists $\epsilon > 0$ and a J-holomorphic map $u : (\mathring{\mathbb{D}}_\epsilon, i) \to (M, J)$ with $u(0) = p$ and $\partial_s u(0) = X$. (We will prove this within the next few weeks.)
- Corollary: All almost complex structures on a surface are integrable.
- The nonlinear Cauchy-Riemann equation for $u : (\Sigma, j) \to (M, J)$ in local holomorphic coordinates (s, t) on Σ: $\partial_s u(s,t) + J(u(s,t))\partial_t u(s,t) = 0$
- Holomorphic vector bundles and linearization of $Tu + J \circ Tu \circ j = 0$ in the integrable case
- Every holomorphic vector bundle $E \to \Sigma$ has a natural linear first-order differential operator
$$\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E) := \Gamma\big(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)\big)$$
that annihilates holomorphic sections (see Exercise 3.1).
- The linearized Cauchy-Riemann operator
$$\mathbf{D}_u : \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$$
of a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ in the non-integrable case, and the formula
$$\mathbf{D}_u \eta = \nabla\eta + J(u) \circ \nabla\eta \circ j + \nabla_\eta J \circ Tu \circ j$$
for ∇ any symmetric connection on M (see Exercise 3.2)
- Linear Cauchy-Riemann type operators $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ on a general complex vector bundle E over a Riemann surface (Σ, j); the Leibniz rule
$$\mathbf{D}(f\eta) = f\mathbf{D}\eta + \bar\partial f(\cdot)\eta, \qquad f \in C^\infty(\Sigma, \mathbb{R}),\ \eta \in \Gamma(E),\ \bar\partial f := df + i\, df \circ j \in \Omega^{0,1}(\Sigma, \mathbb{C})$$
- The difference between any two linear Cauchy-Riemann type operators is a bundle map
- Local expression of linear Cauchy-Riemann type operators as $\bar\partial + A : C^\infty(\mathcal{O}, \mathbb{C}^m) \to C^\infty(\mathcal{O}, \mathbb{C}^m)$ for $\bar\partial := \partial_s + i\partial_t$ in coordinates $s + it \in \mathcal{O} \subset \mathbb{C}$ and a smooth function $A : \mathcal{O} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m) = \mathrm{GL}(2m, \mathbb{R})$
- Statement of the linear local existence theorem: For any smooth function $A : \mathring{\mathbb{D}} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ and any $v \in \mathbb{C}^m$, the problem
$$(\bar\partial + A)f = 0, \qquad f(0) = v$$
admits a smooth solution $f : \mathring{\mathbb{D}}_\epsilon \to \mathbb{C}^m$ for $\epsilon > 0$ sufficiently small.

Lecture 6 (02.11.2022): Some tools for the analysis of $\bar\partial$.

- Outline (minus technical details on Banach spaces) of the proof of linear local existence
- The spaces of **test functions** $\mathscr{D}(\mathcal{U}) := C_0^\infty(\mathcal{U})$ and **distributions**
$$\mathscr{D}'(\mathcal{U}) := \{\text{continuous linear functionals } \Lambda : \mathscr{D}(\mathcal{U}) \to \mathbb{R} : \varphi \mapsto (\Lambda, \varphi)\}$$
on an open domain $\mathcal{U} \subset \mathbb{R}^n$
- Multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| := \sum_j \alpha_j$, $\partial^\alpha := \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ is a differential operator of order $|\alpha|$, $\mathbf{z}^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n} \in \mathbb{C}$ is a monomial function of $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with degree $|\alpha|$
- Examples of distributions:
  (1) Locally integrable functions $f \in L^1_{\mathrm{loc}}(\mathcal{U})$: $(f, \varphi) := \int_{\mathcal{U}} f\varphi$
  (2) Dirac δ-function: $(\delta, \varphi) := \varphi(0)$
- Derivatives of distributions:
$$(\partial^\alpha \Lambda, \varphi) := (-1)^{|\alpha|}(\Lambda, \partial^\alpha \varphi).$$
For $f \in L^1_{\mathrm{loc}}(\mathcal{U})$, if the distributional derivative $\partial^\alpha f \in \mathscr{D}'(\mathcal{U})$ is representable as a function, we call it a **weak derivative** of f.

- Examples (see Exercise 3.3):
  (1) $f \in L^1_{\mathrm{loc}}(\mathbb{R}) \subset \mathscr{D}'(\mathbb{R})$ given by $f(x) = |x|$ has a weak first derivative, but its second derivative is not a function
  (2) $f(x) = \ln|x|$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$ and its derivative is a function but is not locally integrable (principal value integral)
  (3) $K(z) := \frac{1}{2\pi z}$ is in $L^1_{\mathrm{loc}}(\mathbb{C})$ and has $\bar{\partial}K = \delta$, but $\partial K := (\partial_s - i\partial_t)K$ is given by a principal value integral
- Products of smooth functions with distributions: $(f\Lambda, \varphi) := (\Lambda, f\varphi)$
- Convolutions of test functions and distributions on $\mathbb{R}^n$: $(f * \Lambda, \varphi) := (\Lambda, f^- * \varphi)$ where $f^-(x) := f(-x)$ and
$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, dy,$$
  and $\Lambda * f := f * \Lambda$
- $\delta * f = f$ for all $f \in \mathscr{D}(\mathbb{R}^n)$ (see Exercise 3.5)
- Lemma: If $\Lambda \in \mathscr{D}'(\mathcal{U})$ has first derivatives $\partial_1\Lambda, \dots, \partial_n\Lambda \in \mathscr{D}'(\mathcal{U})$ that are all representable by continuous functions, then Λ is representable by a (unique) C^1-function. (For the proof, see e.g. [LL01].)
- Lemma: For $f \in C_0^\infty(\mathbb{R}^n)$ and $\Lambda \in \mathscr{D}'(\mathbb{R}^n)$, $f * \Lambda$ is represented by the smooth function
$$(f * \Lambda)(x) = (\Lambda, \tau_x\varphi), \qquad \text{where} \qquad \tau_x\varphi(y) := \varphi(x-y).$$
- Corollary: the equation $\bar{\partial}u = f$ for $f \in C_0^\infty(\mathbb{C})$ has a smooth solution $u := K * f$
- Definition of the Sobolev spaces
$$W^{k,p}(\mathcal{U}) := \left\{ f \in L^p(\mathcal{U}) \;\middle|\; \text{there exist weak derivatives } \partial^\alpha f \in L^p(\mathcal{U}) \text{ for all } |\alpha| \leqslant k \right\},$$
$$\|f\|_{W^{k,p}} := \sum_{|\alpha| \leqslant k} \|\partial^\alpha f\|_{L^p}$$
- Theorem: For $1 < p < \infty$, the map $C_0^\infty(\mathring{\mathbb{D}}) \hookrightarrow C_0^\infty(\mathbb{C}) \to C^\infty(\mathring{\mathbb{D}}) : f \mapsto (K * f)\big|_{\mathring{\mathbb{D}}}$ extends to a bounded linear operator $L^p(\mathring{\mathbb{D}}) \to W^{1,p}(\mathring{\mathbb{D}})$, which is then a bounded right inverse of $\bar{\partial} : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$. (Proof next week.)

Suggested reading. Linear Cauchy-Riemann type operators and the derivation of $\mathbf{D}_u$ are discussed in more detail in [Wena, §2.3–2.4]. For the basic theory of distributions, a good source is [LL01]; most of the proofs we skipped here can also be found in my notes from Functional Analysis [Wen20a, §10].

Exercises (for the Übung on 08.11.2022). Most of this week's exercises are straightforward verifications of statements made in lecture, and we will not plan to spend much time on these in the Übung unless explicitly requested. The major exception is Exercise 3.3(d), which is an essential step in the regularity theory for the $\bar{\partial}$-operator, so we will discuss that in detail. Whatever time remains will be devoted to a general review of Fourier transforms in preparation for their use in the next lecture.

Exercise 3.1. Assume Σ is a complex manifold (of any finite dimension) and $E \to \Sigma$ is a complex vector bundle of rank $m \in \mathbb{N}$. A **smooth bundle atlas** for $E \to \Sigma$ is a collection of smooth local trivializations $\{\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}^m\}_{\alpha \in I}$ such that $\Sigma = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$, and the corresponding **transition functions** $g_{\beta\alpha} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{GL}(m, \mathbb{C})$ for $(\alpha, \beta) \in I \times I$ are characterized by the relation
$$\Phi_\beta \circ \Phi_\alpha^{-1}(p, v) = (p, g_{\beta\alpha}(p)v) \qquad \text{for all } p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \text{ and } v \in \mathbb{C}^m.$$

The bundle atlas also associates to any section $\eta : \mathcal{O} \to E$ defined over an open subset $\mathcal{O} \subset \Sigma$ a collection of functions $\eta_\alpha : \mathcal{O} \cap \mathcal{U}_\alpha \to \mathbb{C}^m$ characterized by

$$\Phi_\alpha(\eta(p)) = (p, \eta_\alpha(p)) \qquad \text{for all } p \in \mathcal{O} \cap \mathcal{U}_\alpha,$$

which are related to each other by $\eta_\beta = g_{\beta\alpha}\eta_\alpha$ on $\mathcal{O} \cap \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ for any $(\alpha, \beta) \in I \times I$. If the transition functions are all holomorphic, we say that the bundle atlas defines a **holomorphic structure** on $E \to \Sigma$ and call the latter a **holomorphic vector bundle**; in this case one also calls $\eta : \mathcal{O} \to E$ a **holomorphic section** if the functions η_α are all holomorphic. This notion makes sense due to the fact that products of holomorphic functions are also holomorphic.

Denote by $\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E) \subset \operatorname{Hom}_{\mathbb{R}}(T\Sigma, E)$ the vector bundle whose fiber over each point $p \in \Sigma$ is the space of complex-antilinear maps $T_p\Sigma \to E_p$; sections of this bundle are often called **E-valued $(0,1)$-forms**, and the space of such sections is denoted by[3]

$$\Omega^{0,1}(\Sigma, E) := \Gamma\big(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)\big).$$

One can similarly speak of the space of **$\mathbb{C}^m$-valued $(0,1)$-forms**

$$\Omega^{0,1}(\Sigma, \mathbb{C}^m) \subset \Omega^1(\Sigma, \mathbb{C}^m),$$

i.e. smooth $\mathbb{C}^m$-valued 1-forms $\lambda \in \Omega^1(\Sigma, \mathbb{C}^m)$ such that $\lambda_p : T_p\Sigma \to \mathbb{C}^m$ is complex-antilinear at each point $p \in \Sigma$. The bundle atlas above associates to each $\lambda \in \Omega^{0,1}(\Sigma, E)$ a collection of $\mathbb{C}^m$-valued $(0,1)$-forms $\lambda_\alpha \in \Omega^{0,1}(\mathcal{U}_\alpha, \mathbb{C}^m)$ according to the relation

$$\Phi_\alpha(\lambda(X)) = (p, \lambda_\alpha(X)) \qquad \text{for all} \qquad p \in \mathcal{U}_\alpha,\ X \in T_p\Sigma.$$

With this notation in place, prove the following: every holomorphic structure on $E \to \Sigma$ determines a first-order differential operator

$$\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E) = \Gamma\big(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)\big)$$

such that for each $\eta \in \Gamma(E)$ and each local trivialization $\Phi_\alpha : E|_{\mathcal{U}_\alpha} \to \mathcal{U}_\alpha \times \mathbb{C}^m$ in the given holomorphic bundle atlas,

$$(\mathbf{D}\eta)_\alpha = d\eta_\alpha + i\, d\eta_\alpha \circ j.$$

Moreover, a local section $\eta : \mathcal{O} \to E$ is then holomorphic if and only if $\mathbf{D}\eta \equiv 0$, and $\mathbf{D}$ satisfies the Leibniz rule

$$\mathbf{D}(f\eta) = f\,\mathbf{D}\eta + \bar\partial f(\cdot)\eta \qquad \text{for all } f \in C^\infty(\Sigma, \mathbb{C}) \text{ and } \eta \in \Gamma(E),$$

where $\bar\partial f := df + i\, df \circ j \in \Omega^{0,1}(\Sigma, \mathbb{C})$.

Exercise 3.2. As a warmup, suppose first that $E \to M$ is a finite-dimensional smooth vector bundle with a connection ∇, and $s \in \Gamma(E)$ is a smooth section that vanishes at some point $p \in M$.

(a) Show that the linear map $\nabla s(p) : T_pM \to E_p$ depends on the section s but not on the choice of connection ∇. We sometimes denote this map by

$$Ds(p) : T_pM \to E_p$$

and call it the **linearization** of s at p.

For the rest of the exercise, assume (Σ, j) is a Riemann surface, (M, J) is an almost complex manifold, and $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic curve. In lecture we characterized the associated **linearized Cauchy-Riemann operator**

$$\mathbf{D}_u : \Gamma(u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$$

[3] The bundle $\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$ is also often denoted by $\Lambda^{0,1}T^*\Sigma \otimes E$. The related bundle $\operatorname{Hom}_{\mathbb{C}}(T\Sigma, E) = \Lambda^{1,0}T^*\Sigma \otimes E$ and space of bundle-valued $(1,0)$-forms $\Omega^{1,0}(\Sigma, E) = \Gamma(\operatorname{Hom}_{\mathbb{C}}(T\Sigma, E))$ are obtained by replacing the words "complex-antilinear" with "complex-linear".

via the following property: for any smooth 1-parameter family of maps $\{u_\rho : \Sigma \to M\}_{\rho\in(-\epsilon,\epsilon)}$ with $u_0 = u$ and $\eta := \partial_\rho u_\rho|_{\rho=0} \in \Gamma(u^*TM)$, for each $z \in \Sigma$ and $X \in T_z\Sigma$,

$$(\mathbf{D}_u\eta)(X) = \nabla_\rho \left[Tu_\rho(X) + J \circ Tu_\rho(jX)\right]\big|_{\rho_0}$$

for any connection ∇ on $TM \to M$. We also showed that if the connection ∇ is chosen to be symmetric, then an explicit formula for $\mathbf{D}_u$ is given by

$$\mathbf{D}_u\eta = \nabla\eta + J(u) \circ \nabla\eta \circ j + (\nabla_\eta J) \circ Tu \circ j.$$

(b) Show that the definition of $\mathbf{D}_u$ does not depend on the choice of connection ∇ on $TM \to M$.
(c) Verify that $\mathbf{D}_u$ satisfies the Leibniz rule

$$\mathbf{D}_u(f\eta) = f\,\mathbf{D}\eta + \bar\partial f(\cdot)\eta \qquad \text{for all } f \in C^\infty(\Sigma,\mathbb{R}) \text{ and } \eta \in \Gamma(u^*TM),$$

and is thus a so-called **linear Cauchy-Riemann type** operator on $u^*TM \to \Sigma$.
(d) Show that if J is integrable, then $\mathbf{D}_u$ is the operator arising via Exercise 3.1 from the natural holomorphic structure on the bundle $u^*TM \to \Sigma$. In particular, $\mathbf{D}_u$ is in this case complex linear (which is not true for linear Cauchy-Riemann type operators in general).

Exercise 3.3. Prove:

(a) The locally integrable function $f(x) := |x|$ on $\mathbb{R}$ has weak derivative

$$f'(x) := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

(Note: weak derivatives are only defined almost everywhere, so we do not need to specify a value for f' at 0.)
(b) The derivative of the function f' from part (a) in the sense of distributions is $2\delta \in \mathscr{D}'(\mathbb{R})$.
(c) The function $g(x) := \ln|x|$ is in $L^1_{\text{loc}}(\mathbb{R})$, but its classical derivative (away from 0) is not, and its derivative in the sense of distributions is given by the **principal value** integral

$$(g',\varphi) = \text{p. v.}\int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx := \lim_{\epsilon\to 0^+}\int_{|x|\geqslant\epsilon}\frac{\varphi(x)}{x}\,dx.$$

(d) On $\mathbb{C}$ with coordinates $s + it$, write[4]

$$\bar\partial := \partial_s + i\partial_t = 2\frac{\partial}{\partial\bar z}, \qquad \partial := \partial_s - i\partial_t = 2\frac{\partial}{\partial z},$$

and consider the function $K(z) := \frac{1}{2\pi z}$. Show that $K \in L^1_{\text{loc}}(\mathbb{C})$, $\bar\partial K = \delta$ in the sense of distributions, and ∂K can be written as a principal value integral of a function that is not locally integrable, namely

$$(\partial K,\varphi) = -\frac{1}{\pi}\,\text{p. v.}\int_{\mathbb{C}}\frac{\varphi(z)}{z^2}\,dm := -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{\mathbb{C}\setminus\mathbb{D}_\epsilon}\frac{\varphi(z)}{z^2}\,dm,$$

where "dm" denotes the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

[4] The operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial\bar z}$ are not partial derivatives in any conventional sense since $z = s+it$ and $\bar z = s-it$ cannot be regarded as independent variables, but one obtains formulas for these operators as complex linear combinations of the usual partial derivative operators ∂_s and ∂_t via a formal application of the chain rule, regarding z and $\bar z$ as functions of s and t. We take these formulas as definitions, and they are useful for computations whenever a function on $\mathbb{C}$ is written in terms of z and $\bar z$, e.g. the usual derivative of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is then $\frac{\partial f}{\partial z}$, while $\frac{\partial f}{\partial\bar z}$ vanishes due to the Cauchy-Riemann equations.

Exercise 3.4. Verify that for the product of a smooth function $f \in C^\infty(\mathcal{U})$ with a distribution $\Lambda \in \mathscr{D}'(\mathcal{U})$ on some open domain $\mathcal{U} \subset \mathbb{R}^n$, the partial derivative operators ∂_j satisfy the usual Leibniz rule

$$\partial_j(f\Lambda) = (\partial_j f)\Lambda + f\,\partial_j\Lambda,$$

where $\partial_j f$ can be interpreted as a classical derivative, while $\partial_j\Lambda$ is a distributional derivative.

Exercise 3.5. Check that for a smooth compactly supported function $f \in C_0^\infty(\mathbb{R}^n)$, the formula

$$(f * \Lambda, \varphi) := (\Lambda, f^- * \varphi), \qquad f^-(x) := f(-x)$$

defines a continuous linear operator $\mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n) : \Lambda \mapsto f * \Lambda$ that matches the usual convolution of functions

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

when Λ is representable by a function. Prove also the formula

$$\partial^\alpha(f * \Lambda) = \partial^\alpha f * \Lambda = f * \partial^\alpha\Lambda,$$

where $\partial^\alpha f$ is interpreted as a classical derivative and $\partial^\alpha\Lambda$ as a distributional derivative.

Exercise 3.6. Prove $f * \delta = f$ for all $f \in C_0^\infty(\mathbb{R}^n)$.

4. Week 4

Lecture 7 (08.11.2022): The bounded right-inverse of $\bar\partial$.

- Solving $\bar\partial u = f \in C_0^\infty(\mathbb{C})$ by $u := K * f \in C^\infty(\mathbb{C})$ for $K(z) := \frac{1}{2\pi z}$
- Why linear local existence requires the estimate $\|K * f\|_{W^{1,p}} \leqslant c\|f\|_{L^p}$ for $p > 2$
- Proof of $\|K * f\|_{L^p} \leqslant c\|f\|_{L^p}$ by a variant of Young's inequality (see Exercise 4.2)
- Brief review of the Fourier transform:
  - Schwartz space
    $$\mathscr{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \;\middle|\; x^\alpha\partial^\beta f(x) \text{ is bounded on } \mathbb{R}^n \text{ for all multi-indices } \alpha, \beta \right\}$$
  - The operators $\mathscr{F}, \mathscr{F}^* : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$,
    $$(\mathscr{F}f)(p) := \hat{f}(p) := \int_{\mathbb{R}^n} e^{-2\pi i\langle p,x\rangle} f(x)\,dx, \qquad (\mathscr{F}^*f)(x) := \check{f}(x) := \int_{\mathbb{R}^n} e^{2\pi i\langle p,x\rangle} f(p)\,dx, \qquad .$$
  - $\mathscr{F}^{-1} = \mathscr{F}^*$ and $\langle f, \mathscr{F}g\rangle_{L^2} = \langle \mathscr{F}^*f, g\rangle_{L^2}$ imply **Plancherel's theorem**: $\mathscr{F}$ and $\mathscr{F}^*$ have unique extensions to bounded linear isometries $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$
  - The relation $(\mathscr{F}f, \varphi) = (f, \mathscr{F}\varphi)$ for $f, \varphi \in \mathscr{S}(\mathbb{R}^n)$ and $(f,g) := \int_{\mathbb{R}^n} fg$
- The space of **tempered distributions**
  $$\mathscr{S}'(\mathbb{R}^n) := \{\text{continuous linear functionals } \Lambda : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C} : \varphi \mapsto (\Lambda, \varphi)\},$$
  where $\varphi_j \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$ means $x^\alpha\partial^\beta\varphi_j$ converges uniformly on $\mathbb{R}^n$ to $x^\alpha\partial^\beta\varphi$ for all multi-indices α, β
- Examples of tempered distributions:
  - The δ function
  - Any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ that has at most polynomial growth (e.g. not $f(x) = e^{|x|^2}$)
- The operator $\mathscr{F} : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$,
  $$(\mathscr{F}\Lambda, \varphi) := (\Lambda, \mathscr{F}\varphi).$$
- The relation $\widehat{\partial_j\Lambda} = -2\pi i p_j \hat{\Lambda}$ for $\Lambda \in \mathscr{S}'(\mathbb{R}^n)$ (see Exercise 4.3)
- Lemma: For $f \in C_0^\infty(\mathring{\mathbb{D}})$ and $K(z) := \frac{1}{2\pi z}$, the function $|(K * f)(z)|$ is bounded on $\mathbb{C}$ by $C/|z|$ for some constant $C > 0$. Corollary: $K * f$ is both a smooth function and a tempered distribution on $\mathbb{C}$.

- Proof of the estimate $\|\partial(K * f)\|_{L^2} \leqslant c\|f\|_{L^2}$ via Fourier transform: If $u \in C^\infty(\mathbb{C}) \cap \mathscr{S}'(\mathbb{C})$ and $\bar{\partial} u \in L^2(\mathbb{C})$, then ∂u is also in $L^2(\mathbb{C})$ and $\|\partial u\|_{L^2} = \|\bar{\partial} u\|_{L^2}$.
- Why we call $\bar{\partial}$ an **elliptic** operator: $\mathscr{F}(\bar{\partial} u)(\zeta) = Q(\zeta)\hat{u}(\zeta)$ for a polynomial function $Q : \mathbb{C} \to \mathbb{C}$ such that $Q(\zeta) \neq 0$ for all $\zeta \neq 0$.
- The case $p > 2$: we need an estimate $\|\partial K * f\|_{L^p} \leqslant c\|f\|_{L^p}$ for $f \in C_0^\infty(\mathring{\mathbb{D}})$, where $f \mapsto \partial K * f$ denotes the **singular integral operator**

$$(\partial K * f)(z) := -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|\zeta - z| \geqslant \epsilon} \frac{f(\zeta)}{(z-\zeta)^2} \, dm(\zeta).$$

- Statement (without proof) of the Calderón-Zygmund inequality for singular integral operators
- Corollary: $\bar{\partial} : W^{1,p}(\mathring{\mathbb{D}}) \to L^p(\mathring{\mathbb{D}})$ has a bounded right inverse given by (the unique extension of) $f \mapsto K * f$ for every $p \in (1, \infty)$.

Lecture 8 (09.11.2022): Linear elliptic regularity.

- The Hölder spaces $C^{k,\alpha}(\mathcal{U})$ for $k \in \{0\} \cup \mathbb{N}$ and $\alpha \in (0,1)$,

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + \sum_{|\beta|=k} |\partial^\beta f|_{C^\alpha} \qquad \text{where} \qquad |f|_{C^\alpha} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

  Arzelà-Ascoli $\Rightarrow$ the inclusion $C^{k,\alpha}(\mathcal{U}) \hookrightarrow C^k(\bar{\mathcal{U}})$ is compact, i.e. $C^{k,\alpha}$-bounded sequences have C^k-convergent subsequences.
- Useful properties of Sobolev spaces, part 1:
  Assume $k \geqslant 0$ an integer, $1 \leqslant p < \infty$, $\mathcal{U} \subset \mathbb{R}^n$ open and bounded such that $\partial\bar{\mathcal{U}} \subset \mathbb{R}^n$ is a smooth submanifold.
  (1) Approximation: $C^\infty(\bar{\mathcal{U}})$ is dense in $W^{k,p}(\mathcal{U})$.
  Easier variant: if $\rho_j : \mathbb{R}^n \to [0, \infty)$ is an approximate identity and $f \in W^{k,p}(\mathcal{U})$, then $\rho_j * f$ is defined on each precompact open subset $\mathcal{V} \subset \bar{\mathcal{V}} \subset \mathcal{U}$ for large j, is smooth and $W^{k,p}$-convergent to f on $\mathcal{V}$. (We say $\rho_j * f \to f$ in $W^{k,p}_{\mathrm{loc}}$ on $\mathcal{U}$.)
  Remark: $C_0^\infty(\mathcal{U})$ is *not* dense in $W^{k,p}(\mathcal{U})$ except when $k = 0$, but instead defines a closed subspace

$$W_0^{k,p}(\mathcal{U}) := \text{the } W^{k,p}\text{-closure of } C_0^\infty(\mathcal{U}) \subset W^{k,p}(\mathcal{U}).$$

  (2) Extension: Given $\mathcal{V} \subset \bar{\mathcal{V}} \subset \mathcal{U}$ with $\partial\bar{\mathcal{V}}$ smooth, there exist bounded linear operators $E : W^{k,p}(\mathcal{V}) \to W^{k,p}(\mathcal{U})$ such that $Ef|_{\mathcal{V}} = f$.
  (3) Sobolev embedding theorem (case $kp > n$): For any $\alpha \in (0,1)$ satisfying $\alpha \leqslant k - \frac{n}{p}$, there is a continuous linear inclusion map

$$W^{k,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\mathcal{U}),$$

  and therefore also

$$W^{k+d,p}(\mathcal{U}) \hookrightarrow C^{d,\alpha}(\mathcal{U}) \qquad \text{for all integers } d \geqslant 0.$$

  Corollary: $\bigcap_{k \geqslant 0} W^{k,p}(\mathcal{U}) = C^\infty(\bar{\mathcal{U}})$
  (4) There is a continuous bilinear product map

$$C^k(\bar{\mathcal{U}}) \times W^{k,p}(\mathcal{U}) \to W^{k,p}(\mathcal{U}) : (f,g) \mapsto fg,$$

  hence an estimate of the form $\|fg\|_{W^{k,p}} \leqslant \|f\|_{C^k} \cdot \|g\|_{W^{k,p}}$.

- Fundamental elliptic estimate: For each $p \in (1, \infty)$, $\|u\|_{W^{1,p}} \leqslant c\|\bar{\partial}u\|_{L^p}$ holds for all $u \in C_0^\infty(\mathring{\mathbb{D}})$.
  Corollary 1 (by density): Also holds for $u \in W_0^{1,p}(\mathring{\mathbb{D}})$.
  Corollary 2 (since $\bar{\partial}\partial^\alpha = \partial^\alpha\bar{\partial}$): $\|u\|_{W^{k,p}} \leqslant c\|\bar{\partial}u\|_{W^{k-1,p}}$ for all $u \in W_0^{k,p}(\mathring{\mathbb{D}})$, $k \in \mathbb{N}$.
  Proof is easy consequence of the boundedness of $L^p \to W^{1,p} : f \mapsto K * f$.
- Linear regularity theorem: Assume $1 < p < \infty$, $m \geqslant k \geqslant 0$ are integers, $A \in C^m(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ and $f \in W^{m,p}(\mathring{\mathbb{D}}, \mathbb{C}^n)$. Then every weak solution $u \in L^p(\mathbb{D}, \mathbb{C}^n)$ to $(\bar{\partial} + A)u = f$ is in $W^{m+1,p}_{\mathrm{loc}}$ on $\mathring{\mathbb{D}}$, and for each $0 < r < r' < 1$, there is a constant $c > 0$ independent of u and f such that
$$\|u\|_{W^{m+1,p}(\mathring{\mathbb{D}}_r)} \leqslant c\|u\|_{W^{m,p}(\mathring{\mathbb{D}}_{r'})} + c\|f\|_{W^{m,p}(\mathring{\mathbb{D}}_{r'})}.$$
- Corollary: If A and f are smooth, so is u.
- Lemma: If $u \in W^{m,p}(\mathring{\mathbb{D}})$ and $\bar{\partial}u = f \in W^{m,p}(\mathring{\mathbb{D}})$, then $u \in W^{m+1,p}(\mathring{\mathbb{D}}_r)$ for each $r < 1$.
  Quick proof: After multiplying with a cutoff function, can assume without loss of generality u has compact support in $\mathring{\mathbb{D}}$. Choose an approximate identity ρ_j and consider $u_j := \rho_j * u \in C_0^\infty(\mathring{\mathbb{D}})$, which converge in $W^{m,p}$ to u. Also $\bar{\partial}u_j = \rho_j * \bar{\partial}u = \rho_j * f =: f_j$ converge in $W^{m,p}$ to f. Bound $\|u_j - u_k\|_{W^{m+1,p}}$ via the fundamental elliptic estimate: implies u_j is also $W^{m+1,p}$-Cauchy, thus $u \in W^{m+1,p}$. (Accounting for the cutoff function, one obtains this result for the original u on a slightly smaller domain.)

Suggested reading. For background details on Young's inequality, approximate identities, Fourier transforms and tempered distributions, see [LL01] or [Wen20a]. A self-contained proof of the Calderón-Zygmund inequality can be found in [Wena, §2.A]; it takes about seven pages.

For proofs of the essential properties of Sobolev spaces on bounded domains with smooth boundary, see e.g. [Eva98]. There is also the more comprehensive treatment in [AF03], which will also tell you what is known in cases where $\mathcal{U} \subset \mathbb{R}^n$ is not bounded.

Our presentation of the linear regularity theorem follows [Wenc, §2.4.1], except that instead of using difference quotients (which are not well suited for studying weak solutions of class L^p), we used mollifiers. The mollifier argument is a very easy special case of something that is standard in the theory of elliptic operators on closed manifolds, e.g. a more general and harder version of it can be found in [Ebe]. We *will* use difference quotients next week to handle the nonlinear case.

Exercises (for the Übung on 15.11.2022). The most important exercises this week are the first and the last, followed by Exercise 4.5 (on the Sobolev embedding theorem). If there is time to spare in the Übung, we will use it to discuss a few useful consequences of the Sobolev embedding theorem involving continuous product pairings.

Exercise 4.1. In lecture we proved that for any smooth function $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$, the equation $(\bar{\partial} + A)u = 0$ admits solutions $u : \mathbb{D}_\epsilon \to \mathbb{C}^n$ on sufficiently small disks $\mathbb{D}_\epsilon \subset \mathbb{D}$ having arbitrary values at the point $0 \in \mathbb{D}_\epsilon$; by elliptic regularity, such a solution will necessarily be smooth. Show that if we allow *weak* solutions (i.e. solutions in the sense of distributions), then the local existence result remains true under the weaker assumption that $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is of class L^p for some $p > 2$, and the solution might not be smooth, but will at least be continuous.
Remark: We will use this generalization next week in order to prove the similarity principle, an important result about the local behavior of solutions to linear Cauchy-Riemann type equations.

Exercise 4.2. Prove the following variant of Young's inequality: If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{U} \subset \mathbb{R}^n$ is a bounded open subset, then for each $p \in [1, \infty]$ there exists a constant $c > 0$ such that the estimate
$$\|f * g\|_{L^p(\mathcal{U})} \leqslant c\|g\|_{L^p}$$

holds for all $g \in L^p(\mathbb{R}^n)$ with support contained in $\mathcal{U}$. Note that on the left hand side, we only consider the L^p-norm over $\mathcal{U}$, not over $\mathbb{R}^n$.
Hint: The case $p = \infty$ is easy. If $p < \infty$, let $q \in (1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and apply Hölder's inequality to the function

$$|f(x-y)g(y)| = |f(x-y)|^{1/p} |g(y)| \cdot |f(x-y)|^{1/q}.$$

Exercise 4.3. Prove the following statements about Fourier transforms of tempered distributions.

(a) For any $\Lambda \in \mathscr{S}'(\mathbb{R}^n)$ and $j = 1, \dots, n$, the relations

$$\widehat{\partial_j \Lambda} = 2\pi i p_j \widehat{\Lambda} \qquad \text{and} \qquad \partial_j \widehat{\Lambda} = \mathscr{F}(-2\pi i x_j \Lambda)$$

hold, where both expressions to the right of the equal sign are interpreted in terms of products of smooth polynomial functions on $\mathbb{R}^n$ with tempered distributions.

(b) $\mathscr{F}(\delta) = 1$ and $\mathscr{F}(1) = \delta$.

(c) The Fourier transform of any polynomial function $P : \mathbb{R}^n \to \mathbb{C}$ is a linear combination of derivatives of the δ-function, and vice versa.

The next part may require some knowledge of a basic result about the support of distributions. By definition, the **support** $\operatorname{supp}(\Lambda) \subset \mathcal{U}$ of a distribution $\Lambda \in \mathscr{D}'(\mathcal{U})$ is the complement of the union of all open subsets $\mathcal{O} \subset \mathcal{U}$ such that $\Lambda(\varphi) = 0$ whenever $\operatorname{supp}(\varphi) \subset \mathcal{O}$. A good example to think about is the δ-function $\delta \in \mathscr{D}'(\mathbb{R}^n)$, which vanishes on every test function supported in $\mathbb{R}^n \backslash \{0\}$, thus $\operatorname{supp}(\delta) = \{0\}$. The same obviously holds for all derivatives of δ. Conversely, [Hör03, Theorem 2.3.4] says that every distribution with support contained in $\{0\}$ is a linear combination of derivatives of δ.

(d) The function $f(z) := 1/z$ belongs to $L^1_{\mathrm{loc}}(\mathbb{C})$ and defines a tempered distribution on $\mathbb{C}$, though globally, it belongs to neither $L^1(\mathbb{C})$ nor $L^2(\mathbb{C})$. Compute its Fourier transform—show, in particular, that $\widehat{f}$ is a very similar function that also has all of these properties.
Hint: f is closely related to our fundamental solution K for the $\bar{\partial}$-equation, which satisfies $\bar{\partial} K = \delta$.

Exercise 4.4. Given two vector bundles $E, F \to M$ over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, a linear map $L : \Gamma(E) \to \Gamma(F)$ is called a **differential operator of order** m (with smooth coefficients) if choosing local trivializations of E and F over any region $\mathcal{U} \subset M$ together with a chart $\mathcal{U} \to \mathcal{O} \subset \mathbb{R}^n$ identifies the map $\Gamma(E|_{\mathcal{U}}) \xrightarrow{L} \Gamma(F|_{\mathcal{U}})$ with a map of the form

$$\sum_{|\alpha| \leqslant m} c_\alpha \partial^\alpha : C^\infty(\mathcal{O}, \mathbb{F}^k) \to C^\infty(\mathcal{O}, \mathbb{F}^\ell)$$

for smooth functions $c_\alpha : \mathcal{O} \to \operatorname{Hom}(\mathbb{F}^k, \mathbb{F}^\ell)$, where we assume at least one of the c_α with $|\alpha| = m$ is nontrivial. The **principal symbol** of such an operator is the unique smooth fiber-preserving[5] map $\sigma^L : T^*M \to \operatorname{Hom}(E, F)$ such that for every $p \in M$, $\lambda \in T_p^*M$, $\eta \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{R})$ satisfying $f(p) = 0$ and $(df)_p = \lambda$,

$$\sigma^L(\lambda)\eta(p) = \frac{1}{m!} L(f^m \eta)(p).$$

We call L **elliptic** if for every $p \in M$ and every $\lambda \neq 0 \in T_p^*M$, the linear map $\sigma^L(\lambda) : E_p \to F_p$ is invertible.

[5]Here the term **fiber-preserving** means that for each $p \in M$, σ^L maps T_p^*M smoothly to the vector space $\operatorname{Hom}(E_p, F_p)$. Note however that the word "linear" was not included; in general, the map $T_p^*M \to \operatorname{Hom}(E_p, F_p)$ will be a homogeneous polynomial of degree m, so it is linear if and only if L is a first-order operator.

(a) Convince yourself that the definition of σ^L described above does not depend on any choices, and moreover, that two operators of order m have the same principal symbol if and only if the difference between them is an operator of order strictly less than m.
Hint: Just write down a formula for σ^L in local coordinates/trivializations.

(b) For M a complex manifold, E a complex vector bundles and $D : \Gamma(E) \to \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(TM, E))$ a linear Cauchy-Riemann type operator, compute σ^D and show that D is elliptic if and only if $\dim_{\mathbb{C}} M = 1$.

Exercise 4.5. Find direct proofs of the following special cases of the Sobolev embedding theorem:

(a) If $2k > n$, then there is a continuous linear inclusion

$$W^{k,2}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n).$$

Hint: Show first that $f \in L^2(\mathbb{R}^n)$ belongs to $W^{k,2}(\mathbb{R}^n)$ if and only if the product of its Fourier transform $\hat{f} \in L^2(\mathbb{R}^n)$ with the polynomial function $\mathbb{R}^n \to \mathbb{R} : p \mapsto (1 + |p|^2)^{k/2}$ is also in $L^2(\mathbb{R}^n)$. Then prove that $\hat{f} \in L^1(\mathbb{R}^n)$ if $2k > n$. (Why is that good enough?)

(b) If $1 < p < \infty$ and $0 < \alpha \leqslant 1 - \frac{1}{p}$, then for any open interval $\mathcal{U} \subset \mathbb{R}$, there is a continuous linear inclusion

$$W_0^{1,p}(\mathcal{U}) \hookrightarrow C^{0,\alpha}(\mathcal{U}).$$

Hint: By density, it suffices to prove that estimates of the form $\|f\|_{C^0} \leqslant c\|f\|_{W^{1,p}}$ and $|f|_{C^\alpha} \leqslant c\|f\|_{W^{1,p}}$ hold for all $f \in C_0^\infty(\mathcal{U})$. Use the fundamental theorem of calculus, and Hölder's inequality.

Exercise 4.6. Assume $u_j : \mathbb{D} \to \mathbb{C}^n$ is a sequence of smooth solutions to equations of the form

$$(\bar{\partial} + A_j)u_j = f_j$$

for C^∞-convergent sequences $A_j \to A \in C^\infty(\mathbb{D}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n))$ and $f_j \to f \in C^\infty(\mathbb{D}, \mathbb{C}^n)$. Show:

(a) If there is a uniform bound $\|u_j\|_{L^p} \leqslant C$ for some $p \in (1, \infty)$, then u_j has a subsequence converging in C^∞_{loc} on the open disk $\mathring{\mathbb{D}}$.

(b) If u_j is L^p-convergent to some function $u \in L^p(\mathbb{D}, \mathbb{C}^n)$, then the converence is also in C^∞_{loc} on $\mathring{\mathbb{D}}$, and u is thus a smooth solution to $(\bar{\partial} + A)u = f$ on $\mathring{\mathbb{D}}$.

Remark: If the convergence in part (a) were in C^∞ on the closed disk $\mathbb{D}$ instead of C^∞_{loc} on $\mathring{\mathbb{D}}$, it would follow (why?) that solution spaces of the equation $(\bar{\partial} + A)u = 0$ on $\mathbb{D}$ are always finite dimensional. But that is false. It will become true when we consider equations of this type over closed Riemann surfaces instead of the disk $\mathbb{D}$.

5. Week 5

Übung (15.11.2022). One extra result was discussed in the problem class this week that then needed to be used in the subsequent lecture: it's an improvement of Lecture 8's linear local regularity theorem, stating that if $u \in L^p$ $(1 < p < \infty)$ is a weak solution to $(\bar{\partial} + A)u = 0$ with $A \in C^m$ on the disk $\mathbb{D} \subset \mathbb{C}$, then u is of class $W^{m+1,q}$ on all smaller disks for every $q \in (1, \infty)$, and therefore (by the Sobolev embedding theorem) also of class C^m. The point of this statement is that we need not have $q = p$, and the set of weak solutions of class L^p to the equation $(\bar{\partial} + A)u = 0$ is therefore the same for any $p \in (1, \infty)$. The proof (see e.g. [Wenc, Corollary 2.23]) uses both the $kp > n$ and $kp \leqslant n$ cases of the Sobolev embedding theorem; the latter gives inclusions $W^{k,p} \hookrightarrow L^q$ for $q \geqslant p$ in a certain range.

Lecture 9 (15.11.2022): The Fredholm property.

- Global corollary of linear regularity: for a linear Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ on a complex vector bundle $E \to \Sigma$, every weak solution of class L^p_{loc} to $\mathbf{D}\eta = 0$ is smooth.
- Similarity principle: Given a solution $\eta \in \Gamma(E)$ to $\mathbf{D}\eta = 0$, every point $z_0 \in \Sigma$ has a neighborhood on which E admits a continuous local trivialization identifying η with a holomorphic function. (Proof via local existence for complex-linear Cauchy-Riemann type operators of class L^∞_{loc}; cf. Exercise 4.1.)
- Corollary: Unless $\eta \equiv 0$, zeroes of η are isolated, and have positive order in the case $\operatorname{rank}_{\mathbb{C}}(E) = 1$.
- Corollary of the corollary: For a closed Riemann surface Σ and a complex line bundle $E \to \Sigma$ with $c_1(E) < 0$, every Cauchy-Riemann type operator on E is injective.
- The Banach spaces $C^k(E)$ and $W^{k,p}(E)$ of sections of a vector bundle $E \to \Sigma$ over a closed manifold
- Cauchy-Riemann type operators $\mathbf{D} : C^{m+1}(E) \to C^m(F)$ of class C^m for $0 \leqslant m \leqslant \infty$, where $F := \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$; locally $\mathbf{D} = \bar{\partial} + A$ for $A \in C^m(\mathbb{D}, \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n))$
- Bundle metrics $\langle\ ,\ \rangle$, L^2-pairings $\langle \xi, \eta \rangle_{L^2} := \int_\Sigma \langle \xi, \eta \rangle \, d\operatorname{vol}$ and the formal adjoint operator $\mathbf{D}^* : C^{m+1}(F) \to C^m(E)$ for a Cauchy-Riemann type operator $\mathbf{D} : C^{m+1}(E) \to C^m(F)$ of class C^m:
$$\langle \xi, \mathbf{D}\eta \rangle_{L^2} = \langle \mathbf{D}^*\xi, \eta \rangle_{L^2} \qquad \text{for all } \eta \in \Gamma(E),\ \xi \in \Gamma(F).$$
- $\mathbf{D}^*$ exists, is unique, and is also a Cauchy-Riemann type operator (see Exercise 5.2)
- Main theorem on the Fredholm property: For Σ a closed Riemann surface, $E \to \Sigma$ a complex vector bundle and $\mathbf{D}$ a Cauchy-Riemann type operator of class C^m on E, the kernels of the bounded linear operators $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$ and $\mathbf{D}^* : W^{k,p}(F) \to W^{k-1,p}(E)$ for $k \in \{1, \ldots, m+1\}$ and $p \in (1, \infty)$ are finite-dimensional spaces of C^m-sections that do not depend on the choice of k and p. Moreover, the targets of these operators have splittings into closed L^2-orthogonal subspaces
$$W^{k-1,p}(F) = \operatorname{im}(\mathbf{D}) \oplus \ker(\mathbf{D}^*) \qquad \text{and} \qquad W^{k-1,p}(E) = \operatorname{im}(\mathbf{D}^*) \oplus \ker(\mathbf{D}).$$
- Corollary: $\mathbf{D}$ is a Fredholm operator for each k, p, i.e. $\ker(\mathbf{D})$ and $\operatorname{coker}(\mathbf{D}) := W^{k-1,p}(F)/\operatorname{im}(\mathbf{D}) \cong \ker(\mathbf{D}^*)$ are finite dimensional, and its kernel and **Fredholm index**
$$\operatorname{ind}(\mathbf{D}) := \dim\ker(\mathbf{D}) - \dim\operatorname{coker}(\mathbf{D}) = \dim\ker(\mathbf{D}) - \dim\ker(\mathbf{D}^*)$$
are independent of k and p.
- Proof that $\ker(\mathbf{D})$ is independent of k and p: If $m = \infty$ and $\mathbf{D}\eta = 0$ for $\eta \in W^{k,p}(E)$, local regularity implies $\eta \in \bigcap_{k \in \mathbb{N}} W^{k,p}(E) = C^\infty(E) = \bigcap_{k,q} W^{k,q}(E)$. For $m < \infty$, the proof requires the enhanced regularity result covered in the Übung, which implies $\eta \in \bigcap_{1<q<\infty} W^{m+1,q}(E)$.
- Main tool: local regularity gave the estimate $\|u\|_{W^{k,p}(\mathring{\mathbb{D}}_r)} \leqslant c\|(\bar{\partial}+A)u\|_{W^{k-1,p}(\mathring{\mathbb{D}})} + c\|u\|_{W^{k-1,p}(\mathring{\mathbb{D}})}$ for $u \in W^{k,p}(\mathring{\mathbb{D}})$ and $r < 1$, which implies the global estimate
$$\|\eta\|_{W^{k,p}(E)} \leqslant c\|\mathbf{D}\eta\|_{W^{k-1,p}(F)} + c\|\eta\|_{W^{k-1,p}(E)} \qquad \text{for all } \eta \in W^{k,p}(E).$$
- Useful properties of Sobolev spaces, part 1.5 (Rellich-Kondrachov):
On a bounded open domain $\mathcal{U} \subset \mathbb{R}^n$ with $\partial\bar{\mathcal{U}}$ smooth, the inclusions $W^{k,p}(\mathcal{U}) \hookrightarrow W^{k-1,p}(\mathcal{U})$ are compact.
Global corollary: For a vector bundle E over a closed manifold, the inclusions $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ are compact.

- Lemma: Suppose X, Y, Z are Banach spaces, $T : X \to Y$ is a bounded linear operator and $K : X \to Z$ is a compact linear operator, such that an estimate of the form

$$\|x\|_X \leqslant c\|Tx\|_Y + c\|Kx\|_Z \qquad \text{for all } x \in X$$

  holds. Then $\ker T \subset X$ is finite dimensional and $\operatorname{im} T \subset Y$ is closed. (Proof by showing bounded sequences in $\ker T$ have convergent subsequences.)
- Corollary: $\mathbf{D}$ and $\mathbf{D}^*$ have finite-dimensional kernels.

Lecture 10 (16.11.2022): Nonlinear regularity.

- Proof that $W^{k-1,p}(F) = \operatorname{im}(\mathbf{D}) \oplus \ker(\mathbf{D}^*)$ for a linear Cauchy-Riemann type operator $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}\big(\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)\big)$ of class C^m $(m \geqslant k-1)$ on a complex vector bundle E over a closed Riemann surface Σ.
- Useful properties of Sobolev spaces, part 2 (nonlinear):
  Under the same assumptions as in part 1...
  (1) Product pairings: If $kp > n$, $k \geqslant m \geqslant 0$ are integers and $k - \frac{n}{p} \geqslant m - \frac{n}{q}$, then there is a continuous bilinear map

$$W^{k,p}(\mathcal{U}) \times W^{m,q}(\mathcal{U}) \to W^{m,q}(\mathcal{U}) : (f, g) \mapsto fg.$$

  (2) Compositions: If $kp > n$, $\Omega \subset \mathbb{R}^N$ is an open set and we define

$$W^{k,p}(\mathcal{U}, \Omega) := \left\{ f \in W^{k,p}(\mathcal{U}, \mathbb{R}^N) \;\middle|\; f(\bar{\mathcal{U}}) \subset \Omega \right\},$$

  which is an open subset of $W^{k,p}(\mathcal{U}, \mathbb{R}^N)$ by the Sobolev embedding theorem, then there is a continuous map

$$C^k(\Omega) \times W^{k,p}(\mathcal{U}, \Omega) \to W^{k,p}(\mathcal{U}) : (f, g) \mapsto f \circ g.$$

- Regularity theorem for the inhomogeneous nonlinear Cauchy-Riemann equation

$$\partial_s u(z) + J(z, u(z))\partial_t u = f(z), \qquad \text{given } J : \mathring{\mathbb{D}} \times \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n) \text{ and } f : \mathring{\mathbb{D}} \to \mathbb{C}^n.$$

  Assume $1 < p < \infty$ and $k \in \mathbb{N}$ satisfy $kp > 2$, $m \geqslant 0$ is an integer, $J, J_1, J_2, \ldots$ are of class C^m and $f, f_1, f_2, \ldots$ are of class $W^{m,p}$. Then:
  (1) Every solution $u \in W^{k,p}(\mathring{\mathbb{D}})$ to $\partial_s u + J(\cdot, u)\partial_t u = f$ is of class $W^{m+1,p}_{\mathrm{loc}}$.
  (Corollary: if J and f are smooth, so is u.)
  (2) Given $J_j \to J$ in the C^m-topology and a sequence $u_j \in W^{k,p}(\mathring{\mathbb{D}})$ of solutions to $\partial_s u_j + J_j(\cdot, u_j)\partial_t u_j = f_j$:
    (a) If $\|u_j\|_{W^{k,p}}$ and $\|f_j\|_{W^{m,p}}$ are uniformly bounded on $\mathring{\mathbb{D}}$, then $\|u_j\|_{W^{k+1,p}}$ is uniformly bounded on all compact subsets.
    (Corollary: If $J_j \to J$ and $f_j \to f$ in C^∞, then u_j converges in C^∞_{loc} to a solution u to $\partial_s u + J(\cdot, u)\partial_t u = f$.)
    (b) If $u_j \to u$ in $W^{k,p}$ and $f_j \to f$ in $W^{m,p}$, then u_j also converges to u in $W^{m+1,p}_{\mathrm{loc}}$.
    (Corollary: All reasonable topologies on a moduli space of J-holomorphic curves are equivalent.
- Proof of the $W^{m+1,p}_{\mathrm{loc}}$-bound on u_j by a rescaling trick: for any $z_0 \in \mathring{\mathbb{D}}$, choosing coordinates such that $u(z_0) = 0$ and $J(z_0, 0) = i$, and then replacing u_j with the function $\hat{u}_j(z) := \frac{u_j(z_0 + \epsilon z)}{\epsilon^\alpha}$ for some $\alpha \in (0, 1)$ with $\alpha \leqslant k - \frac{2}{p}$ transforms $\partial_s u_j + J(\cdot, u_j)\partial_t u_j = f_j$ into an equation whose nonlinear part becomes vanishingly small.
- Sketch of the proof that $u \in W^{m+1,p}_{\mathrm{loc}}$ via difference quotients

Suggested reading.

- Similarity principle: [Wena, §2.8]
- Banach spaces of sections of a bundle over a closed manifold: [Wenb, §A.2]
- The Fredholm property for Cauchy-Riemann type operators: [Wena, §3.3]; the presentation there only considers operators of class C^∞, but there are no substantial differences in the C^m case with $m < \infty$.
- Further useful properties of Sobolev spaces (products, composition, rescaling, difference quotients): [Wenb, §A.1.3–A.1.4]
- Nonlinear regularity theorem: [Wenc, §2.4.2]

Exercises (for the Übung on 22.11.2022).

Exercise 5.1. Use the similarity principle to prove the following local results about J-holomorphic maps $u : (\Sigma, j) \to (M, J)$ from a connected Riemann surface into an almost complex manifold.

(a) *(Unique continuation)* If $u, v : (\Sigma, j) \to (M, J)$ are two J-holomorphic maps with an intersection point $u(z_0) = v(z_0)$ such that the derivatives of u and v at z_0 match to all orders in some choice of local coordinates, then $u \equiv v$.

Solution:
Since the question is purely local, we can choose coordinates near z_0 on Σ and near $u(z_0) = v(z_0)$ on M in order to assume without loss of generality that u and v are maps $\mathbb{D} \to \mathbb{C}^n$ satisfying

$$\partial_s u + J(u)\partial_t u = \partial_s v + J(v)\partial_t v = 0 \qquad \text{and} \qquad u(0) = v(0) = 0,$$

where $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ is a smooth almost complex structure on $\mathbb{C}^n$ with $J(0) = i$. (The latter is possible because every complex structure on a vector space matches the standard one under a suitable choice of basis, but since the almost complex structure on M might not be integrable, we cannot assume J matches i at more than one point. We *are* using the knowledge that j is integrable, since $\dim \Sigma = 2$, i.e. the coordinates we have chosen on Σ near z_0 are holomorphic, so that $j\partial_s \equiv \partial_t$ and $j\partial_t \equiv -\partial_s$.)

Now add the assumption that in the chosen coordinates,

$$\partial^\alpha u(0) = \partial^\alpha v(0)$$

for all multi-indices α; one can show that if this is true in some particular choice of coordinates, then it is also true for all other choices. The function $h := v - u : \mathbb{D} \to \mathbb{C}^n$ now satisfies $\partial^\alpha h(0) = 0$ for all α, so Taylor's formula implies that for every $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that

$$|h(z)| \leqslant C_k |z|^k \qquad \text{for all } z \in \mathbb{D}. \tag{5.1}$$

We will now show that h also satisfies a linear Cauchy-Riemann type equation, making it subject to the similarity principle. Indeed, for each $z \in \mathbb{D}$, we find

$$\begin{aligned}
\partial_s h(z) + J(u(z))\partial_t h(z) &= [\partial_s v(z) + J(v(z))\partial_t v(z)] - [\partial_s u(z) + J(u(z))\partial_t u(z)] \\
&\qquad - [J(v(z)) - J(u(z))]\,\partial_t v(z) \\
&= -[J(v(z)) - J(u(z))]\,\partial_t v(z) \\
&= -\left(\int_0^1 \frac{\partial}{\partial\tau} J(u(z) + \tau h(z))\,d\tau\right)\partial_t v(z) \\
&= -\left(\int_0^1 DJ(u(z) + \tau h(z))h(z)\,d\tau\right)\partial_t v(z) =: -A(z)h(z),
\end{aligned}$$

where at the end we are defining the smooth function $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ by

$$A(z)w := \left(\int_0^1 DJ(u(z) + \tau h(z))w \, d\tau \right) \partial_t v(z) \in \mathbb{C}^n.$$

If we also define $\bar{J} : \mathbb{D} \to \mathcal{J}(\mathbb{C}^n)$ by $\bar{J}(z) := J(u(z))$, this calculation shows that h satisfies the linear PDE

$$\partial_s h(z) + \bar{J}(z)\partial_t h(z) + A(z)h(z) = 0. \tag{5.2}$$

If $\bar{J}$ were identically equal to the standard complex structure i, then (5.2) would precisely match $(\bar{\partial} + A)h = 0$, our usual expression for a linear Cauchy-Riemann type equation written in a local trivialization. The following observation shows that (5.2) is in fact equivalent to an equation of that form. We can view h as a section of the trivial real vector bundle $\mathbb{D} \times \mathbb{C}^n \to \mathbb{D}^n$, which is also a complex vector bundle on which we define the complex structure on the fiber over $z \in \mathbb{D}$ to be $\bar{J}(z) : \mathbb{C}^n \to \mathbb{C}^n$. As a complex vector bundle, it admits a complex local trivialization on some neighborhood $\mathbb{D}_\epsilon \subset \mathbb{D}$ of 0, which literally means a smooth map of the form

$$\Phi : \mathbb{D}_\epsilon \times \mathbb{C}^n \to \mathbb{D}_\epsilon \times \mathbb{C}^n : (z, v) \mapsto (z, \Psi(z)v)$$

such that for each $z \in \mathbb{D}$, $\Psi(z)$ is a complex-linear isomorphism from $(\mathbb{C}^n, \bar{J}(z))$ to $(\mathbb{C}^n, i)$. In other words, $\Psi(z) : \mathbb{C}^n \to \mathbb{C}^n$ is real-linear, invertible, and satisfies $\Psi(z)\bar{J}(z) = i\Psi(z)$. This local trivialization identifies our section h with the function

$$f(z) := \Psi(z)h(z) \in \mathbb{C}^n,$$

which then satisfies

$$\begin{aligned} \bar{\partial} f &= \partial_s (\Psi h) + i\partial_t (\Psi h) = \Psi \, \partial_s h + i\Psi \, \partial_t h + (\partial_s \Psi + i \, \partial_t \Psi) \, h \\ &= \Psi \, \partial_s h + \Psi \bar{J} \, \partial_t h + (\partial_s \Psi + i \, \partial_t \Psi) \, h = -\Psi A \Psi^{-1} f + (\partial_s \Psi + i \, \partial_t \Psi) \, \Psi^{-1} f, \end{aligned}$$

thus we obtain $(\bar{\partial} + B)f = 0$ if we set $B := \Psi A \Psi^{-1} - (\bar{\partial}\Psi)\Psi^{-1}$. This shows that h is indeed in the kernel of a linear Cauchy-Riemann type operator on the bundle $\mathbb{D} \times \mathbb{C}^n \to \mathbb{D}$, so by the similarity principle, one can find another local trivialization as described above such that the function $\Psi : \mathbb{D}_\epsilon \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is continuous and $f = \Psi h$ is holomorphic. Since $\Psi(z)$ is invertible for all z, h is then identically zero if and only if f is identically zero. If it is not, then f has a zero of finite order at 0, implying that it satisfies an estimate of the form

$$|f(z)| \geqslant c|z|^m$$

on some neighborhood of $0 \in \mathbb{D}$ for some $c > 0$ and $m \in \mathbb{N}$. Since Ψ is continuous, it follows that h also satisfies an estimate of this form, which contradicts (5.1).

(b) *(Critical points are isolated)* If $u : (\Sigma, j) \to (M, J)$ is J-holomorphic and its first derivative $T_{z_0}u : T_{z_0}\Sigma \to T_{u(z_0)}M$ vanishes at some point $z_0 \in \Sigma$, then either u is constant or z_0 has a neighborhood $\mathcal{U} \subset \Sigma$ such that u is immersed on $\mathcal{U}\backslash\{z_0\}$.

Solution:
As in part (a), we can assume after choosing coordinates that u is a map $\mathbb{D} \to \mathbb{C}^n$ satisfying $\partial_s u + J(u)\partial_t u = 0$ with $u(0) = 0$, where $J : \mathbb{C}^n \to \mathcal{J}(\mathbb{C}^n)$ satisfies $J(0) = i$. The complex-linearity of $Du(z)$ at each point implies that $\partial_t u$ vanishes wherever $\partial_s u$ does, so the goal is to show that if $\partial_s u(0) = 0$ but $\partial_s u$ is not identically zero, then the zero of $\partial_s u$ at 0 is isolated. Applying ∂_s to the equation $\partial_s u + J(u)\partial_t u = 0$ gives

$$\partial_s(\partial_s u) + J(u)\partial_t(\partial_s u) + [DJ(u)\partial_s u] \, \partial_t u = 0,$$

which can be rewritten in the form $\partial_s(\partial_s u) + \bar{J}(\partial_s u) + A\partial_s u$ for smooth functions $\bar{J} : \mathbb{D} \to \mathcal{J}(\mathbb{C}^n)$ and $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. By the same argument used in part (a), the similarity principle now implies that $\partial_s u$ is either identically zero or has isolated zeroes.

Exercise 5.2. Assume $E \to \Sigma$ is a complex vector bundle over a Riemann surface (Σ, j), and let

$$\Lambda^{1,0}T^*\Sigma, \qquad \Lambda^{0,1}T^*\Sigma$$

denote the complex line bundles over Σ whose fibers at a point $z \in \Sigma$ are the spaces of complex-linear and complex-antilinear maps $T_z\Sigma \to \mathbb{C}$ respectively. The complex dual bundle of $T\Sigma$ is thus $\Lambda^{1,0}T^*\Sigma$, and there are natural complex vector bundle isomorphisms

$$F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E) \cong \Lambda^{0,1}T^*\Sigma \otimes E, \qquad \text{and} \qquad \mathrm{Hom}_{\mathbb{C}}(T\Sigma, E) \cong \Lambda^{1,0}T^*\Sigma \otimes E.$$

Choose Hermitian bundle metrics $\langle\ ,\ \rangle$ on E and $T\Sigma$ and observe that they determine real bundle metrics

$$\langle\ ,\ \rangle^{\mathbb{R}} := \mathrm{Re}\langle\ ,\ \rangle$$

that are invariant under the respective complex structures on these bundles, e.g. we have $\langle X, Y\rangle^{\mathbb{R}} = \langle jX, jY\rangle^{\mathbb{R}}$ for all $X, Y \in T_z\Sigma$ at a point $z \in \Sigma$. Viewing $\langle\ ,\ \rangle^{\mathbb{R}}$ as a Riemannian metric on Σ, the induced area form $d\,\mathrm{vol} \in \Omega^2(\Sigma)$ is then given by

$$d\,\mathrm{vol}(X, Y) := \langle iX, Y\rangle^{\mathbb{R}}.$$

(a) Show that the bundle metric on $T\Sigma$ determines a natural complex vector bundle isomorphism $F \cong T\Sigma \otimes E$, and thus determines a natural Hermitian bundle metric $\langle\ ,\ \rangle$ on F.

Solution:
To avoid confusion in the following, we will always write $\langle\ ,\ \rangle_V$ for the chosen Hermitian bundle metric on any given complex vector bundle V such as E, F or $T\Sigma$, and $\langle\ ,\ \rangle_V^{\mathbb{R}} := \mathrm{Re}\langle\ ,\ \rangle_V$ for its real part. We will also write $\langle\ ,\ \rangle_{\mathbb{C}^n}$ for the standard Hermitian inner product on $\mathbb{C}^n$, whose real part $\langle\ ,\ \rangle_{\mathbb{C}^n}^{\mathbb{R}}$ is then the standard Euclidean inner product on $\mathbb{R}^{2n} = \mathbb{C}^n$.

We already have a natural isomorphism $F \cong \Lambda^{0,1}T^*\Sigma \otimes E$, so it suffices to observe that the Hermitian bundle metric $\langle\ ,\ \rangle_{T\Sigma}$ determines an isomorphism[6]

$$T\Sigma \to \Lambda^{0,1}T^*\Sigma : X \mapsto \langle \cdot, X\rangle_{T\Sigma}.$$

The resulting isomorphism $T\Sigma \otimes E \to \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$ identifies $X \otimes \eta$ for $X \in T_z\Sigma$ and $\eta \in E_z$ at a point $z \in \Sigma$ with the complex-antilinear map

$$T_z\Sigma \to E_z : Y \mapsto \langle Y, X\rangle_{T\Sigma} \cdot \eta.$$

The tensor product $T\Sigma \otimes E$ inherits a natural Hermitian bundle metric from the bundle metrics on $T\Sigma$ and E such that

$$\langle X \otimes \eta, Y \otimes \xi\rangle_{T\Sigma\otimes E} = \langle X, Y\rangle_{T\Sigma} \cdot \langle \eta, \xi\rangle_E \in \mathbb{C}$$

for $X, Y \in T_z\Sigma$ and $\eta, \xi \in E_z$ at any point $z \in \Sigma$. This determines a bundle metric $\langle\ ,\ \rangle_F$ on F via the isomorphism above.

[6]Our convention here is that $\langle X, Y\rangle$ is complex linear with respect to Y and antilinear with respect to X. The bijective real-linear bundle map $T\Sigma \to \Lambda^{1,0}T^*\Sigma : X \mapsto \langle X, \cdot\rangle_{T\Sigma}$ is thus not a complex bundle isomorphism, as it is not complex linear. Real vector bundles are always isomorphic to their dual bundles, but this is not true of complex bundles in general; one can show that it is not true in particular whenever the first Chern class is nonzero.

In the following, we use the real parts of the bundle metrics on E and (via part (a)) F together with the area form on Σ to define real-valued L^2-pairings for sections of E or F, which determine the notion of formal adjoint operators. With this understood, suppose $\mathbf{D} : C^{m+1}(E) \to C^m(F)$ is a linear Cauchy-Riemann type operator of class C^m on E.

(b) Show that a formal adjoint $\mathbf{D}^* : C^{m+1}(F) \to C^m(E)$ for $\mathbf{D}$ exists, is unique, and can be identified in suitable local trivializations and coordinates with operators of the form

$$-\partial + A : C^{m+1}(\mathbb{D}, \mathbb{C}^n) \to C^m(\mathbb{D}, \mathbb{C}^n),$$

where $\partial := \partial_s - i\partial_t$ and $A \in C^m(\mathbb{D}, \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n))$.

Solution:
The existence and uniqueness of the formal adjoint is a general fact about differential operators on Euclidean vector bundles. In the present context, one can see it as follows. As preparation, we need to be fairly explicit about how to identify $\mathbf{D}$ locally with an operator of the form $\bar{\partial} + A$. Suppose $\mathcal{U} \subset \Sigma$ is a region that can be identified with the unit disk $\mathbb{D} \subset \mathbb{C}$ via a choice of holomorphic coordinate, and that there also exists a trivialization of E over $\mathcal{U}$ that identifies $\langle\ ,\ \rangle_E$ with the standard inner product $\langle\ ,\ \rangle_{\mathbb{C}^n}$. These choices identify $\Gamma(E|_{\mathcal{U}})$ with $C^\infty(\mathbb{D}, \mathbb{C}^n)$ and $\Gamma(F|_{\mathcal{U}})$ with $\Omega^{0,1}(\mathbb{D}, \mathbb{C}^n)$, i.e. we shall view the restriction of $\mathbf{D}$ to the region $\mathcal{U} \subset \Sigma$ as a Cauchy-Riemann type operator of class C^m on the trivial vector bundle $\mathbb{D} \times \mathbb{C}^n \to \mathbb{D}$. The operator

$$\mathbf{D}_0 : C^\infty(\mathbb{D}, \mathbb{C}^n) \to \Omega^{0,1}(\mathbb{D}, \mathbb{C}^n), \qquad \mathbf{D}_0 f := df + i\, df \circ i$$

is a smooth Cauchy-Riemann type operator on this trivial bundle, so it follows that $\mathbf{D}$ and $\mathbf{D}_0$ differ by a zeroth-order term, meaning

$$\mathbf{D} f = \mathbf{D}_0 f + A f\, d\bar{z}$$

for some function $A : \mathbb{D} \to \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n)$. At this stage we have not yet fully trivialized the bundle F over $\mathcal{U}$, but our coordinates and trivialization of E naturally determine such a trivialization: it amounts to identifying $\Omega^{0,1}(\mathbb{D}, \mathbb{C}^n)$ with $C^\infty(\mathbb{D}, \mathbb{C}^n)$ via the correspondence $g\, d\bar{z} \mapsto g$, and since

$$\begin{aligned}\bar{\partial} f\, d\bar{z} &= (\partial_s f + i\partial_t f)(ds - i\, dt) = \partial_s f\, ds + \partial_t f\, dt + i\,(\partial_t f\, ds - \partial_s f\, dt) \\ &= df + i\, df \circ i = \mathbf{D}_0 f,\end{aligned}$$

the expression above for $\mathbf{D}$ on the trivial bundle $\mathbb{D} \times \mathbb{C}^n \to \mathbb{D}$ is now identified with the operator

$$\mathbf{D} = \bar{\partial} + A : C^{m+1}(\mathbb{D}, \mathbb{C}^n) \to C^m(\mathbb{D}, \mathbb{C}^n),$$

where the assumption that $\mathbf{D}$ is of class C^m means that $A : \mathbb{D} \to \operatorname{End}_{\mathbb{R}}(\mathbb{C}^n)$ is a C^m-function. So far this is nothing new, but I wanted to be explicit about the local trivialization that we are using on F.

We assumed above that the trivialization of $E|_{\mathcal{U}}$ is chosen to identify $\langle\ ,\ \rangle_E$ with $\langle\ ,\ \rangle_{\mathbb{C}^n}$, but we are not free to make any similar assumption about the bundle metrics on $T\Sigma$ and F, e.g. this would impose an extra condition on our holomorphic coordinate over $\mathcal{U}$, which might not be attainable. The good news however is that since $T\Sigma$ has complex rank 1, all Hermitian inner products at any point are real multiples of each other, so under the chosen identification of $\Gamma(T\Sigma|_{\mathcal{U}})$ with $C^\infty(\mathbb{D}, \mathbb{C})$, we can write

$$\langle\ ,\ \rangle_{T\Sigma} = h \langle\ ,\ \rangle_{\mathbb{C}}$$

for some smooth function $h : \mathbb{D} \to (0, \infty)$. Since $\langle i\cdot, \cdot \rangle_{\mathbb{C}}^{\mathbb{R}} = ds \wedge dt$ on $\mathbb{C}$, this same function gives us a formula for the area form $d\,\mathrm{vol}$, namely

$$d\,\mathrm{vol} = h\, ds \wedge dt.$$

Similarly, under the isomorphism $\Gamma(T\Sigma|_{\mathcal{U}}) = C^\infty(\mathbb{D}, \mathbb{C})$, the vector field represented by a function $g : \mathbb{D} \to \mathbb{C}$ is sent by the bundle isomorphism $T\Sigma \to \Lambda^{0,1}T^*\Sigma : X \mapsto \langle \cdot, X\rangle_{\mathbb{C}}$ to the $(0,1)$-form $\lambda \in \Omega^{0,1}(\mathbb{D}, \mathbb{C})$ given by

$$\lambda_z(Y) = h(z)\langle \cdot, g(z)\rangle_{\mathbb{C}} = h(z)g(z)\langle \cdot, 1\rangle_{\mathbb{C}} = h(z)g(z)\, d\bar{z},$$

so in particular, this isomorphism identifies the $(0,1)$-form $d\bar{z}$ with the vector field $1/h$. Using the prescription of part (a) for defining $\langle\ ,\ \rangle_F$, it follows that in our chosen trivialization of $F|_{\mathcal{U}}$, with E-valued $(0,1)$-forms written as functions $f, g : \mathbb{D} \to \mathbb{C}^n$,

$$\langle f, g\rangle_F = \langle (1/h) \otimes f, (1/h) \otimes g\rangle_{T\Sigma\otimes E} = \langle 1/h, 1/h\rangle_{T\Sigma} \cdot \langle f, g\rangle_E = \frac{1}{h}\langle f, g\rangle_{\mathbb{C}^n}.$$

Putting all this together, if we restrict the defining relation $\int_\Sigma \langle \xi, \mathbf{D}\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol} = \int_\Sigma \langle \mathbf{D}^*\xi, \eta\rangle_E^{\mathbb{R}}\, d\,\mathrm{vol}$ for the formal adjoint operator to sections with compact support in $\mathcal{U}$ and write it in our chosen trivializations, it takes the form

$$\int_{\mathbb{D}} \frac{1}{h}\langle f, (\bar{\partial} + A)g\rangle_{\mathbb{C}^n}^{\mathbb{R}}\, h\, ds \wedge dt = \int_{\mathbb{D}} \langle \mathbf{D}^* f, g\rangle_{\mathbb{C}^n}^{\mathbb{R}}\, h\, ds \wedge dt \qquad \text{for } f, g \in C_0^\infty(\mathbb{D}, \mathbb{C}^n).$$

Writing $\partial := \partial_s - i\partial_t$, integration by parts allows us to rewrite the left hand side as

$$\int_{\mathbb{D}} \langle (-\partial + A^T)f, g\rangle_{\mathbb{C}^n}^{\mathbb{R}}\, ds \wedge dt = \int_{\mathbb{D}} \left\langle \frac{1}{h}(-\partial + A^T)f, g \right\rangle_{\mathbb{C}^n}^{\mathbb{R}} h\, ds \wedge dt,$$

where for each $z \in \mathbb{D}$, $A^T(z) \in \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ denotes the transpose of $A(z)$ with respect to the standard Euclidean inner product. This can only be true for all f and g if $\mathbf{D}^* : \Gamma(F|_{\mathcal{U}}) \to \Gamma(E|_{\mathcal{U}})$ is given by the formula

$$\mathbf{D}^* = \frac{1}{h}(-\partial + A^T) : C^{m+1}(\mathbb{D}, \mathbb{C}^n) \to C^m(\mathbb{D}, \mathbb{C}^n). \tag{5.3}$$

In this way, the uniqueness of $\mathbf{D}^*$ is established, and its existence can now also be deduced by using a partition of unity to write any given pair of sections $\eta \in \Gamma(E)$ and $\xi \in \Gamma(F)$ as finite sums of sections with compact supports in sufficiently small neighborhoods on which suitable coordinates and trivializations are defined, then piecing together finitely-many copies of (5.3) to produce a global formula for $\mathbf{D}^*$.

Finally, note that if we adjust our chosen local trivialization of E by multiplication with the positive function h, then the local formula (5.3) for $\mathbf{D}^*$ becomes $-\partial + A^T$, and clearly A^T is of class C^m if A is.

For the rest of this exercise, let us assume for simplicity that $m = \infty$. The **conjugate** $\bar{E} \to \Sigma$ of a complex vector bundle $E \to \Sigma$ is defined as the same *real* vector bundle (since every complex bundle is also a real bundle) but with the complex structure $J : \bar{E} \to \bar{E}$ defined by multiplication with $-i$ instead of i.

(c) Show that the conjugate $\bar{E}$ of a complex vector bundle E is always isomorphic to the complex dual bundle E^*. What choices need to be made in order to define an explicit isomorphism?

Solution:
Here is a useful notational device for dealing with conjugate bundles. Whenever $\eta \in E$, let

us write the same element as $\bar{\eta} \in \bar{E}$ whenever it is regarded as an element of the conjugate bundle. In this way, the map

$$E \to \bar{E} : \eta \mapsto \bar{\eta}$$

can be understood as the identity map since E and $\bar{E}$ are the same set, and it is also a *real* bundle isomorphism, but not a complex bundle isomorphism since the complex structures on E and $\bar{E}$ are different. In fact, $\eta \mapsto \bar{\eta}$ is a complex-antilinear map $E_z \to \bar{E}_z$ for each $z \in \Sigma$, so it satisfies the easy-to-remember formula

$$\overline{c\eta} = \bar{c}\bar{\eta} \qquad \text{for all } c \in \mathbb{C},\ \eta \in E,$$

where $\bar{c}$ just means the usual complex conjugate of a complex number $c \in \mathbb{C}$. With this notation in place, we can choose a Hermitian bundle metric on E, define a real-linear bundle isomorphism by

$$\bar{E} \to E^* : \bar{\eta} \mapsto \langle \eta, \cdot \rangle_E,$$

and observe that it is also complex linear since for any $c \in \mathbb{C}$, it sends $c\bar{\eta} = \overline{\bar{c}\eta}$ to $\langle \bar{c}\eta, \cdot \rangle_E = c\langle \eta, \cdot \rangle$.

(d) Show that the complex line bundle $\Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma$ is trivial.

Solution:
By part (c), $\Lambda^{1,0}T^*\Sigma$ is isomorphic to $\overline{T\Sigma}$, thus it suffices to show that $\overline{T\Sigma} \otimes \Lambda^{0,1}T^*\Sigma$ is trivial. The pairing

$$\overline{T\Sigma} \otimes \Lambda^{0,1}T^*\Sigma \to \mathbb{C} : \bar{X} \otimes \lambda \mapsto \lambda(X)$$

defines a complex-linear bundle map to the trivial complex line bundle; it is complex linear due to the fact that both λ and $\bar{X} \mapsto X$ are complex antilinear. This bundle map is manifestly surjective on every fiber, and since $\overline{T\Sigma} \otimes \Lambda^{0,1}T^*\Sigma$ is a line bundle, it is therefore also injective on every fiber.

(e) Find an isomorphism between the complex vector bundles $\bar{E}$ and $\overline{\mathrm{Hom}_{\mathbb{C}}}(T\Sigma, \bar{F})$ that identifies the map $-\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ with a linear Cauchy-Riemann type operator $\Gamma(\bar{F}) \to \Omega^{0,1}(\Sigma, \bar{F})$.

Solution:
To start with, we have a natural isomorphism $F \cong \Lambda^{0,1}T^*\Sigma \otimes E$. Here are some easy observations:

- Analogously to part (c), any choice of Hermitian bundle metric on a bundle E determines an isomorphism

$$E \to \Lambda^{0,1}E^* : \eta \mapsto \langle \cdot, \eta \rangle_E,$$

  where $\Lambda^{0,1}E^*$ is the bundle whose fiber over $z \in \Sigma$ is the space of complex-antilinear maps $E_z \to \mathbb{C}$. (Similarly, $\Lambda^{1,0}E^*$ is just fancy notation for the dual bundle E^*.) A special case of this is the observation from our solution to part (a) that the bundle metric $\langle\ ,\ \rangle_{T\Sigma}$ determines a bundle isomorphism

$$T\Sigma \to \Lambda^{0,1}T^*\Sigma.$$

- There is a natural isomorphism between the conjugate bundle of $\Lambda^{0,1}T^*\Sigma$ and $\Lambda^{1,0}T^*\Sigma$, and vice versa. Indeed, $\Lambda^{0,1}T^*\Sigma$ is isomorphic to $T\Sigma$ and $\Lambda^{1,0}T^*\Sigma$ is the dual space or $T\Sigma$, which by part (c) is isomorphic to $\overline{T\Sigma}$. Both of these isomorphisms depend on the choice of bundle metric $\langle\ ,\ \rangle_{T\Sigma}$, but it turns out that the resulting isomorphism $\overline{\Lambda^{0,1}T^*\Sigma} \to \Lambda^{1,0}T^*\Sigma$ does not depend on this choice. To see this, recall that the isomorphism $T\Sigma \to \Lambda^{0,1}T^*\Sigma$ sends X to $\langle \cdot, X \rangle_{T\Sigma}$, while the isomorphism

$\overline{T\Sigma} \to \Lambda^{1,0}T^*\Sigma$ sends $\bar{X}$ to $\langle X, \cdot\rangle_{T\Sigma}$, so if we write $\lambda := \langle \cdot, X\rangle_{T\Sigma} \in \Lambda^{0,1}T^*\Sigma$, the isomorphism $\overline{\Lambda^{0,1}T^*\Sigma} \to \Lambda^{1,0}T^*\Sigma$ sends $\bar{\lambda}$ to the dual vector $\mu \in \Lambda^{1,0}T^*\Sigma$ given by

$$\mu(Y) = \langle X, Y\rangle_{T\Sigma} = \overline{\langle Y, X\rangle_{T\Sigma}} = \overline{\lambda(Y)},$$

giving rise to the choice-independent formula

$$\Phi : \overline{\Lambda^{0,1}T^*\Sigma} \to \Lambda^{1,0}T^*\Sigma, \quad \Phi(\bar{\lambda})Y := \overline{\lambda(Y)} \quad \text{for } \lambda \in \Lambda^{0,1}T_z^*\Sigma,\ Y \in T_z\Sigma,\ z \in \Sigma.$$

Similarly, an isomorphism from the conjugate of $\Lambda^{1,0}T^*\Sigma$ to $\Lambda^{0,1}T^*\Sigma$ is given by

$$\Psi : \overline{\Lambda^{1,0}T^*\Sigma} \to \Lambda^{0,1}T^*\Sigma, \quad \Psi(\bar{\mu})Y := \overline{\mu(Y)} \quad \text{for } \mu \in \Lambda^{1,0}T_z^*\Sigma,\ Y \in T_z\Sigma,\ z \in \Sigma.$$

- For any two complex bundles E, F over Σ, there is a natural isomorphism

$$\overline{E \otimes F} \to \bar{E} \otimes \bar{F} : \overline{\eta \otimes \xi} \mapsto \bar{\eta} \otimes \bar{\xi}.$$

In light of the natural isomorphism $\operatorname{Hom}_{\mathbb{C}}(T\Sigma, F) \cong \Lambda^{1,0}T^*\Sigma \otimes F$ and the fact from part (d) that $\Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma$ is a trivial line bundle, we have

$$\operatorname{Hom}_{\mathbb{C}}(T\Sigma, F) \cong \Lambda^{1,0}T^*\Sigma \otimes F \cong \Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma \otimes E \cong E,$$

and thus

$$\bar{E} \cong \overline{\Lambda^{1,0}T^*\Sigma \otimes F} \cong \overline{\Lambda^{1,0}T^*\Sigma} \otimes \bar{F} \cong \Lambda^{0,1}T^*\Sigma \otimes \bar{F} \cong \overline{\operatorname{Hom}}_{\mathbb{C}}(\Sigma, \bar{F})$$

as claimed. For the moment, let us use the isomorphism $E \cong \operatorname{Hom}_{\mathbb{C}}(T\Sigma, F)$ to view $-\mathbf{D}^* : \Gamma(F) \to \Gamma(E)$ as an operator

$$-\mathbf{D}^* : \Gamma(F) \to \Gamma(\operatorname{Hom}_{\mathbb{C}}(T\Sigma, F)) =: \Omega^{1,0}(\Sigma, F).$$

We claim that from this perspective, $-\mathbf{D}^*$ satisfies the modified Leibniz rule

$$-\mathbf{D}^*(f\xi) = f(-\mathbf{D}^*)\xi + (\partial f)\xi \qquad \text{for all } \xi \in \Gamma(F),\ f \in C^\infty(\Sigma, \mathbb{R}),$$

where we abbreviate

$$\partial f := df - i\, df \circ j \in \Omega^{1,0}(\Sigma, \mathbb{C})$$

so that $(\partial f)\xi$ can be understood as an F-valued $(1,0)$-form. Operators satisfying this type of Leibniz rule are sometimes called **anti-Cauchy-Riemann type** operators on the bundle F. We will then show in a second step that an anti-Cauchy-Riemann type operator on F is equivalent to a Cauchy-Riemann type operator on its conjugate bundle.

To prove the claim, suppose $\eta \in \Gamma(E)$, $\xi \in \Gamma(F)$ and $f \in C^\infty(\Sigma, \mathbb{R})$. By the defining relation of the formal adjoint and the Leibniz rule for $\mathbf{D}$, we have

$$\begin{aligned}
\int_\Sigma \langle -\mathbf{D}^*(f\xi), \eta\rangle_E^{\mathbb{R}}\, d\,\mathrm{vol} &= -\int_\Sigma \langle f\xi, \mathbf{D}\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol} = -\int_\Sigma \langle \xi, f\mathbf{D}\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol} \\
&= -\int_\Sigma \langle \xi, \mathbf{D}(f\eta) - (\bar{\partial} f)\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol} \\
&= -\int_\Sigma \langle \mathbf{D}^*\xi, f\eta\rangle_E^{\mathbb{R}}\, d\,\mathrm{vol} + \int_\Sigma \langle \xi, (\bar{\partial} f)\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol} \\
&= \int_\Sigma \langle f(-\mathbf{D}^*)\xi, \eta\rangle_E^{\mathbb{R}}\, d\,\mathrm{vol} + \int_\Sigma \langle \xi, (\bar{\partial} f)\eta\rangle_F^{\mathbb{R}}\, d\,\mathrm{vol}\,.
\end{aligned}$$

The Leibniz rule $-\mathbf{D}^*(f\xi) = f(-\mathbf{D}^*)\xi + (\partial f)\xi$ will now follow if and only if the last integral in this expression can be rewritten as $\int_\Sigma \langle (\partial f)\xi, \eta\rangle_E^{\mathbb{R}}\, d\,\mathrm{vol}$, where we use the isomorphism $\Lambda^{1,0}T^*\Sigma \otimes F \cong E$ to identify $(\partial f)\xi \in \Omega^{1,0}(\Sigma, F)$ with a section of E. Observe that since f is real-valued, the complex-valued 1-forms $\bar{\partial} f$ and ∂f are related to each other by complex conjugation. The desired relation is thus immediate from the following formula relating the bundle metrics on E and F under the natural isomorphisms in our picture: for all $z \in \Sigma$, $\lambda \in \Lambda^{1,0}T_z^*\Sigma$, $v \in E_z$ and $w \in F_z$,

$$\langle w, \bar{\lambda} \otimes v \rangle_F = \langle \lambda \otimes w, v \rangle_E,$$

where on the left hand side, complex conjugation gives $\bar{\lambda} \in \Lambda^{0,1}T_z^*\Sigma$ and we use the natural isomorphism $\Lambda^{0,1}T_z^*\Sigma \otimes E_z \cong F_z$ to interpret $\bar{\lambda} \otimes v$ as an element of F_z, while on the right hand side, the isomorphism $\Lambda^{1,0}T_z^*\Sigma \otimes F_z \cong E_z$ interprets $\lambda \otimes w$ as an element of E_z. To verify this formula, note that $\lambda = \langle X, \cdot \rangle_{T\Sigma}$ for a unique $X \in T_z\Sigma$, and we can also write $w = \bar{\mu} \otimes u \in \Lambda^{0,1}T_z^*\Sigma \otimes E_z \cong F_z$ for some $\mu \in \Lambda^{1,0}T_z^*\Sigma$ and $u \in E_z$, where $\mu = \langle Y, \cdot \rangle_{T\Sigma}$ and thus $\bar{\mu} = \langle \cdot, Y \rangle_{T\Sigma}$ for a unique $Y \in T_z\Sigma$. This yields

$$\langle w, \bar{\lambda} \otimes v \rangle_F = \langle \bar{\mu} \otimes u, \bar{\lambda} \otimes v \rangle_{\Lambda^{0,1}T^*\Sigma \otimes E} = \langle \bar{\mu}, \bar{\lambda} \rangle_{\Lambda^{0,1}T^*\Sigma} \cdot \langle u, v \rangle_E = \langle Y, X \rangle_{T\Sigma} \cdot \langle u, v \rangle_E.$$

Turning attention to the right hand side and recalling how the trivialization of $\Lambda^{1,0}T^*\Sigma \otimes \Lambda^{0,1}T^*\Sigma$ was defined, we have

$$\langle \lambda \otimes w, v \rangle_E = \langle \lambda \otimes \bar{\mu} \otimes u, v \rangle_E = \langle \lambda(Y) u, v \rangle_E = \overline{\lambda(Y)} \langle u, v \rangle_E,$$

and by definition $\overline{\lambda(Y)} = \overline{\langle X, Y \rangle_{T\Sigma}} = \langle Y, X \rangle_{T\Sigma}$, thus the two sides match as claimed, completing the proof that $-\mathbf{D}^* : \Gamma(F) \to \Omega^{1,0}(\Sigma, F)$ is an anti-Cauchy-Riemann type operator.

For the final step, we claim that on any complex vector bundle $E \to \Sigma$, there is a canonical bijection relating Cauchy-Riemann type operators on E to anti-Cauchy-Riemann type operators on $\bar{E}$, which will therefore change $-\mathbf{D}^*$ into a Cauchy-Riemann type operator on $\bar{F}$. To see the correspondence, write $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E) \cong \Lambda^{0,1}T^*\Sigma \otimes E$ as usual, and associate to any linear map $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ the linear map $\bar{\mathbf{D}} : \Gamma(\bar{E}) \to \Gamma(\bar{F})$ defined by

$$\bar{\mathbf{D}}\bar{\eta} := \overline{\mathbf{D}\eta}.$$

In light of the natural isomorphism $\bar{F} \cong \overline{\Lambda^{0,1}T^*\Sigma \otimes E} \cong \overline{\Lambda^{0,1}T^*\Sigma} \otimes \bar{E} \cong \Lambda^{1,0}T^*\Sigma \otimes \bar{E} \cong \mathrm{Hom}_{\mathbb{C}}(T\Sigma, \bar{E})$, $\bar{\mathbf{D}}$ can then be interpreted as a map $\Gamma(\bar{E}) \to \Omega^{1,0}(\Sigma, \bar{E})$, and from this perspective, it is straightforward to show that $\bar{\mathbf{D}}$ is an anti-Cauchy-Riemann type operator if and only if $\mathbf{D}$ is of Cauchy-Riemann type.

(f) Compute a formula relating $c_1(E)$ and $c_1(\bar{F})$. (You will need this in Exercise 5.4 below.)

Solution:

We need a few basic facts about the first Chern number as preparation. We shall only consider bundles over a closed Riemann surface, since that is what is required for our purposes, but most of the following is also true more generally.

First, we claim that for any two complex line bundles $E, F \to \Sigma$,

$$c_1(E \otimes F) = c_1(E) + c_1(F).$$

Proof: just choose generic smooth sections $\eta \in \Gamma(E)$ and $\xi \in \Gamma(F)$, preferably with disjoint zero sets, and count (with signs!) the zeroes of $\eta \otimes \xi \in \Gamma(E \otimes F)$.

Caution: Unlike the very similar formula for the first Chern number of a direct sum, the formula for $c_1(E \otimes F)$ is not true in general for bundles of higher rank!

Second, we claim that the first Chern numbers of any complex vector bundle E and its dual bundle E^* are related by

$$c_1(E^*) = -c_1(E).$$

If E is a line bundle, we can conclude this from the observation that the pairing

$$E^* \otimes E \to \mathbb{C} : \lambda \otimes \eta \mapsto \lambda(\eta)$$

defines an isomorphism from $E^* \otimes E$ to the trivial complex line bundle; the latter has first Chern number 0, so this implies

$$0 = c_1(E^* \otimes E) = c_1(E^*) + c_1(E)$$

due to the first claim above. Now, if E has higher rank, then it can be split into a direct sum of line bundles $E_1 \oplus \ldots \oplus E_m$.[7] There is then a natural isomorphism $E^* \cong E_1^* \oplus \ldots \oplus E_m^*$, so the direct sum property of c_1 together with the rank 1 case gives

$$c_1(E^*) = c_1(E_1^*) + \ldots + c_1(E_m^*) = -c_1(E_1) - \ldots - c_1(E_m) = -c_1(E).$$

Finally, we note that $\bar{E}$ is always isomorphic to E^*, thus

$$c_1(\bar{E}) = -c_1(E).$$

Now let's compute $c_1(F)$. We can write $E = E_1 \oplus \ldots \oplus E_m$ for suitable line bundles $E_1, \ldots, E_m \to \Sigma$, and using the isomorphism

$$F \cong \Lambda^{0,1}T^*\Sigma \otimes E \cong T\Sigma \otimes E \cong T\Sigma \otimes (E_1 \oplus \ldots \oplus E_m) \cong \bigoplus_{j=1}^m T\Sigma \otimes E_j,$$

we use the direct sum property and the tensor product formula for line bundles to deduce

$$\begin{aligned} c_1(F) &= \sum_{j=1}^m c_1(T\Sigma \otimes E_j) = \sum_{j=1}^m [c_1(T\Sigma) + c_1(E_j)] = mc_1(T\Sigma) + \sum_{j=1}^m c_1(E_j) \\ &= mc_1(T\Sigma) + c_1(E) = m\chi(\Sigma) + c_1(E), \end{aligned}$$

where in the last step we have appealed to the Poincaré-Hopf theorem to introduce the Euler characteristic. We conclude

$$c_1(\bar{F}) = -\operatorname{rank}_{\mathbb{C}}(E) \cdot \chi(\Sigma) - c_1(E).$$

Exercise 5.3. In lecture we proved that the bounded linear map

$$\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}\big(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)\big)$$

defined via a linear Cauchy-Riemann type operator $\mathbf{D}$ of class C^m for $m \geqslant k-1$ and $p \in (1, \infty)$ on a complex vector bundle $E \to \Sigma$ over a closed Riemann surface Σ has finite-dimensional kernel and cokernel. Prove that when $m = \infty$, this is also true for the linear map $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$, and $\operatorname{ind}(\mathbf{D}) := \dim\ker(\mathbf{D}) - \dim\operatorname{coker}(\mathbf{D}) \in \mathbb{Z}$ is the same as in the Sobolev space setting.
Hint: One cannot repeat the same arguments we used in lecture to prove this, because $\Gamma(E)$ and $\Omega^{0,1}(\Sigma, E)$ are not Banach spaces. Try instead combining the result from lecture with regularity results.

Solution:
Abbreviate $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$. The linear regularity theorem proved in lecture implies that the kernel of the map $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$ is the same for every $k \in \mathbb{N}$ and $p \in (1, \infty)$, and is identical to the kernel of its restriction $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ to the space of smooth sections. The same of course applies to the formal adjoint $\mathbf{D}^*$, and we claim

$$\Gamma(F) = \operatorname{im}(\mathbf{D}) \oplus \ker(\mathbf{D}^*),$$

[7] This is not true for arbitrary complex vector bundles, but is true whenever the base is a Riemann surface. The reason is that if the base is only 2-dimensional, but the bundle has rank more than 1, then the total space has real dimension strictly greater than 4, and generic perturbations of any section will therefore avoid intersecting the zero-section, meaning that one can always find a nowhere-zero section. Such a section generates a rank 1 subbundle $E_1 \subset E$, so after choosing a bundle metric, one obtains a splitting $E = E_1 \oplus E_1^\perp$. If $E_1^\perp$ still has rank more than 1, the same argument can now be repeated for $E_1^\perp$ to split it further.

where $\mathrm{im}(\mathbf{D})$ is understood as the set of all $\mathbf{D}\eta$ for *smooth* sections $\eta \in \Gamma(E)$. Indeed, given $\xi \in \Gamma(F)$, ξ also belongs to each of the Sobolev spaces $W^{k,p}(F)$, for which we established a similar splitting in lecture, thus we can write $\xi = \mathbf{D}\eta + \alpha$ for some $\eta \in W^{k+1,p}(E)$ and a uniquely determined $\alpha \in \ker(\mathbf{D}^*)$. By regularity, α is smooth, so the right hand side of the equation $\mathbf{D}\eta = \xi - \alpha$ is also smooth, and the same regularity theorem then implies that η is smooth, which proves the claim. We therefore obtain a natural isomorphism $\mathrm{coker}(\mathbf{D}) = \Gamma(F)/\mathrm{im}(\mathbf{D}) \cong \ker(\mathbf{D}^*)$, and since the latter is the same space in the smooth case as in the Sobolev setting, $\dim \mathrm{coker}(\mathbf{D})$ is therefore also the same.

Exercise 5.4. The **Riemann-Roch formula** states that for any linear Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ on a complex vector bundle $E \to \Sigma$ of rank $n \in \mathbb{N}$ over a closed connected Riemann surface Σ of genus $g \geqslant 0$,[8]

$$\mathrm{ind}(\mathbf{D}) = (2-2g)n + 2c_1(E) \in \mathbb{Z},$$

where $c_1(E)$ in this context is an abbreviation for the first Chern number $\langle c_1(E), [\Sigma]\rangle \in \mathbb{Z}$. We will prove this in full generality within the next few weeks, but the goal of this exercise is to prove the case where Σ has genus zero.

(a) Show that $\mathrm{ind}(\mathbf{D}_0) = \mathrm{ind}(\mathbf{D}_1)$ for any two linear Cauchy-Riemann type operators $\mathbf{D}_0, \mathbf{D}_1 : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ on the same bundle.
Hint: The following result from functional analysis may serve as a black box. If $T : X \to Y$ is a Fredholm operator between two Banach spaces and $K : X \to Y$ is a compact operator, then $T + K$ is also Fredholm and has the same index as T.

Solution:
We can write $\mathbf{D}_1\eta = \mathbf{D}_0\eta + A\eta$ where $A : E \to F$ is a bundle map. (For simplicity let's assume $\mathbf{D}_0$ and $\mathbf{D}_1$ are both of class C^∞, so that A is a *smooth* bundle map, but this isn't strictly necessary.) For any choice of $k \in \mathbb{N}$ and $p \in (1,\infty)$, the difference $\mathbf{D}_1 - \mathbf{D}_0 : W^{k,p}(E) \to W^{k-1,p}(F)$ thus takes the form

$$W^{k,p}(E) \to W^{k-1,p} : \eta \mapsto A\eta,$$

which in local trivializations looks like the product of a vector-valued function $\eta : \mathbb{D} \to \mathbb{C}^n$ of class $W^{k,p}$ with a smooth matrix-valued function $A : \mathbb{D} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. This map can also be presented as the composition of the inclusion $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ with the bounded linear map

$$W^{k-1,p}(E) \to W^{k-1,p} : \eta \mapsto A\eta,$$

and this composition is a compact operator since, by the Rellich-Kondrashov theorem, the inclusion $W^{k,p}(E) \hookrightarrow W^{k-1,p}(E)$ is compact. By the general functional-analytic fact stated in the hint, it follows that $\mathrm{ind}(\mathbf{D}_0) = \mathrm{ind}(\mathbf{D}_1)$.

(b) Use the similarity principle to prove that if $g = 0$ and $n = 1$, then $\mathbf{D}$ is always injective or surjective, depending on the value of $c_1(E)$.
Hint: Use the formal adjoint $\mathbf{D}^*$ to characterize the surjectivity of $\mathbf{D}$.

Solution:
If $c_1(E) \leqslant -1$, then the similarity principle implies immediately that $\mathbf{D}$ is injective, because any nontrivial solution $\eta \in \Gamma(E)$ to $\mathbf{D}\eta = 0$ would need to have isolated zeroes

[8] Since $\mathbf{D}$ is in general real linear but not complex linear, the dimensions in our definition of $\mathrm{ind}(\mathbf{D})$ are real dimensions. In complex algebraic geometry, where Cauchy-Riemann type operators are always complex linear, the Fredholm index is normally defined in terms of complex dimensions, thus the Riemann-Roch formula often appears instead as $\mathrm{ind}(\mathbf{D}) = n(1-g) + c_1(E)$.

that all count positively, implying $c_1(E) \geqslant 0$. If on the other hand $c_1(E) \geqslant -1$, then the calculation in Exercise 5.2(f) implies

$$c_1(\bar{F}) = -\chi(S^2) - c_1(E) = -2 - c_1(E) \leqslant -1,$$

and since $-\mathbf{D}^*$ is equivalent to a Cauchy-Riemann type operator on $\bar{F}$, the same application of the similarity principle implies that $\mathbf{D}^*$ is injective, and $\mathbf{D}$ is therefore surjective. (Observe that in the case of a line bundle over S^2 with $c_1(E) = -1$, we are now done with the calculation: $\mathbf{D}$ must in this case be an isomorphism, hence $\operatorname{ind}(\mathbf{D}) = 0$.)

(c) Deduce the Riemann-Roch formula for $g = 0$ and $n = 1$ by constructing for each integer $k \geqslant -1$ a holomorphic vector bundle $E_k \to S^2 = \mathbb{C} \cup \{\infty\}$ with $c_1(E_k) = k$ and computing explicitly the dimension of its space of holomorphic sections.

Solution:
The following bit of topological background knowledge will be important: two complex line bundles $E_0, E_1 \to \Sigma$ over a closed connected Riemann surface Σ are isomorphic if and only if $c_1(E_0) = c_1(E_1)$. If you haven't seen this fact before, here is a quick proof for the case of bundles over $S^2 = \mathbb{C} \cup \{\infty\}$. First, we claim that if $E \to S^2$ has $c_1(E) = 0$, then E is trivial. Indeed, choose a section $\eta \in \Gamma(E)$ that is nonzero at (and therefore also in a neighborhood of) the point at ∞. Over $\mathbb{C} = S^2 \backslash \{\infty\}$, one can choose a connection and use parallel transport along rays from the origin to construct a trivialization of E, so $E|_{\mathbb{C}}$ is trivial and a choice of trivialization identifies the restricted section $\eta|_{\mathbb{C}}$ with a function $\eta : \mathbb{C} \to \mathbb{C}$ that has no zeroes outside of some compact disk $\mathbb{D}_R \subset \mathbb{C}$ for $R > 0$ large. If η is chosen so that its zero-set is finite, then its algebraic count of zeroes is then given by the winding number of η around the circle $\partial\mathbb{D}_R$, which must therefore be zero if $c_1(E) = 0$. But this winding condition implies that η can be modified in some compact subset of the interior of $\mathbb{D}_R$ to a function that has no zeroes at all, thus producing a global nowhere-zero section $\eta \in \Gamma(E)$. On a line bundle, the existence of such a section implies that the bundle is trivial, proving the claim. Now for any two line bundles $E_0, E_1 \to S^2$ with $c_1(E_0) = c_1(E_1)$, we have $c_1(E_0 \otimes E_1^*) = c_1(E_0) - c_1(E_1) = 0$, implying that $E_0 \otimes E_1^*$ is trivial. Since $E_1^* \otimes E_1$ is also trivial (cf. the solution to Exercise 5.2(f)), it follows that

$$E_0 \cong E_0 \otimes (E_1^* \otimes E_1) \cong (E_0 \otimes E_1^*) \otimes E_1 \cong E_1.$$

Now recall that by part (a), the index of a Cauchy-Riemann type operator on any complex vector bundle depends only on the bundle, not on the choice of operator. It follows that the Riemann-Roch formula for line bundles over S^2 will be established if we can show that for every $k \in \mathbb{Z}$, there exists a specific line bundle $E_k \to S^2$ with $c_1(E_k) = k$ with a specific Cauchy-Riemann type operator for which the formula is correct.

Here is a construction of such a bundle for arbitrary $k \in \mathbb{Z}$. Let $E^{(1)}$ and $E^{(2)}$ denote two copies of the trivial holomorphic line bundle $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and define

$$E_k := (E^{(1)} \sqcup E^{(2)}) / (z, v) \sim \Phi_k(z, v),$$

where $\Phi_k : E^{(1)}|_{\mathbb{C}\backslash\{0\}} \to E^{(2)}|_{\mathbb{C}\backslash\{0\}}$ is a bundle isomorphism covering the biholomorphic map $z \mapsto 1/z$ and defined by $\Phi_k(z, v) = (1/z, g_k(z)v)$, with

$$g_k(z)v := \frac{1}{z^k} v.$$

The function $g_k(z)$ is a holomorphic transition map, so E_k has a natural holomorphic structure and thus carries a complex-linear Cauchy-Riemann operator $\mathbf{D}_k$ whose kernel is the space of holomorphic sections. Regarding a function $f : \mathbb{C} \to \mathbb{C}$ as a section of $E^{(1)}$, we have

$$\Phi_k(1/z, f(1/z)) = (z, z^k f(1/z)),$$

which means that f extends to a smooth section of E_k if and only if the function $g(z) = z^k f(1/z)$ extends smoothly to $z = 0$. It follows that $c_1(E_k) = k$, as one can choose $f(z) = 1$ for z in the unit disk and then modify $g(z) = z^k$ to a smooth function that algebraically has k zeroes at 0 (note that an actual modification is necessary only if $k < 0$). Similarly, the holomorphic sections of E_k can be identified with the entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $z^k f(1/z)$ extends holomorphically to $z = 0$; if $k < 0$ this implies $f \equiv 0$, and if $k \geqslant 0$ it means $f(z)$ is a polynomial of degree at most k, hence $\dim \ker \mathbf{D}_k = 2+2k$. Now if $k \geqslant -1$, the solution to part (b) tells us that $\mathbf{D}_k$ must be surjective, and its index is therefore

$$\operatorname{ind}(\mathbf{D}_k) = \dim \ker(\mathbf{D}_k) = 2 + 2k = \chi(S^2) + 2c_1(E_k).$$

This establishes the Riemann-Roch formula for all line bundles over S^2 with $c_1 \geqslant -1$. If on the other hand we are given a line bundle $E \to S^2$ with $c_1(E) \leqslant -1$, then writing $F := \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)$, Exercise 5.2(f) gives $c_1(\bar{F}) = -\chi(S^2) - c_1(E) \geqslant -2 + 1 = -1$, and since $-\mathbf{D}^*$ is a Cauchy-Riemann type operator on $\bar{F}$, we conclude that $\mathbf{D}^*$ satisfies the Riemann-Roch formula, and thus

$$\begin{aligned}\operatorname{ind}(\mathbf{D}) &= -\operatorname{ind}(\mathbf{D}^*) = -\left[\chi(S^2) + 2c_1(\bar{F})\right] = -\chi(S^2) - 2\left[-\chi(S^2) - c_1(E)\right] \\ &= \chi(S^2) + 2c_1(E).\end{aligned}$$

(d) Deduce the case $g = 0$ and $n > 1$ by splitting an arbitrary higher-rank bundle into a sum of line bundles.

Solution:
Assume $E = E_1 \oplus \ldots \oplus E_n$ for line bundles $E_1, \ldots, E_n \to S^2$, and $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(S^2, E)$ is a Cauchy-Riemann type operator. Since $\operatorname{ind}(\mathbf{D})$ depends only on the bundle and not the choice of operator, we are free to replace $\mathbf{D}$ with a different Cauchy-Riemann type operator for convenience, and we can do so by choosing on each of the line bundles $E_j \to S^2$ a Cauchy-Riemann type operator $\mathbf{D}_j : \Gamma(E_j) \to \Omega^{0,1}(S^2, E_j)$. Writing $\Gamma(E) = \Gamma(E_1) \oplus \ldots \oplus \Gamma(E_n)$ and $\Omega^{0,1}(S^2, E) = \Omega^{0,1}(S^2, E_1) \oplus \ldots \oplus \Omega^{0,1}(S^2, E_n)$, it is easy to check that the operator $\Gamma(E) \to \Omega^{0,1}(S^2, E)$ defined in block form by

$$\mathbf{D} := \begin{pmatrix} \mathbf{D}_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathbf{D}_n \end{pmatrix} : \Gamma(E_1) \oplus \ldots \oplus \Gamma(E_n) \to \Omega^{0,1}(S^2, E_1) \oplus \ldots \oplus \Omega^{0,1}(S^2, E_n)$$

is also a Cauchy-Riemann type operator on E, with index

$$\operatorname{ind}(\mathbf{D}) = \operatorname{ind}(\mathbf{D}_1) + \ldots + \operatorname{ind}(\mathbf{D}_n).$$

Applying the rank 1 case of the Riemann-Roch formula, we conclude

$$\operatorname{ind}(\mathbf{D}) = \sum_{j=1}^{n} \left[\chi(S^2) + 2c_1(E_j)\right] = n\chi(S^2) + 2c_1(E).$$

6. Week 6

This week was cancelled because I got Covid-19. I cannot recommend it.

7. Week 7

This week was also cancelled because I still had Covid-19, and I still cannot recommend it.

8. Week 8

Lecture 11 (06.12.2022): Local existence of J-holomorphic curves.

- Nonlinear local existence theorem: Assume $J \in \mathcal{J}(\mathbb{C}^n)$ is a smooth almost complex structure such that (without loss of generality) $J(0) = i$, and $a_1, \dots, a_m \in \mathbb{C}^n$ are constants for some $m \geqslant 0$. Then:
  1. For any $\epsilon > 0$ sufficiently small, there exists a J-holomorphic map $u : (\mathbb{D}_\epsilon, i) \to (\mathbb{C}^n, J)$ satisfying $u(0) = 0$ and $\frac{\partial^k u}{\partial z^k}(0) = a_k$ for each $k = 1, \dots, m$.
  2. Given a map u as in the statement above and a C^∞_{loc}-convergent sequence of almost complex structures $J_j \to J$ on $\mathbb{C}^n$, there also exists for sufficiently large j a sequence $u_j : (\mathbb{D}_\epsilon, i) \to (\mathbb{C}^n, J_j)$ of J_j-holomorphic maps satisfying the same conditions at 0 as u and converging in C^∞_{loc} to u.
- Remarks:
  1. In light of the Cauchy-Riemann equation, the partial derivatives $\frac{\partial^k u}{\partial z^k}(0)$ for $k = 0, \dots, m$ (excluding derivatives with respect to $\bar{z}$) determine $\partial^\alpha u(0)$ for all multi-indices with $|\alpha| \leqslant m$. (See Exercise 8.1.)
  2. There is no uniqueness. For holomorphic functions, specifying $\frac{\partial^k u}{\partial z^k}(0)$ for $k = 0, \dots, m$ still allows an infinite-dimensional space of solutions; specifying them for all $k \geqslant 0$ would produce uniqueness but kill existence (the Taylor series might never converge).
- Preparation 1: $\bar{\partial} : W^{k,p}(\mathbb{D}) \to W^{k-1,p}(\mathbb{D})$ has a bounded right inverse for every $k \in \mathbb{N}$ and $1 < p < \infty$. Proof follows from $k = 1$ case and linear regularity using a bounded extension operator $W^{k-1,p}(\mathbb{D}) \to W^{k-1,p}_0(\mathbb{D}_{1+\epsilon})$.
- Preparation 2: Differential calculus in Banach spaces
  - Definition of the derivative for maps $X \supset \mathcal{U} \xrightarrow{f} Y$ where X, Y are Banach spaces and $\mathcal{U} \subset X$ is open
  - Maps $f : \mathcal{U} \to Y$ of class C^k
  - Inverse function theorem (the case $Df(x_0) : X \to Y$ invertible)
  - Implicit function theorem (the case $Df(x_0) : X \to Y$ surjective with a bounded right inverse); why having a bounded right inverse is important
- Continuous multilinear maps are smooth
- Useful lemma: Assume $\mathcal{U} \subset \mathbb{R}^n$ is open and bounded, and for any finite-dimensional vector space V, the symbol $X(\mathcal{U}, V)$ denotes a Banach space whose elements are continuous functions $\bar{\mathcal{U}} \to V$, possibly satisfying additional conditions, such that the following properties hold:
  1. (C^0-inclusion) The inclusion $X(\mathcal{U}, V) \hookrightarrow C^0(\mathcal{U}, V)$ is a continuous map
  2. (Banach algebra) Pointwise multiplication defines a continuous bilinear map
  $$X(\mathcal{U}, \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^N)) \times X(\mathcal{U}, \mathbb{R}^m) \to X(\mathcal{U}, \mathbb{R}^N) : (A, u) \mapsto Au$$
  3. (C^k-continuity) For any convex open set $\Omega \subset \mathbb{R}^m$ and every function $f : \Omega \to \mathbb{R}^N$ of class C^k for some integer $k \geqslant 0$, there is a continuous map
  $$\Phi_f : X(\mathcal{U}, \Omega) \to X(\mathcal{U}, \mathbb{R}^N) : u \mapsto f \circ u, \tag{8.1}$$
  where we denote $X(\mathcal{U}, \Omega) := \left\{ u \in X(\mathcal{U}, \mathbb{R}^m) \;\middle|\; u(\bar{\mathcal{U}}) \subset \Omega \right\}$. (By the C^0-inclusion property, this is an open subset of $X(\mathcal{U}, \mathbb{R}^m)$.)

Then if f is additionally of class C^{k+r} for some $r \in \mathbb{N}$, it follows that the map Φ_f in (8.1) is of class C^r, and its first derivative $D\Phi_f(u) : X(\mathcal{U}, \mathbb{R}^m) \to X(\mathcal{U}, \mathbb{R}^N)$ at any point $u \in X(\mathcal{U}, \Omega)$ is given by

$$D\Phi_f(u)\eta = (Df \circ u)\eta \in X(\mathcal{U}, \mathbb{R}^N) \qquad \text{for } \eta \in X(\mathcal{U}, \mathbb{R}^m),$$

an expression that makes sense due to the C^k-continuity ($Df : \Omega \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^N)$ is of class C^k) and Banach algebra properties.
Quick proof sketch: By induction, it suffices to consider the case $r = 1$ and prove the stated formula for $D\Phi_f(u)$. We use the fundamental theorem of calculus to write down a formula for the remainder $\|\eta\| \cdot R(\eta) = \Phi_f(u + \eta) - \Phi_f(u) - (Df \circ u)\eta$ as an integral, and then deduce from the C^k-continuity property that $\lim_{\eta \to 0} R(\eta) = 0$.
Example: The lemma can be applied with $X := W^{k,p}$ if $kp > n$.

- Remark: With similar methods and a little extra effort, one can also often show that the map $C^{k+r}(\bar{\Omega}, \mathbb{R}^N) \times X(\mathcal{U}, \Omega) \to X(\mathcal{U}, \mathbb{R}^N) : (f, u) \mapsto f \circ u$ is of class C^r.
- Proof of the nonlinear local existence theorem: After rescaling to zoom in around $0 \in \mathbb{D}$, it suffices to look for solutions defined on the unit disk under the assumption that J is arbitrarily C^∞-close to i. Then pick $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$ and consider the "local moduli space"

$$\mathcal{M} := \left\{ (J, u) \in C^{k+m+1}(\mathbb{D}^{2n}, \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)) \times W^{k+m,p}(\mathbb{D}, \mathbb{C}^n) \;\middle|\; F(J, u) := \partial_s u + (J \circ u)\partial_t u = 0 \right\}.$$

The map $F : C^{k+m+1} \times W^{k+m,p} \to W^{k+m-1,p}$ is of class C^1 and its derivative at a point $(i, u) \in \mathcal{M}$ contains the surjective term $\bar{\partial} : W^{k+m,p} \to W^{k+m-1}$, so by the implicit function theorem, $\mathcal{M}$ is a C^1-Banach submanifold near the set of points (i, u) with $u : \mathbb{D} \to \mathbb{C}^n$ holomorphic. One then uses the surjectivity of $\bar{\partial}$ again and the existence of i-holomorphic maps $\mathbb{D} \to \mathbb{C}^n$ with arbitrary derivatives at 0 with respect to z up to order m to show that the map

$$\mathcal{M} \to C^{k+m+1} \times \mathbb{C}^{n(m+1)} : (J, u) \mapsto \left(J, u(0), \frac{\partial u}{\partial z}(0), \ldots, \frac{\partial^m u}{\partial z^m}(0) \right)$$

is a submersion near such points.

Lecture 12 (07.12.2022): Moduli spaces and bubbling analysis.

- Definition of energy for a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ in a symplectic manifold (M, ω) with $J \in \mathcal{J}_\tau(M, \omega)$:

$$E_\omega(u) := \int_\Sigma u^*\omega.$$

Tameness implies $E_\omega(u) \geqslant 0$, with equality if and only if u is (locally) constant.
- Definitions of moduli spaces:

$$\mathcal{M}_{g,m}(J) := \{(\Sigma, j, \zeta, u)\} / \sim$$

where (Σ, j) is a closed connected Riemann surface of genus g, $\zeta = (\zeta_1, \ldots, \zeta_m)$ are distinct points in Σ, $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic map, and $(\Sigma, j, \zeta, u) \sim (\Sigma', j', \zeta', u')$ if and only if there is a biholomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$ with $\varphi(\zeta) = \zeta'$ (preserving the order) and $u' \circ \varphi = u$. Also for $A \in H_2(M)$,

$$\mathcal{M}_{g,m}(J, A) := \left\{ [(\Sigma, j, \zeta, u)] \in \mathcal{M}(J) \;\middle|\; [u] := u_*[\Sigma] = A \right\},$$

and for a subset $X \subset \mathcal{J}(M)$,

$$\mathcal{M}_{g,m}(X) := \left\{ (J, U) \;\middle|\; U \in \mathcal{M}_{g,m}(J) \right\},$$

with $\mathcal{M}_{g,m}(X, A)$ defined similarly.

- Topologies of the moduli spaces: say (J_k, U_k) converges to (J, U) in $\mathcal{M}_{g,m}(\mathcal{J}(M))$ if $J_k \to J$ in C^∞ and there exists a fixed surface Σ with fixed marked points $\zeta = (\zeta_1, \ldots, \zeta_m)$ in Σ such that for large k, $U_k = [(\Sigma, j_k, \zeta, u_k)]$ and $U = [(\Sigma, j, \zeta, u)]$ with $j_k \to j$ and $u_k \to u$ in the C^∞-topology on Σ.
- Statement of Gromov's compactness theorem (vague version): $\mathcal{M}_{g,m}(\mathcal{J}(M))$ embeds naturally as an open subset of a metrizable space $\overline{\mathcal{M}}_{g,m}(\mathcal{J}(M))$ such that for any C^∞-convergent sequence of symplectic forms $\omega_k \to \omega$ on M and C^∞-convergent sequence of tame almost complex structures $\mathcal{J}(M, \omega_k) \ni J_k \to J \in \mathcal{J}(M, \omega)$, any sequence $U_k \in \overline{\mathcal{M}}_{g,m}(J_k)$ with $E_{\omega_k}(U_k)$ uniformly bounded has a subsequence convergent to an element of $\overline{\mathcal{M}}_{g,m}(J)$.
- Corollary (since $E_\omega(u) = \langle [\omega], [u] \rangle$ depends only on $[u] \in H_2(M)$): the spaces $\overline{\mathcal{M}}_{g,m}(J, A)$ are compact.
- Regularity lemma: Choose Riemannian metrics on Σ and M, suppose (Σ, j) is a Riemann surface and $\Sigma_1 \subset \Sigma_2 \subset \ldots \subset \bigcup_{k \in \mathbb{N}} \Sigma_k = \Sigma$ are nested open subsets with complex structures $j_k \in \mathcal{J}(\Sigma_k)$ that converge in $C^\infty_{\mathrm{loc}}(\Sigma)$ to j, $J_1, J_2, \ldots, J \in \mathcal{J}(M)$ are almost complex structures on M with C^∞_{loc}-convergence $J_k \to J$, and $u_k : (\Sigma_k, j_k) \to (M, J_k)$ is a sequence of J_k-holomorphic curves. If the sequence u_k is uniformly C^1-bounded on compact subsets of Σ, then it has a subsequence C^∞_{loc}-convergent to a J-holomorphic curve $u : (\Sigma, j) \to (M, J)$.
  Proof: For any $z_0 \in \Sigma$, we can use nonlinear local existence to compose u_k with a convergent sequence of j_k-holomorphic charts and thus identify a neighborhood of z_0 in (Σ_k, j_k) with $(\mathbb{D}, i)$. Then apply our local nonlinear regularity theorem to turn uniform $W^{1,p}$-bounds for $p > 2$ (which follow from C^1-bounds) into $W^{k,p}_{\mathrm{loc}}$-bounds; then apply the Sobolev embedding theorem and Arzelà-Ascoli.
- Rescaling/bubbling "lemma": Under the same assumptions except that u_k is uniformly C^0- but not C^1-bounded on compact subsets, it follows that a nonconstant J-holomorphic plane $v : (\mathbb{C}, i) \to (M, J)$ "bubbles off".
  Proof: Assuming $|du_k(z_k)| =: R_k \to \infty$, choose $\epsilon_k \to 0$ such that $\epsilon_k R_k \to \infty$, and use C^1-bounds to show that the reparametrized disks $v_k : \mathbb{D}_{\epsilon_k R_k} \to M : z \mapsto u_k(z_k + z/R_k)$ have a subsequence convergent in $C^\infty_{\mathrm{loc}}(\mathbb{C})$ to a map $v : \mathbb{C} \to M$ with $|dv(0)| = 1$. This requires:
- The Hofer lemma: Assume (X, d) is a complete metric space with a continuous function $g : X \to [0, \infty)$, $x_0 \in X$ and $\epsilon_0 > 0$. Then there exist $x \in X$ with $d(x_0, x) \leqslant 2\epsilon_0$ and $\epsilon > 0$ with $\epsilon \leqslant \epsilon_0$ such that $\epsilon g(x) \geqslant \epsilon_0 g(x_0)$ and $g \leqslant 2g(x)$ on the closed ball of radius ϵ around x.
- If $J_k \in \mathcal{J}(M, \omega_k)$ and $J \in \mathcal{J}(M, \omega)$ with $\omega_k \to \omega$ and the energies $E_{\omega_k}(u_k)$ are bounded, it follows that $E_\omega(v) < \infty$.
- Statement of Gromov's removable singularity theorem: If $J \in \mathcal{J}_\tau(M, \omega)$ and $u : (\mathbb{D} \backslash \{0\}, i) \to (M, J)$ is J-holomorphic with bounded image and $E_\omega(u) < \infty$, then u has a smooth (and therefore J-holomorphic) extension to $\mathbb{D}$.
- Improved rescaling/bubbling lemma: Under the same assumptions, if the almost complex structures J_k are tame and the energies $E_{\omega_k}(u_k)$ are bounded, it follows that there exists a nonconstant J-holomorphic sphere $v : (S^2 = \mathbb{C} \cup \{\infty\}, i) \to (M, J)$.

Lecture 13 (07.12.2022): Bubble trees.

- Proof of the continuous extension in the removable singularity theorem: reparametrize via $[0, \infty) \times S^1 \to \mathbb{D} \backslash \{0\} : (s, t) \mapsto e^{-2\pi(s+it)}$ so that u is a J-holomorphic map $([0, \infty) \times S^1, i) \to (M, J)$ with bounded image and $E_\omega(u) < \infty$. Bubbling analysis implies $|du|$ is bounded, since otherwise a nonconstant plane with zero energy would bubble off. Then for any sequence $s_k \to \infty$, $u_k : [-s_k, \infty) \times S^1 \to M : (s,t) \mapsto u(s+s_k,t)$ is C^1-bounded and must have a subsequence converging in C^∞_{loc} to a J-holomorphic map $u_\infty : (\mathbb{R} \times S^1, i) \to (M,J)$ with zero energy, i.e. constant. One then needs to show that for all choices of sequence $s_k \to \infty$ and subsequence, one gets the same constant; this follows from:

- Monotonicity lemma: Assume (M,ω) is a compact symplectic manifold with $J \in \mathcal{J}(M,\omega)$, and introduce the Riemannian metric $g(X,Y) := \frac{1}{2}(\omega(X,JY) + \omega(Y,JX))$. There exist constants $c, R > 0$ such that for every $r \in (0,R)$ and $p \in M$, every proper J-holomorphic curve $u : (\Sigma, j) \to (B_r(p), J)$ passing through p satisfies $\int_\Sigma u^*\omega \geqslant cr^2$.
(This is really a result from minimal surface theory; we are taking it as a black box.)
- Energy quantization lemma: On any closed symplectic manifold (M,ω) with $J \in \mathcal{J}(M,\omega)$, there exists a constant $\hbar > 0$ such that every nonconstant J-holomorphic sphere $u : (S^2, i) \to (M,J)$ satisfies $E_\omega(u) \geqslant \hbar$.
Proof by contradiction using a compactness argument; bubbling is easy to exclude.
- Main result on bubble trees: Assume $\omega_k \to \omega$ are C^∞-convergent symplectic forms on a closed manifold M with C^∞-convergent tame almost complex structures $\mathcal{J}_\tau(M,\omega_k) \ni J_k \to J \in \mathcal{J}_\tau(M,\omega)$, (Σ, j) is a closed Riemann surface with a C^∞-convergent sequence of complex structures $j_k \to j$, and $u_k : (\Sigma, j_k) \to (M, J_k)$ is a sequence of J_k-holomorphic curves with bounded energy $E_{\omega_k}(u_k)$. Then there exists a finite subset $\Gamma \subset \Sigma$ such that after replacing u_k with a subsequence:
  1. u_k converges in C^∞_{loc} on $\Sigma \backslash \Gamma$ to a J-holomorphic curve $u_\infty : (\Sigma\backslash\Gamma, j) \to (M,J)$ with $E_\omega(u_\infty) < \infty$, which therefore has a smooth extension to Σ.
  2. Each $\zeta \in \Gamma$ corresponds to a finite "tree" of J-holomorphic spheres $v^1_\zeta, \dots, v^{N_\zeta}_\zeta : (S^2, i) \to (M,J)$, called "bubbles", such that for large k,
  $$[u_k] = [u_\infty] + \sum_{\zeta \in \Gamma} \sum_{i=1}^{N_\zeta} [v^i_\zeta] \in H_2(M).$$

  Proof sketch: Energy quantization implies bubbles can form by rescaling near at most finitely many points $\Gamma \subset \Sigma$, so away from this set, there are C^1-bounds and u_k thus has a convergent subsequence. Each $\zeta \in \Gamma$ has a positive *mass* that measures the amount of energy getting concentrated in arbitrarily small neighborhoods of ζ as $k \to \infty$. Given $z_k \to \zeta$ with $|du_k(z_k)| \to \infty$, do the rescaling around z_k so that $v_k : \mathbb{D}_{\epsilon_k R_k} \to M$ has $\int_{\mathbb{D}} v_k^*\omega_k = m_\zeta - \frac{\hbar}{2}$. Now there is a finite set $\Gamma_1 \subset \mathbb{C}$ such that v_k is C^1-bounded away from Γ_1, and each point in Γ_1 likewise has a positive mass, and one can rescale in the same manner around each. Since each nontrivial bubble has energy at least $\hbar$ and the energies of u_k are bounded, this process must eventually stop, giving finitely many bubbles.
- The moduli space $\mathcal{M}_{g,m}$ of marked Riemann surfaces (i.e. taking (M,J) to be a point), identification of $\mathcal{M}_{g,m}$ with $\mathcal{J}(\Sigma)/\operatorname{Diff}(\Sigma,\zeta)$ for a model genus g surface Σ with marked points $\zeta_1, \dots, \zeta_m \in \Sigma$ and
$$\operatorname{Diff}(\Sigma,\zeta) := \left\{ \varphi \in \operatorname{Diff}(\Sigma) \;\middle|\; \varphi(\zeta_i) = \zeta_i \text{ for all } i = 1, \dots, m \right\}.$$
- Proposition: If $g \geqslant 1$ or $m \geqslant 3$, the action of $\operatorname{Diff}(\Sigma,\zeta)$ on $\mathcal{J}(\Sigma)$ is proper.
Proof: Need to show that if $\varphi_k : (\Sigma, j'_k) \to (\Sigma, j_k)$ are degree 1 holomorphic maps with $j_k \to j$ and $j'_k \to j'$ in $\mathcal{J}(\Sigma)$, then φ_k has a convergent subsequence.
Case $g > 0$: $\pi_2(\Sigma) = 0$ implies (Σ, j) admits no nonconstant holomorphic spheres, so there can be no bubbling.
Case $g = 0$ and $m \geqslant 3$: There may be bubbling, but at most one bubble and it absorbs all the energy, so φ_k converges outside of one point to a constant, which is impossible since each φ_k fixes at least three distinct points.

- Corollary: In these cases, the automorphism group

$$\mathrm{Aut}(\Sigma, j, \zeta) := \left\{ \varphi \in \mathrm{Diff}(\Sigma, \zeta) \;\middle|\; \varphi : (\Sigma, j) \to (\Sigma, j) \text{ holomorphic} \right\}$$

  is always compact. (Notice that this is not true for (S^2, i) with fewer than three marked points.)

Suggested reading.

- Background on differential calculus in Banach spaces: [Lan93, Chapters XIII–XIV], and see also [Lan99, Chapters II–III] for the basic notions involving Banach manifolds and Banach space bundles
- Differentiability of maps between Banach spaces: [Wena, §2.12]; the "useful lemma" about $u \mapsto f \circ u$ is Lemma 2.12.5, and its extension for the map $(f, u) \mapsto f \circ u$ is Lemma 2.12.7. These results are based on a more abstractly-stated lemma in the paper [Elĭ67], which is one of the standard references for constructions of Banach manifold structures on spaces of maps from one manifold to another.
- Nonlinear local existence: [Wena, 2.13]
- Bubbling analysis: The basic rescaling argument (including the Hofer lemma) that our bubbling analysis is based on is covered in [Wena, §5.3], where it is applied toward proving a specific compactness result needed for Gromov's nonsqueezing theorem. That presentation doesn't go into the topic of bubble trees, but [MS12, §4.6–4.7] gives a more detailed presentation on that. For Gromov's removable singularity theorem, see [Wenc, §9.1].

Exercises. There will again be no Übung next week, but these exercises may be discussed in the first Übung after the holidays. Exercise 8.2 is somewhat hard, but if you recognize that a parametric local existence result for J-holomorphic curves is required, then you've understood the main idea.

Exercise 8.1. For functions of a complex variable $z = s + it$, recall the formal partial differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right).$$

Show that if $J \in \mathcal{J}(\mathbb{C}^n)$ is an almost complex structure on $\mathbb{C}^n$ with $J(0) = i$ and $u, v : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ are two J-holomorphic maps with $u(0) = v(0) = 0$ and

$$\frac{\partial^k u}{\partial z^k}(0) = \frac{\partial^k v}{\partial z^k}(0) \qquad \text{for all } k = 1, \dots, m,$$

then also $\partial^\alpha u(0) = \partial^\alpha v(0)$ for all multi-indices of order $|\alpha| \leqslant m$.

Exercise 8.2. Suppose (M, J) is an almost complex manifold of real dimension 4 and $\Sigma \subset M$ is the image of an embedded closed J-holomorphic curve with trivial normal bundle. Prove that Σ then has a neighorhood $\mathcal{U} \subset M$ that can be identified diffeomorphically with $\Sigma \times \mathring{\mathbb{D}}$ so that Σ itself is identified with $\Sigma \times \{0\}$ and the almost complex structure J on $\Sigma \times \mathbb{D}$ takes the form

$$J(w, z) = \begin{pmatrix} j(w, z) & 0 \\ y(w, z) & i \end{pmatrix} \qquad \text{on} \qquad T_{(w,z)}(\Sigma \times \mathring{\mathbb{D}}) = T_w\Sigma \oplus \mathbb{C}$$

where $j(w, z) : T_w\Sigma \to T_w\Sigma$ and $y(w, z) : T_w\Sigma \to \mathbb{C}$ satisfy $[j(w, z)]^2 = -\mathbb{1}$ and $y(w, z)j(w, z) + iy(w, z) = 0$. Would you expect a result like this to hold if $\dim M > 4$?

9. Week 9

Lecture 14 (13.12.2022): Deligne-Mumford compactness.

- Uniformization theorem (classical Riemann surface theory): every simply connected Riemann surface is biholomorphically equivalent to either (1) $(S^2 = \mathbb{C} \cup \{\infty\}, i)$, (2) $(\mathbb{C}, i)$, or (3) $(\mathbb{H} := \{\operatorname{Im} z > 0\} \subset \mathbb{C}, i)$
- Corollaries (1): For all $j \in \mathcal{J}(S^2)$, $(S^2, j) \cong (S^2, i)$. Since

$$\begin{aligned}\operatorname{Aut}(S^2, i) &= \left\{ \varphi : S^2 \to S^2 \;\middle|\; \varphi(z) = \frac{az+b}{cz+d} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \right\} \\ &\cong \mathrm{SL}(2,\mathbb{C})/\{\pm \mathbb{1}\} =: \mathrm{PSL}(2,\mathbb{C})\end{aligned}$$

  contains a unique map that sends any given three points to any other given three points, this implies $\mathcal{M}_{0,k}$ is a one-point space for $k = 0, 1, 2, 3$.
- Corollaries (3): One can show that $\mathbb{H} = \{ s + it \mid s \in \mathbb{R},\ t > 0 \}$ with the Riemannian metric

$$g_P := \frac{1}{t^2}(ds^2 + dt^2)$$

  is isometric to the hyperbolic plane. It follows that this metric is complete with constant Gaussian curvature $K_G \equiv -1$ and defines the standard conformal structure of $(\mathbb{H}, i)$. Moreover,

$$\operatorname{Isom}(\mathbb{H}, g_P) = \operatorname{Aut}(\mathbb{H}, i) \cong \mathrm{PSL}(2, \mathbb{R}),$$

  i.e. the isometries of $(\mathbb{H}, g_P)$ are precisely the fractional linear transformations on S^2 that preserve $\mathbb{H}$. It follows that any Riemann surface whose universal cover is $(\mathbb{H}, i)$ admits a metric of constant negative curvature; by Gauss-Bonnet, this excludes e.g. $\mathbb{T}^2$.
- Corollaries (2): If (Σ, j) has universal cover $(\mathbb{C}, i)$, then $(\Sigma, j) = (\mathbb{C}/\Gamma, i)$ for a freely acting discrete subgroup $\Gamma \subset \operatorname{Aut}(\mathbb{C}, i)$; since $\operatorname{Aut}(\mathbb{C}, i)$ consists of affine maps and most of them have a fixed point, Γ consists only of translations. There are three options:
  1. Γ trivial: then $(\Sigma, j) \cong (\mathbb{C}, i) = (S^2 \backslash \{\infty\}, i)$.
  2. Γ cyclic: then $(\Sigma, j) \cong (\mathbb{C}/w\mathbb{Z}, i)$ for some $w \in \mathbb{C}\backslash\{0\}$, and this is equivalent to $\mathbb{C}/i\mathbb{Z} = (\mathbb{R} \times S^1, i) \cong (S^2 \backslash \{0, \infty\}, i)$.
  3. Γ a lattice: then $(\Sigma, j) \cong (\mathbb{C}/(a\mathbb{Z} + b\mathbb{Z}), i)$ for some real-linearly independent $a, b \in \mathbb{C}$, and this is equivalent to $\mathbb{C}/(\mathbb{Z} + w\mathbb{Z})$ for some $w \in \mathbb{H}$, which is equivalent to a torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with a translation-invariant complex structure. This proves there is a surjective map

$$\mathbb{H} \to \mathcal{M}_{1,0} : w \mapsto [(\mathbb{C}/(\mathbb{Z} + w\mathbb{Z}), i, \varnothing)].$$

- Corollaries (3), continued: if (Σ, j) has universal cover $(\mathbb{H}, i)$, then $(\Sigma, j) = (\mathbb{H}/\Gamma, i)$ for a freely acting discrete subgroup $\Gamma \subset \operatorname{Aut}(\mathbb{H}, i) = \operatorname{Isom}(\mathbb{H}, g_P)$, thus (Σ, j) admits a **Poincaré metric** g_j, which is complete, defines the same conformal structure as j, and has constant curvature -1. (Exercise 9.1 below shows that g_j is unique.)
- Most important corollary/definition: Suppose (Σ, j) is a closed connected Riemann surface and $\zeta \subset \Sigma$ is a finite set. Call (Σ, j, ζ) **stable** if $\chi(\Sigma\backslash\zeta) < 0$, i.e. $2g + m \geqslant 3$. In this case, $(\Sigma\backslash\zeta, j)$ has a unique Poincaré metric. Stability excludes only the four cases $g = 0$ with $m < 3$ and $g = 1$ with $m = 0$.
- Definition: a **pair-of-pants decomposition** of a stable marked Riemann surface (Σ, j, ζ) is a finite collection of disjoint closed geodesics $C \subset \Sigma\backslash\zeta$ (with respect to the Poincaré metric) such that each component of $\Sigma\backslash(\zeta \cup C)$ has the homotopy type of a disk with two holes.

- Observe: the Euler characteristic of a pair of pants is -1, thus a pair-of-pants decomposition of (Σ, j, ζ) contains $-\chi(\Sigma \backslash \zeta)$ pieces. (Stability is obviously necessary.)
- Fact from hyperbolic geometry: on a pair of pants (P, g) with complete metric g of constant curvature -1 and geodesic boundary, the lengths of the boundary components determine g up to diffeomorphism. (This also allows "boundary components of length 0", understood as cusps/punctures at which P is noncompact.)
- Important theorem of Bers (quoted without proof): For each $g, m \geqslant 0$ with $2g + m \geqslant 3$, there exists a universal constant $C = C(g, m) > 0$ such that every marked Riemann surface (Σ, j, ζ) with genus g and m marked points and Poincaré metric g_j admits a pair-of-pants decomposition defined by closed geodesics of length $\leqslant C$.
- Corollary: If $2g + m \geqslant 3$ and $[(\Sigma_k, j_k, \zeta_k)] \in \mathcal{M}_{g,m}$ is a sequence admitting pair-of-pants decompositions whose geodesics all satisfy $\delta \leqslant$ length $\leqslant C$ for some $\delta > 0$, then a subsequence converges to some element of $\mathcal{M}_{g,m}$.
- The alternative: some geodesics in the pair-of-pants decompositions collapse in the limit to length 0. (illustrated with pictures)
- Definition: a **nodal marked Riemann surface** with m marked points is a tuple (S, j, ζ, Δ), where (S, j) is a closed (but possibly disconnected) Riemann surface, ζ is an ordered set of m distinct points in S, and Δ is an unordered set of unordered pairs $\big\{\{z_1^+, z_1^-\}, \dots, \{z_N^+, z_N^-\}\big\}$ of distinct points in $S \backslash \zeta$; each matched pair $\{z_i^+, z_i^-\} \in \Delta$ is called a **node**, and the individual points $z_i^\pm$ are called **nodal points**. We say $(S, j, \zeta, \Delta) \sim (S', j', \zeta', \Delta')$ if there exists a biholomorphic map $\varphi : (S, j) \to (S', j')$ that sends ζ to ζ' preserving the order and sends each node of Δ to a node of Δ'. We call (S, j, ζ, Δ) **stable** if every connected component $\Sigma \subset S \backslash (\zeta \cup \Delta)$ has $\chi(\Sigma) < 0$. If the closed surface $\widehat{S}$ formed by performing connected sums on S at each of its nodal pairs $\{z_i^+, z_i^-\}$ is connected with genus g, then we say (S, j, ζ, Δ) has **arithmetic genus** g.
- Definition: for $2g + m \geqslant 3$,

$$\overline{\mathcal{M}}_{g,m} := \{\text{stable nodal Riemann surfaces of arithmetic genus } g \text{ with } m \text{ marked points}\} \,/\sim .$$

  There is a natural inclusion

$$\mathcal{M}_{g,m} \hookrightarrow \overline{\mathcal{M}}_{g,m} : [(\Sigma, j, \zeta)] \mapsto [(\Sigma, j, \zeta, \varnothing)].$$

- Deligne-Mumford compactness theorem: $\overline{\mathcal{M}}_{g,m}$ admits a natural topology as a compact metrizable space such that the inclusion $\mathcal{M}_{g,m} \hookrightarrow \overline{\mathcal{M}}_{g,m}$ is a continuous map onto an open and dense subset.
- Remark: There is no definition of $\overline{\mathcal{M}}_{g,m}$ for $2g + m < 3$. Thanks to uniformization, $\mathcal{M}_{g,m}$ in these cases is easy enough to understand without compactifying.

Lecture 15 (14.12.2022): Gromov compactness.

- Topology of $\overline{\mathcal{M}}_{g,m}$: defining a neighborhood base of $[(S, j, \zeta, \Delta)]$ by pre-gluing to replace neighborhoods of nodes with long "necks" $([-R, R] \times S^1, i)$.
- Remark: If we did not require stability for elements of $\overline{\mathcal{M}}_{g,m}$, it would not be Hausdorff. (Non-stable sphere components with one or two nodal points could be added arbitrarily to limits of sequences.)
- Nodal J-holomorphic curves in (M, J): (S, j, ζ, Δ, u) such that (S, j, ζ, Δ) is a nodal marked Riemann surface (not necessarily stable) and $u : (S, j) \to (M, J)$ is a J-holomorphic curve with $u(z^+) = u(z^-)$ for each node $\{z^+, z^-\} \in \Delta$. Say $(S, j, \zeta, \Delta, u) \sim (S', j', \zeta', \Delta', u')$ if there is an equivalence of nodal marked Riemann surfaces $\varphi : (S, j, \zeta, \Delta) \to (S', j', \zeta', \Delta')$ such that $u' \circ \varphi = u$. Call (S, j, ζ, Δ, u) **stable** if for every connected component $\Sigma \subset S \backslash (\zeta \cup \Delta)$, either $\chi(\Sigma) < 0$ or $u|_\Sigma$ is nonconstant.

- Definition:

$$\overline{\mathcal{M}}_{g,m}(J) := \{\text{stable nodal } J\text{-holomorphic curves of arithmetic genus } g \text{ with } m \text{ marked points}\} / \sim .$$

There is a natural inclusion

$$\mathcal{M}_{g,m}(J) \hookrightarrow \overline{\mathcal{M}}_{g,m}(J) : [(\Sigma, j, \zeta, u)] \mapsto [(\Sigma, j, \zeta, \varnothing, u)]$$

if we exclude from $\mathcal{M}_{g,m}(J)$ all elements with $2g + m < 3$ and u constant. We will always exclude these henceforth.
- Convergence in $\overline{\mathcal{M}}_{g,m}(J)$: given $[(S, j, \zeta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(J)$, let $\widehat{S}$ denote a closed surface of genus g with a finite collection of disjoint circles $C \subset \widehat{S}$ such that $\widehat{S} \backslash C$ can be identified diffeomorphically with $S \backslash \Delta$. (One obtains such a surface by performing connected sums on S at the nodes.) We can view j as a smooth complex structure on $\widehat{S} \backslash C$ that degenerates along C, and $u|_{S \backslash \Delta} = u|_{\widehat{S} \backslash C}$ has a continuous extension over $\widehat{S}$ that is constant on each component of C. We say a sequence $[(\Sigma_k, j_k, \zeta_k, u_k)] \in \mathcal{M}_{g,m}(J)$ converges to $[(S, j, \zeta, \Delta, u)]$ if for large k, $[(\Sigma_k, j_k, \zeta_k, u_k)] = [(\widehat{S}, j'_k, \zeta, u'_k)]$ for sequences of complex structures $j'_k \to j$ converging in $C^\infty_{\mathrm{loc}}(\widehat{S} \backslash C)$ and maps $u'_k \to u$ converging in both $C^\infty_{\mathrm{loc}}(\widehat{S} \backslash C)$ and $C^0(\widehat{S})$.
- Energy: for $J \in \mathcal{J}_\tau(M, \omega)$ and $U = [(S, j, \zeta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(J)$, define

$$E_\omega(U) := \int_S u^*\omega = \langle [\omega], [u] \rangle$$

where $[u] := u_*[S] \in H_2(M)$. For each $A \in H^2(M)$ write

$$\overline{\mathcal{M}}_{g,m}(J, A) := \left\{ [(S, j, \zeta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(J) \;\middle|\; [u] = A \right\}.$$

- Gromov's compactness theorem: Assume M is closed with a C^∞-convergent sequence $\omega_k \to \omega$ of symplectic forms on M and a C^∞-convergent sequence of tame almost complex structures $\mathcal{J}_\tau(M, \omega_k) \ni J_k \to J \in \mathcal{J}_\tau(M, \omega)$. If $U_k \in \overline{\mathcal{M}}_{g,m}(J_k)$ has uniformly bounded energy $E_{\omega_k}(U_k)$, then it has a subsequence convergent to an element of $\overline{\mathcal{M}}_{g,m}(J)$.
Corollary: $\overline{\mathcal{M}}_{g,m}(J, A)$ is compact for each $J \in \mathcal{J}_\tau(M, \omega)$ and $A \in H_2(M)$.
- Proof sketch for a sequence of non-nodal curves $[(\Sigma_k, j_k, \zeta_k, u_k)] \in \mathcal{M}_{g,m}(J)$:
  - Step 1: Bubbling analysis and energy quantization $\Rightarrow$ there exists a number $N \geqslant 0$ and a sequence $\Theta_k \subset \Sigma_k \backslash \zeta_k$ of sets of N extra marked points such that each $(\Sigma_k, j_k, \zeta_k \cup \Theta_k)$ is stable, and for the Poincaré metric g_k on $\Sigma_k \backslash (\zeta_k \cup \Theta_k)$, the maps u_k satisfy a bound

    $$|du_k(z)| \cdot \mathrm{injrad}_{g_k}(z) \leqslant C$$

    with a constant $C > 0$ independent of k and $z \in \Sigma_k \backslash (\zeta_k \cup \Theta_k)$. (Here we use the injectivity radius to control how close z is to a marked point or a collapsing geodesic.)
  - Step 2: By Deligne-Mumford compactness, a subsequence of $[(\Sigma_k, j_k, \zeta_k \cup \Theta_k)] \in \mathcal{M}_{g,m+N}$ converges to some $[(S, j, \zeta \cup \Theta, \Delta)] \in \overline{\mathcal{M}}_{g,m+N}$. Writing $\widehat{S}$ for a closed surface of genus g with a finite collection of disjoint circles $C \subset \widehat{S}$ such that $\widehat{S} \backslash C \cong S \backslash \Delta$, we can now assume without loss of generality that for large k, $\Sigma_k = \widehat{S}$, $\zeta_k = \zeta$, $\Theta_k = \Theta$, $j_k \to j$ in $C^\infty_{\mathrm{loc}}(\widehat{S} \backslash C)$ and $|du_k|$ is uniformly bounded on compact subsets of $\widehat{S} \backslash (C \cup \zeta \cup \Theta)$. Regularity then gives a subsequence such that u_k converges in C^∞_{loc} on $\widehat{S} \backslash (C \cup \zeta \cup \Theta) \cong S \backslash (\Delta \cup \zeta \cup \Theta)$ to a J-holomorphic map $u_\infty : (S \backslash (\Delta \cup \zeta \cup \Theta), j) \to (M, J)$, which has finite energy due to the uniform energy bound. Gromov's removable singularity theorem thus extends u_∞ smoothly over S.

– Step 3: $|du_k(z_k)|$ can still blow up along sequences $z_k \in \widehat{S} \backslash (C \cup \zeta \cup \Theta)$ accumulating at at most finitely many points in $C \cup \zeta \cup \Theta$. One then finds bubble trees at these points, with only finitely many bubbles due to the uniform energy bound and energy quantization. Bubbles add extra spherical components to the nodal Riemann surface $(S, j, \zeta \cup \Theta, \Delta)$ and may make it non-stable, but the bubbling analysis guarantees that every bubble with fewer than three marked or nodal points is a nonconstant J-holomorphic sphere, producing a limiting nodal J-holomorphic curve that is stable.

Lecture 16 (14.12.2022): The Riemann-Roch formula.

- Assume (Σ, j) is a closed connected Riemann surface, $E \to \Sigma$ is a complex vector bundle of rank n, $F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, E)$ and $\mathbf{D} : \Gamma(E) \to \Gamma(F) = \Omega^{0,1}(\Sigma, E)$ is a linear Cauchy-Riemann type operator. We showed in Lecture 9 that $\mathbf{D}$ is a Fredholm operator, and we now want to show that its index is
$$\mathrm{ind}(\mathbf{D}) = n\chi(\Sigma) + 2c_1(E).$$
Note that $\mathbf{D}$ is in general a *real*-linear (not complex-linear) operator, and this is its *real* Fredholm index.
- Observe: if we can prove the formula for $n = 1$, then the general case follows just by considering direct sums of line bundles. We assume from now on that $E \to \Sigma$ is a line bundle.
- Striking coincidence: the line bundle $\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)$ is isomorphic to $T\Sigma \otimes E \otimes E$ and thus has $c_1(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)) = \chi(\Sigma) + 2c_1(E)$, the same integer that appears in the Riemann-Roch formula. This number is thus the algebraic count of zeroes of a generic complex-antilinear map bundle map $E \to F$.
- Idea of Taubes: by a zeroth-order (and thus compact) perturbation, we can replace $\mathbf{D}$ with $\mathbf{D} + \tau A$ for some section $A \in \overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)$ without changing the index. Choose A to have only finitely many positive zeroes Z_+ and negative zeroes $Z_- \subset \Sigma$, all with order ± 1. Main theorem: for $\tau \gg 0$, the kernel of $\mathbf{D} + \tau A$ has a basis consisting of one section $\eta_\zeta \in \Gamma(E)$ for each positive zero $\zeta \in Z_+$, such that η_ζ looks like a Gaussian in coordinates near ζ and is very small away from ζ. A similar statement holds for the formal adjoint $(\mathbf{D} + \tau A)^*$ and the negative zeroes $\zeta \in Z_-$. The Riemann-Roch formula follows since

$$\mathrm{ind}(\mathbf{D}) = \mathrm{ind}(\mathbf{D} + \tau A) = \dim\ker(\mathbf{D} + \tau A) - \dim\ker((\mathbf{D} + \tau A)^*) = \#Z_+ - \#Z_- = c_1(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F)).$$

- Warmup case: take $\Sigma = \mathbb{T}^2$ and $E = \mathbb{T}^2 \times \mathbb{C} \cong F$ with $\mathbf{D} = \bar{\partial} = \partial_s + i\partial_t : C^\infty(\mathbb{T}^2, \mathbb{C}) \to C^\infty(\mathbb{T}^2, \mathbb{C})$ and an antilinear zeroth-order perturbation $A : C^\infty(\mathbb{T}^2, \mathbb{C}) \to C^\infty(\mathbb{T}^2, \mathbb{C})$, which can be written as $A\eta = \beta\bar{\eta}$ for some function $\beta : \mathbb{T}^2 \to \mathbb{C}$. Since the bundle is trivial, we are free to assume β is nowhere zero, so there is an estimate $|A\eta| \geqslant c|\eta|$ at every point. Claim: $\mathbf{D}_\tau := \bar{\partial} + \tau A$ is an isomorphism for all $\tau \gg 0$, and thus has index $\chi(\mathbb{T}^2) + 2c_1(E) = 0$. It suffices to prove that $\mathbf{D}_\tau$ is injective, since the same argument then applies to its formal adjoint. This is proved using the Weitzenböck formula
$$\mathbf{D}_\tau^* \mathbf{D}_\tau \eta = \bar{\partial}^* \bar{\partial}\eta + \tau^2 A^* A\eta - \tau(\partial\beta)\bar{\eta}.$$
- General Weitzenböck formula for a Cauchy-Riemann type operator $\mathbf{D} : \Gamma(E) \to \Gamma(F)$ and antilinear perturbation $\mathbf{D}_\tau := \mathbf{D} + \tau A$ with $A \in \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(E, F))$: There exists a real-linear bundle map $B : E \to E$ such that
$$\mathbf{D}_\tau^* \mathbf{D}_\tau \eta = \mathbf{D}^* \mathbf{D}\eta + \tau^2 A^* A\eta + \tau B\eta.$$

Proof: We can write $A\eta = \beta\bar{\eta}$ for a section $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$, where $\bar{E}$ is the conjugate bundle of E and $E \to \bar{E} : \eta \mapsto \bar{\eta}$ denotes the canonical complex-antilinear bundle isomorphism, i.e. the identity map (see Exercise 5.2). If $\mathbf{D}$ is complex linear, then it defines a holomorphic vector bundle structure on E, and similarly, $-\mathbf{D}^* : \Gamma(F) \to \Gamma(E) \cong \Omega^{1,0}(\Sigma, E)$ and

$$\bar{\mathbf{D}} : \Gamma(\bar{E}) \to \Gamma(\bar{F}) \cong \Omega^{1,0}(\Sigma, \bar{E}) : \bar{\eta} \mapsto \overline{\mathbf{D}\eta}$$

are complex-linear anti-Cauchy-Riemann operators (again see Exercise 5.2), meaning they satisfy the same Leibniz rule as a Cauchy-Riemann type operator but with $\bar{\partial} f \in \Omega^{0,1}(\Sigma, \mathbb{C})$ replaced by $\partial f := df - i\, df \circ j \in \Omega^{1,0}(\Sigma, \mathbb{C})$. Such operators make the underlying vector bundles into *antiholomorphic* vector bundles, meaning transition functions are complex conjugates of holomorphic functions. It follows that $\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)$ also inherits a natural antiholomorphic bundle structure and thus carries a natural anti-Cauchy-Riemann type operator ∂_H satisfying the Leibniz rule

$$-\mathbf{D}^*(\Phi\bar{\eta}) = (\partial_H \Phi)\bar{\eta} + \Phi(\bar{\mathbf{D}}\bar{\eta}) \qquad \text{for } \Phi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F)),\ \bar{\eta} \in \Gamma(\bar{E}).$$

Using this rule, an easy computation proves that the Weitzenböck formula holds with $B\eta := -(\partial_H \beta)\bar{\eta}$. If $\mathbf{D}$ is not complex linear, one can split it into its complex-linear part $\mathbf{D}^{\mathbb{C}}$ and complex-antilinear part C, where $\mathbf{D}^{\mathbb{C}}$ is another Cauchy-Riemann type operator and C is a zeroth-order term, then compute further based on the Weitzenböck formula for $\mathbf{D}^{\mathbb{C}}$.

- Further assumptions (for convenience): $A\eta = \beta\bar{\eta}$ where $\beta \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(\bar{E}, F))$ has finite zero set $Z = Z_+ \sqcup Z_-$, and for each $\zeta \in Z_\pm$ there are holomorphic coordinates identifying a neighborhood $(\mathcal{D}(\zeta), j)$ of ζ with $(\mathbb{D}, i)$, and trivializations of E and F in which $\mathbf{D}$ looks like $\bar{\partial} = \partial_s + i\partial_t$ and β takes the form

$$\beta(z) = z \quad \text{for } \zeta \in Z_+, \qquad \beta(z) = \bar{z} \quad \text{for } \zeta \in Z_-.$$

  In particular, the equation $\mathbf{D}_\tau \eta = 0$ then looks like $\bar{\partial}\eta + \tau z \bar{\eta} = 0$ on $\mathcal{D}(\zeta)$ if $\zeta \in Z_+$, or $\bar{\partial}\eta + \tau \bar{z}\bar{\eta} = 0$ if $\zeta \in Z_-$.
- Energy concentration lemma: Suppose $\tau_k \to \infty$ and $\eta_k \in \ker \mathbf{D}_{\tau_k}$ satisfies a uniform L^2-bound, and for each $\zeta \in Z_\pm$, use the coordinates and trivializations chosen above on $\mathcal{D}(\zeta)$ to define the functions

$$\eta_k^\zeta : \mathbb{D}_{\sqrt{\tau_k}} \to \mathbb{C}, \qquad \eta_k^\zeta(z) := \frac{1}{\sqrt{\tau_k}} \eta_k(z/\sqrt{\tau_k}).$$

  Then:
  (1) $\|\eta_k^\zeta\|_{L^2(\mathbb{D}_{\sqrt{\tau_k}})} = \|\eta_k\|_{L^2(\mathcal{D}(\zeta))}$.
  (2) η_k^ζ satisfies the equation

$$\bar{\partial}\eta_k^\zeta + z\bar{\eta}_k^\zeta = 0 \quad \text{if } \zeta \in Z_+, \qquad \bar{\partial}\eta_k^\zeta + \bar{z}\bar{\eta}_k^\zeta = 0 \quad \text{if } \zeta \in Z_-.$$

  (3) η_k^ζ has a subsequence C^∞_{loc}-convergent on $\mathbb{C}$ to a function $\eta_\infty^\zeta \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$.
  (4) For any other sequence $\xi_k \in \ker \mathbf{D}_{\tau_k}$ satisfying the same assumptions,

$$\langle \eta_k, \xi_k \rangle_{L^2} \to \sum_{\zeta \in Z} \langle \eta_\infty^\zeta, \xi_\infty^\zeta \rangle_{L^2(\mathbb{C})}$$

  as $k \to \infty$. In particular, taking $\xi_k = \eta_k$ and interpreting $\|\eta_k\|_{L^2(\mathcal{U})}$ as the "energy" of η_k over a region $\mathcal{U} \subset \Sigma$, this shows that the energy of η_k is concentrated in a neighborhood of Z as $\tau_k \to \infty$, and outside this neighborhood it becomes arbitrarily small.

Proof: (1) and (2) are straightforward computations, and (3) then follows from linear elliptic regularity theory. Statement (4) follows after using the Weitzenböck formula to prove that on the region $\Sigma_\epsilon := \Sigma \setminus \bigcup_{\zeta \in Z} \mathcal{D}(\zeta)$, $\|\eta_k\|_{L^2(\Sigma_\epsilon)} \to 0$ as $k \to \infty$. This also uses the fact that on Σ_ϵ, there is a pointwise estimate of the form $|A\eta| \geqslant c|\eta|$ for some $c > 0$.

- Define $\mathbf{D}_\zeta := \mathbf{D}_\pm$ for $\zeta \in Z_\pm$, where $\mathbf{D}_\pm : C^\infty(\mathbb{C}, \mathbb{C}) \to C^\infty(\mathbb{C}, \mathbb{C})$ are the Cauchy-Riemann type operators
$$\mathbf{D}_+ f := \bar{\partial} f + z\bar{f}, \qquad \mathbf{D}_- f := \bar{\partial} f + \bar{z}\bar{f}.$$
With a little more work, the energy concentration lemma implies that there is an isomorphism
$$\ker \mathbf{D}_\tau \cong \bigoplus_{\zeta \in Z} \left\{ f \in L^2(\mathbb{C}) \;\middle|\; \mathbf{D}_\zeta f = 0 \right\}$$
for sufficiently large $\tau > 0$.
- Proposition: All solutions $f \in L^2(\mathbb{C})$ to $\mathbf{D}_- f = 0$ are trivial, and all solutions $f \in L^2(\mathbb{C})$ to $\mathbf{D}_+ f = 0$ are constant real multiples of the Gaussian function $e^{-\frac{1}{2}|z|^2}$.
Proof for $\mathbf{D}_-$: the formal adjoint of $\bar{\partial} = \partial_s + i\partial_t$ is $\bar{\partial}^* = -\partial = -\partial_s + i\partial_t$, and $\bar{\partial}^*\bar{\partial} = -\partial_s^2 - \partial_t^2 =: \Delta$ is then the standard Laplace operator. One can derive a Weitzenböck formula
$$\mathbf{D}_-^* \mathbf{D}_- f = \bar{\partial}^* \bar{\partial} f + |z|^2 f = \Delta f + |z|^2 f,$$
and use it to prove that for any f satisfying $\mathbf{D}_- f = 0$,
$$\Delta |f|^2 = -2|z|^2|f|^2 - 2|\nabla f|^2 \leqslant 0,$$
meaning $|f|^2 : \mathbb{C} \to \mathbb{R}$ is a subharmonic function. It therefore satisfies the mean value property: for any $z_0 \in \mathbb{C}$ and $r > 0$,
$$|f(z_0)|^2 \leqslant \frac{1}{\pi r^2} \int_{B_r(z_0)} |f|^2,$$
and if $f \in L^2(\mathbb{C})$, this implies $|f(z_0)|^2 \leqslant \frac{1}{\pi r^2}\|f\|_{L^2}^2$ for every $r > 0$ and thus $f \equiv 0$.
Proof for $\mathbf{D}_+$: a similar Weitzenböck formula can be used to show that if $f \in L^2(\mathbb{C})$ satisfies $\mathbf{D}_+ f = 0$ then $\operatorname{Im} f \equiv 0$. We can thus write $f(z) = g(z)e^{-\frac{1}{2}|z|^2}$ for a unique function $g : \mathbb{C} \to \mathbb{R}$. Since $\mathbf{D}_+(e^{-\frac{1}{2}|z|^2}) = 0$, the Leibniz rule then implies $\bar{\partial} g = 0$, so g is a real-valued holomorphic function on $\mathbb{C}$, implying it is constant.

Suggested reading. Our presentation of the Deligne-Mumford and Gromov compactness theorems is very similar to [Wenc, Lecture 9], especially §9.3.3 and 9.4.1, though the latter contains some features that you can ignore because it works in the more general setting of punctured J-holomorphic curves in noncompact symplectic cobordisms. Many of the details of hyperbolic geometry that we needed to cite without proof are covered nicely in [Hum97] and [SS92]. Unfortunately, the version of Gromov's compactness theorem that appeared in [Gro85] was rather preliminary and not adequate for modern use; e.g. it never mentioned the notion of *stability*, which was later recognized by Kontsevich as being important of you want your compactification to be Hausdorff. The only reference I know for a complete proof of Gromov compactness as we stated it in lecture is [BEH+03], which proves a much more general result of which Gromov compactness is a special case. The proof we sketched is also based on the argument written there, and a more detailed account of it can be found in [Abb14].

The proof of the Riemann-Roch formula we presented was first sketched by Taubes in a brief appendix to [Tau96], and the details are worked out in [Wenc, Lecture 5], in which you can ignore §5.8 because it also deals with a more general setting than we are considering.

Exercises (for the Übung on 4.01.2023). Note that starting in January, the Übung will take place on **Wednesdays at 15:15 in 1.114** instead of the usual Tuesday morning time slot.

Exercise 9.1. Prove that there is at most one Riemannian metric on $\mathbb{H}$ that is complete, conformally equivalent to the standard Euclidean metric, and has constant curvature -1. Deduce the uniqueness of the Poincaré metric on any Riemann surface whose universal cover is $(\mathbb{H}, i)$.

Exercise 9.2. One consequence of the uniqueness of complex structures on S^2 is that the map

$$S^2 \backslash \{0, 1, \infty\} \to \mathcal{M}_{0,4} : \zeta \mapsto [(S^2, i, (0, 1, \infty, \zeta))]$$

is a homeomorphism. Show that this map extends to a homeomorphism $S^2 \to \overline{\mathcal{M}}_{0,4}$, and describe the three stable nodal Riemann surfaces that occur as the images of the points $0, 1, \infty$.

Exercise 9.3. Recall from Exercise 1.5 that $\mathrm{Aut}(S^2, i) \cong \mathrm{PSL}(2, \mathbb{C})$ is naturally isomorphic to the moduli space $\mathcal{M}_{0,3}(i, [S^2])$ of degree 1 holomorphic spheres in (S^2, i) with three marked points, whose evaluation map defines a homeomorphism onto the complement of the fat diagonal $\Delta \subset S^2 \times S^2 \times S^2$. Show that the continuous extension of this map to[9]

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,3}(i, [S^2]) \to S^2 \times S^2 \times S^2$$

is surjective, and describe specific stable nodal holomorphic curves corresponding to each element of the fat diagonal. Show in particular:

(a) $\mathrm{ev}^{-1}(w_1, w_2, w_3) \subset \overline{\mathcal{M}}_{0,3}(i, [S^2])$ contains a unique element whenever two of the $w_i \in S^2$ are identical and the third is different.
(b) $\mathrm{ev}^{-1}(w, w, w) \subset \overline{\mathcal{M}}_{0,3}(i, [S^2])$ is homeomorphic to S^2 for each individual $w \in S^2$.

Exercise 9.4. Using the standard complex structure i on S^2, $S^2 \times S^2$ inherits an integrable complex structure J that is compatible with any symplectic form of the form $\alpha \oplus \beta \in \Omega^2(S^2 \times S^2)$ for two positive area forms $\alpha, \beta \in \Omega^2(M)$. Consider the family of embedded J-holomorphic curves

$$u_c : (S^2, i) \to (S^2 \times S^2, J), \qquad u_c(z) := (z, cz)$$

for $c \in \mathbb{C}\backslash\{0\}$, which define a family of elements in the moduli space $\mathcal{M}_{0,0}(J, A)$ for $A := [S^2 \times \{\mathrm{const}\}] + [\{\mathrm{const}\} \times S^2] \in H_2(S^2 \times S^2)$. What do these curves converge to in $\overline{\mathcal{M}}_{0,0}(J, A)$ as $c \to 0$ or $c \to \infty$?
Caution: Make sure that whichever nodal curves you describe represent the right homology class!

10. Week 10

Lecture 17 (3.01.2023): Functional-analytic setup for $\bar{\partial}_J$.

- The moduli space of parametrized J-holomorphic curves

  $$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(j, J, A) := \left\{ u \in C^\infty(\Sigma, M) \;\middle|\; Tu \circ j = J \circ Tu \text{ and } [u] := u_*[\Sigma] = A \right\}$$

  for given $J \in \mathcal{J}(M)$, $j \in \mathcal{J}(\Sigma)$ and $A \in H_2(M)$, and evaluation map

  $$\mathrm{ev} : \widetilde{\mathcal{M}} \to M^{\times m} : u \mapsto (u(\zeta_1), \ldots, u(\zeta_m))$$

  for a given ordered set of distinct points $\zeta = (\zeta_1, \ldots, \zeta_m)$ in Σ. (Assume Σ is closed with genus g and M has dimension $2n$.)

[9] This exercise originally claimed that the extension of ev to the compactification is a homeomorphism onto $S^2 \times S^2 \times S^2$, but it was noticed during the Übung that that is false.

- The space

$$W^{k,p}(\Sigma, M) := \left\{ u \in C^0(\Sigma, M) \;\middle|\; u \text{ is of class } W^{k,p}_{\mathrm{loc}} \text{ in all local charts on } \Sigma \text{ and } M \right\}$$

for $k \in \mathbb{N}$ and $p \in (1, \infty)$ with $kp > 2$, and subset

$$\mathcal{B} := \left\{ u \in W^{k,p}(\Sigma, M) \;\middle|\; [u] = A \right\}.$$

- Proposition (see [Elĭ67]): $W^{k,p}(\Sigma, M)$ has a natural smooth Banach manifold structure such that for each $f \in C^\infty(\Sigma, M)$ and each choice of connection on M (for defining the exponential map), there is a chart of the form

$$W^{k,p}(\Sigma, M) \ni \exp_f \eta \mapsto \eta \in W^{k,p}(f^*TM)$$

identifying open subsets of $W^{k,p}(\Sigma, M)$ with open neighborhoods of 0 in the Banach space of sections $W^{k,p}(f^*TM)$. Moreover, for each $u \in W^{k,p}(\Sigma, M)$ there is a natural isomorphism $T_u W^{k,p}(\Sigma, M) = W^{k,p}(u^*TM)$. (Here $u^*TM \to \Sigma$ is a vector bundle of class $W^{k,p}$, for which sections of class $W^{m,p}$ can be defined for any $m \leqslant k$ since the condition $kp > 2$ implies there is a continuous product pairing $W^{k,p} \times W^{m,p} \to W^{m,p}$.)
Proof of smoothness: transition maps take the form $\eta \mapsto F \circ \eta$ for smooth fiber-preserving maps F. (See the "useful lemma" in Lecture 11.)
- The Banach space bundle $\mathcal{E} \to \mathcal{B}$ with fibers

$$\mathcal{E}_u = W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$$

and smooth section

$$\bar{\partial}_J : \mathcal{B} \to \mathcal{E} : u \mapsto Tu + J \circ Tu \circ j,$$

such that $\widetilde{\mathcal{M}} = \bar{\partial}_J^{-1}(0)$. (By elliptic regularity, the C^∞ and $W^{k,p}$-topologies on this set match.)
- The linearization

$$D\bar{\partial}_J(u) = \mathbf{D}_u : T_u\mathcal{B} \to \mathcal{E}_u : \eta \mapsto \nabla\eta + J \circ \nabla\eta \circ j + \nabla_\eta J \circ Tu \circ j$$

(∇ any symmetric connection on M) at a zero $u \in \bar{\partial}_J^{-1}(0)$. By Riemann-Roch, $\mathbf{D}_u$ is a Fredholm operator with

$$\mathrm{ind}(\mathbf{D}_u) = n\chi(\Sigma) + 2c_1(u^*TM) = n(2 - 2g) + 2c_1(A),$$

where $c_1(A) := \langle c_1(TM), A \rangle$.
- Proposition (via the implicit function theorem):
(1) If $\mathbf{D}_u$ is surjective, then a neighborhood of u in $\widetilde{\mathcal{M}}$ is a smooth submanifold of $\mathcal{B}$ with finite dimension $n\chi(\Sigma) + 2c_1(A)$, and $\mathrm{ev} : \widetilde{\mathcal{M}} \to M^{\times m}$ on this neighborhood is a smooth map. (Note: the implicit function theorem applies because surjective + Fredholm $\Rightarrow$ there is a bounded right inverse.)
(2) Let

$$W^{k,p}_\zeta = W^{k,p}_\zeta(u^*TM) := \left\{ \eta \in W^{k,p}(u^*TM) \;\middle|\; \eta(\zeta_i) = 0 \text{ for each } i = 1, \dots, m \right\}.$$

If the restricted operator $W^{k,p}_\zeta \xrightarrow{\mathbf{D}_u} \mathcal{E}_u$ is also surjective, then $\mathrm{ev} : \widetilde{\mathcal{M}} \to M^{\times m}$ is a submersion near u.
Smoothness of ev follows mainly from the fact that for any $f \in C^\infty(\Sigma, M)$, since $kp > 2$,

$$W^{k,p}(f^*TM) \to T_{f(\zeta_1)}M \times \ldots \times T_{f(\zeta_m)}M : \eta \mapsto (\eta(\zeta_1), \dots, \eta(\zeta_m))$$

is a continuous linear (and therefore smooth) map.

- Taking $(M, J) := (\Sigma, j)$ and $A := [\Sigma] \in H_2(\Sigma)$ gives the automorphism groups
$$\mathrm{Aut}(\Sigma, j) = \bar{\partial}_j^{-1}(0)$$
and
$$\mathrm{Aut}(\Sigma, j, \zeta) = \mathrm{ev}^{-1}(\zeta_1, \ldots, \zeta_m) \subset \bar{\partial}_j^{-1}(0).$$
- Linear Cauchy-Riemann operator on $T\Sigma$:
$$\mathbf{D}_\Sigma := D\bar{\partial}_j(\mathrm{Id}) : W^{k,p}(T\Sigma) = T_{\mathrm{Id}}\mathcal{B} \to \mathcal{E}_{\mathrm{Id}} = W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$$
and the restriction
$$\mathbf{D}_{\Sigma,\zeta} := \mathbf{D}_\Sigma\big|_{W^{k,p}_\zeta} : W^{k,p}_\zeta(T\Sigma) \to W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)).$$
- Theorem: $\mathrm{Aut}(\Sigma, j, \zeta)$ is a Lie group with Lie algebra
$$\mathfrak{aut}(\Sigma, j, \zeta) = \ker \mathbf{D}_{\Sigma,\zeta} \subset \Gamma_\zeta(T\Sigma) := \left\{ X \in \Gamma(T\Sigma) \;\middle|\; X(\zeta_i) = 0 \text{ for } i = 1, \ldots, m \right\} \subset W^{k,p}_\zeta(T\Sigma),$$
and moreover, it is discrete if (Σ, j, ζ) is stable.
Corollary (due to the proper action of $\mathrm{Diff}(\Sigma, \zeta)$ on $\mathcal{J}(\Sigma)$): In the stable cases, $\mathrm{Aut}(\Sigma, j, \zeta)$ is finite.
Proof: Each case follows from the implicit/inverse function theorem after bounding the dimension of $\ker \mathbf{D}_{\Sigma,\zeta}$, which one does by using the similarity principle to prove that
$$\ker \mathbf{D}_{\Sigma,\zeta} \to T_{w_1}\Sigma \times \ldots \times T_{w_N}\Sigma : X \mapsto (X(w_1), \ldots, X(w_N))$$
is injective for suitable finite sets of distinct points $w_1, \ldots, w_N \in \Sigma$, depending on $c_1(T\Sigma) = \chi(\Sigma)$.
Case $g = 0$ with $m \leqslant 3$: $\dim \ker \mathbf{D}_{\Sigma,\zeta} \leqslant 2(3 - m) = \mathrm{ind}\, \mathbf{D}_{\Sigma,\zeta}$ implies $\mathbf{D}_{\Sigma,\zeta}$ is surjective.
Case $2g + m \geqslant 3$ (stable): $\mathbf{D}_{\Sigma,\zeta}$ is injective.
Case $g = 1$ and $m = 0$: $\dim \ker \mathbf{D}_{\Sigma,\zeta} \leqslant 2$, and this must be an equality since $\mathrm{Aut}(\mathbb{T}^2, j)$ always contains a 2-dimensional family of translations.

Lecture 18 (4.01.2023): Teichmüller slices and Fredholm regular curves.

- If $\chi(\Sigma \backslash \zeta) < 0$, then the action of the identity component $\mathrm{Diff}_0(\Sigma, \zeta) \subset \mathrm{Diff}(\Sigma, \zeta)$ on $\mathcal{J}(\Sigma)$ is free (as well as proper).
Proof: If $\varphi \neq \mathrm{Id} \in \mathrm{Diff}(\Sigma, \zeta)$ is biholomorphic and homotopic to the identity, then it has $\#\,\mathrm{Fix}(\varphi) = \chi(\Sigma)$ by the Lefschetz fixed point theorem, and its fixed points are isolated and count positively. Then $\chi(\Sigma) \geqslant m$ since φ fixes the marked points.
- Definition: **Teichmüller space** of genus g Riemann surfaces with m marked points:
$$\mathcal{T}(\Sigma, \zeta) := \mathcal{J}(\Sigma)/\mathrm{Diff}_0(\Sigma, \zeta).$$
The **mapping class group** of (Σ, ζ) is the discrete group
$$M(\Sigma, \zeta) := \mathrm{Diff}(\Sigma, \zeta)/\mathrm{Diff}_0(\Sigma, \zeta),$$
and we have
$$\mathcal{M}_{g,m} \cong \mathcal{J}(\Sigma)/\mathrm{Diff}(\Sigma, \zeta) = \mathcal{T}(\Sigma, \zeta)/M(\Sigma, \zeta).$$
- A d-dimensional **orbifold** is a space X such that for every $x \in X$, there is a finite group G_x (the **isotropy** group at x) with a linear action on $\mathbb{R}^d$ and a G_x-invariant open subset $\mathcal{U} \subset \mathbb{R}^d$ such that some neighborhood of x in X is homeomorphic to $\mathcal{U}/G_x$. With some care (though the correct definitions are a bit non-obvious), one can also speak of *smooth orbifolds*. A slight enhancement of the usual slice theorem for free and proper group actions then gives quotients M/G natural smooth orbifold structures whenever G is a Lie group acting smoothly and properly on M with finite stabilizer subgroups at every point. We will see below that $\mathcal{T}(\Sigma, \zeta)$ is a smooth finite-dimensional manifold, so presenting $\mathcal{M}_{g,m}$

as the quotient of this manifold by a proper action of the discrete group $M(\Sigma,\zeta)$ makes $\mathcal{M}_{g,m}$ a smooth orbifold of the same dimension.

- Informally, think of $\mathrm{Diff}_0(\Sigma,\zeta)$ as an infinite-dimensional Lie group acting freely and properly on the infinite-dimensional manifold $\mathcal{J}(\Sigma)$, and the orbits $\mathrm{Diff}_0(\Sigma,\zeta)\cdot j$ are then infinite-dimensional submanifolds of $\mathcal{J}(\Sigma)$. Recalling the Cauchy-Riemann operator $\mathbf{D}_{\Sigma,\zeta} : \Gamma_\zeta(T\Sigma) \to \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$, we then have
$$T_j\left(\mathrm{Diff}_0(\Sigma,\zeta)\cdot j\right) = \operatorname{im}\mathbf{D}_{\Sigma,\zeta} \subset \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)) = T_j\mathcal{J}(\Sigma).$$
Proof: for a smooth family $\{\varphi_\tau \in \mathrm{Diff}_0(\Sigma,\zeta)\}_{\tau\in(-\epsilon,\epsilon)}$ with $\varphi_0 = \mathrm{Id}$ and $j_\tau := \varphi_\tau^* j$, covariantly differentiate the expression
$$\bar\partial_{j_\tau,j}(\varphi_\tau) := T\varphi_\tau + j\circ T\varphi_\tau\circ j_\tau = 0$$
at $\tau = 0$.
- Teichmüller slice theorem: $\mathcal{T}(\Sigma,\zeta)$ naturally admits the structure of a smooth finite-dimensional manifold such that for each $j\in\mathcal{J}(\Sigma)$, there is a natural isomorphism
$$T_{[j]}\mathcal{T}(\Sigma,\zeta) = \operatorname{coker}\mathbf{D}_{\Sigma,\zeta}.$$
Moreover:
  1. For any finite-dimensional smoothly embedded[10] family $\mathcal{T}\subset\mathcal{J}(\Sigma)$ of complex structures containing $j\in\mathcal{J}(\Sigma)$ such that $T_j\mathcal{T}\subset T_j\mathcal{J}(\Sigma) = \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ is complementary to $\operatorname{im}\mathbf{D}_{\Sigma,\zeta}$, the map
$$\mathcal{T}\to\mathcal{T}(\Sigma,\zeta) : j' \mapsto [j']$$
is a local diffeomorphism near j. (We refer to any family with this property as a **Teichmüller slice through** j.)
  2. The Teichmüller slice $\mathcal{T}\subset\mathcal{J}(\Sigma)$ through j can always be chosen to have the following additional properties:
     - (a) $\mathcal{T}$ is invariant under the action of $\mathrm{Aut}(\Sigma,j,\zeta)$ by $\varphi\cdot j' := \varphi^* j'$.
     - (b) Every $j'\in\mathcal{T}$ is identical to j on some neighborhood of ζ.

Proof: (1) Imitate the proof of the finite-dimensional slice theorem, using suitable Sobolev completions of $\mathrm{Diff}_0(\Sigma,\zeta)$ and $\mathcal{J}(\Sigma)$ so that one can speak of Banach manifolds and use the inverse/implicit function theorems.
(2) Choosing any complement $T_j\mathcal{T}\subset\Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ of $\operatorname{im}\mathbf{D}_{\Sigma,\zeta}$ gives rise to a Teichmüller slice of the form
$$\mathcal{T} := \left\{ \left(\mathbb{1} + \frac{1}{2}jy\right) j \left(\mathbb{1}+\frac{1}{2}jy\right)^{-1} \;\middle|\; y\in T_j\mathcal{T} \text{ close to } 0\right\}.$$
If $G := \mathrm{Aut}(\Sigma,j,\zeta)$ is finite, we can choose a G-invariant L^2-pairing on $\Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ and define $T_j\mathcal{T}$ to be the G-invariant L^2-orthogonal complement of $\operatorname{im}\mathbf{D}_{\Sigma,\zeta}$, then modify it by an L^p-small change so that every $y\in T_j\mathcal{T}$ vanishes on some small G-invariant neighborhood of ζ.
- Corollary: For all $j\in\mathcal{J}(\Sigma)$,
$$\dim\mathrm{Aut}(\Sigma,j,\zeta) - \dim\mathcal{T}(\Sigma,\zeta) = \operatorname{ind}(\mathbf{D}_{\Sigma,\zeta}) = \operatorname{ind}(\mathbf{D}_\Sigma) - 2m = 3\chi(\Sigma) - 2m. \tag{10.1}$$
In particular, if (Σ,j,ζ) is stable, then $\dim\mathcal{T}(\Sigma,\zeta) = 6g-6+2m$.

[10]By "finite-dimensional smoothly embedded family", we really mean a smooth family $\{j_\tau\in\mathcal{J}(\Sigma)\}_{\tau\in X}$ parametrized by a finite-dimensional manifold X, such that the map $X\to\mathcal{J}(\Sigma):\tau\mapsto j_\tau$ is injective and for each $\tau\in X$, the linearization $T_\tau X\to\Gamma(\overline{\mathrm{End}}(T\Sigma,j_\tau)) : v\mapsto \partial_s j_{\gamma(s)}\big|_{s=0}$ defined by choosing a smooth path $\gamma(s)\in X$ with $\dot\gamma(0) = v$ is also injective. The image of this injective linear map is what we call the "tangent space" of $\mathcal{T}$ at j_τ and denote by $T_{j_\tau}\mathcal{T}\subset\Gamma(\overline{\mathrm{End}}(T\Sigma,j_\tau))$.

- Remark: If $\mathcal{T} \subset \mathcal{J}(\Sigma)$ is chosen to be $\mathrm{Aut}(\Sigma, j, \zeta)$-invariant, then in the stable case, it follows that the map
$$\mathcal{T}/\mathrm{Aut}(\Sigma, j, \zeta) \to \mathcal{M}_{g,m} : [j'] \mapsto [(\Sigma, j', \zeta)]$$
is a local homeomorphism near $[j]$; this is one way of defining "orbifold charts" for $\mathcal{M}_{g,m}$.
- Local structure of the moduli space $\mathcal{M}_{g,m}(J, A)$: given $[(\Sigma, j_0, \zeta, u_0)] \in \mathcal{M}_{g,m}(J, A)$, set $G := \mathrm{Aut}(\Sigma, j_0, \zeta)$, choose a G-invariant Teichmüller slice $\mathcal{T} \subset \mathcal{T}(\Sigma)$ through j_0, and define $\mathcal{B} \subset W^{k,p}(\Sigma, M)$ as the component with $[u] = A$. There is a Banach space bundle $\mathcal{E} \to \mathcal{T} \times \mathcal{B}$ with fibers
$$\mathcal{E}_{(j,u)} = W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}((T\Sigma, j), (u^*TM, J)))$$
and a smooth section
$$\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \to \mathcal{E} : (j, u) \mapsto Tu + J \circ Tu \circ j$$
that is G-equivariant: $\bar{\partial}_J(\varphi^* j, u \circ \varphi) = \varphi^* \bar{\partial}_J(j, u)$ for $\varphi \in G$. Its linearization at $(j, u) \in \bar{\partial}_J^{-1}(0)$ is
$$D\bar{\partial}_J(j, u) : T_j\mathcal{T} \times T_u\mathcal{B} \to \mathcal{E}_{(j,u)} : (y, \eta) \mapsto \mathbf{D}_u\eta + J \circ Tu \circ y.$$
Since $\mathbf{D}_u$ is Fredholm and $T_j\mathcal{J}$ is finite-dimensional, this is a Fredholm operator of index
$$\mathrm{ind}\, D\bar{\partial}_J(j, u) = \mathrm{ind}(\mathbf{D}_u) + \dim \mathcal{T}(\Sigma, \zeta). \tag{10.2}$$
- Definition: For $(j, u) \in \bar{\partial}_J^{-1}(0)$, the element $[(\Sigma, j, \zeta, u)] \in \mathcal{M}_{g,m}(J, A)$ is called **Fredholm regular** if the operator $D\bar{\partial}_J(j, u) : T_j\mathcal{T} \times T_u\mathcal{B} \to \mathcal{E}_{(j,u)}$ is surjective. (See Exercise 10.6 below on why this does not depend on the various choices.) The set of Fredholm regular curves defines an open (though possibly empty) subset
$$\mathcal{M}^{\mathrm{reg}}_{g,m}(J, A) \subset \mathcal{M}_{g,m}(J, A).$$
- Lemma (based on the slice theorem): The map
$$\bar{\partial}_J^{-1}(0)/G \to \mathcal{M}_{g,m}(J, A) : [(j, u)] \mapsto [(\Sigma, j, \zeta, u)]$$
is a local homeomorphism near $[(j_0, u_0)]$.
- Theorem (by the implicit function theorem and slice theorem): $\mathcal{M}^{\mathrm{reg}}_{g,m}(J, A)$ admits a natural smooth orbifold structure of finite dimension
$$\dim \mathcal{M}^{\mathrm{reg}}_{g,m}(J, A) = \text{vir-dim}\, \mathcal{M}_{g,m}(J, A) := (n - 3)(2 - 2g) + 2c_1(A) + 2m,$$
which is also called the **virtual dimension** of $\mathcal{M}_{g,m}(J, A)$.[11]
Proof of the dimension formula: using (10.1) and (10.2) and the Riemann-Roch formula for $\mathrm{ind}(\mathbf{D}_u)$, we have
$$\begin{aligned}\dim\left(\bar{\partial}_J^{-1}(0)/G\right) &= \mathrm{ind}\, D\bar{\partial}_J(j, u) - \dim \mathrm{Aut}(\Sigma, j_0, \zeta) = \mathrm{ind}(\mathbf{D}_u) - \mathrm{ind}(\mathbf{D}_{\Sigma,\zeta}) \\ &= n\chi(\Sigma) + 2c_1(A) - 3\chi(\Sigma) + 2m = (n - 3)\chi(\Sigma) + 2c_1(A) + 2m.\end{aligned}$$

[11]Philosophically: the virtual dimension of a moduli space is an integer determined by topological conditions that can be interpreted as the dimension that the space "wants" to have, and will have in particular whenever certain transversality conditions are satisfied. For example, if M is a smooth n-manifold and $f : M \to \mathbb{R}^k$ is a smooth map, then the set $f^{-1}(0)$ has virtual dimension $n - k$, and is actually a smooth manifold of that dimension if 0 is a regular value. (Note that if the latter is not the case, then $f^{-1}(0)$ may fail to be smooth, or it may coincidentally be a smooth manifold but with dimension larger than its virtual dimension.)

Suggested reading. As I mentioned in class, anyone who gets serious about using Banach manifolds like $W^{k,p}(\Sigma, M)$ in research should read the paper [Elĭ67] *exactly* once. Otherwise, most of what we covered this week about Teichmüller slices, Fredholm regularity and the local structure of $\mathcal{M}^{\text{reg}}_{g,m}(J, A)$ can be found in [Wena, §4.3] and (in a more general context) [Wenc, Lecture 7]; in particular, these sources go into more detail on the slightly tedious issue of verifying that all the charts we construct on $\mathcal{T}(\Sigma, \zeta)$ and $\mathcal{M}^{\text{reg}}_{g,m}(J, A)$ are smoothly compatible.

It's natural to be curious about orbifolds when you see them for the first time, but for now I'm going to refrain from recommending anything to read about them, because as soon as one wants to discuss fundamental notions like smooth maps between smooth orbifolds, the basic definitions become more complicated than one would expect, and not all sources agree completely on what these definitions should be. It has been generally agreed in recent years that the most elegant approach is to recast the definition of an orbifold in the language of *proper étale groupoids*, which are a class of categories, so that maps between them should be regarded as functors, and if that kind of language doesn't make you nervous, feel free to google for more. In this course, most of the orbifolds we have to think about will actually turn out to be manifolds, thus it will not matter.

Exercises (for the Übung on 11.01.2023).

Exercise 10.1 (requested several weeks ago by Naageswaran)**.** We saw this week how to define smooth structures on moduli spaces by presenting them as zero-sets of smooth nonlinear Fredholm sections of infinite-dimensional Banach space bundles. The definitions of those bundles always require some choices, e.g. the Sobolev parameters k and p satisfying $kp > 2$, and ideally, one would also like to know that the smooth structures inherited by our moduli space do not depend on those choices. Consider in particular the space $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}(j, J, A)$ of parametrized J-holomorphic curves, which we saw can be identified with a smooth finite-dimensional submanifold

$$\widetilde{\mathcal{M}}^{k,p} := \bar{\partial}_J^{-1}(0) \subset W^{k,p}(\Sigma, M)$$

whenever the Fredholm operator $\mathbf{D}_u : W^{k,p}(u^*TM) \to W^{k-1,p}(\overline{\text{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ is known to be surjective. We know already from elliptic regularity theory that the index and kernel of $\mathbf{D}_u$ do not depend on k and p, thus neither does the surjectivity condition. Assuming this condition holds, show that for any $k, m \in \mathbb{N}$ and $p, q \in (1, \infty)$ satisfying

$$k \geqslant m, \qquad p \leqslant q \qquad \text{and} \qquad k - \frac{2}{p} \geqslant m - \frac{2}{q} > 0,$$

one has $\widetilde{\mathcal{M}}^{k,p} \subset \widetilde{\mathcal{M}}^{m,q}$ and the inclusion $\widetilde{\mathcal{M}}^{k,p} \hookrightarrow \widetilde{\mathcal{M}}^{m,q}$ is a diffeomorphism.

Exercise 10.2. For the Riemann surface $(\Sigma, j) = (S^2, i)$ with $m \leqslant 2$ marked points ζ, use results from complex analysis to determine the group $\text{Aut}(\Sigma, j, \zeta)$ explicitly, and compare its dimension with the formula we computed in lecture for $\dim \ker \mathbf{D}_{\Sigma,\zeta}$.

Exercise 10.3. The uniformization theorem implies that $\mathcal{M}_{0,m}$ is a one-point space for $m \leqslant 3$, and with a little knowledge of the mapping class group on the sphere, one can show that the corresponding Teichmüller space is also trivial. But here is a way to prove that without any knowledge of uniformization: show that for any smooth family $\{j_\tau \in \mathcal{J}(S^2)\}_{\tau \in [0,1]}$ with $j_0 = i$, the space

$$\widetilde{\mathcal{M}} := \left\{(\tau, \varphi) \in [0, 1] \times C^\infty(S^2, S^2) \;\middle|\; \varphi : (S^2, i) \to (S^2, j_\tau) \text{ biholomorphic with } \varphi(\zeta) = \zeta \text{ for } \zeta = 0, 1, \infty\right\}$$

is a compact connected 1-manifold with a natural smooth structure such that the map $\widetilde{\mathcal{M}} \to [0, 1] : (\tau, \varphi) \mapsto \tau$ is a diffeomorphism. (Why does that imply the claim above about Teichmüller space?)

Exercise 10.4 (harder, but worthwhile)**.** Our proof in lecture of the Teichmüller slice theorem focused on the stable case, but the theorem is true in all cases. Fill in the gaps in the proof for the non-stable cases:

(a) $g = 0$ with $m \leqslant 2$ (thanks to Exercise 10.3 there is not much to do here)
(b) $g = 1$ with $m = 0$.

For the torus with no marked points, a good starting point is the corollary of uniformization that $(\mathbb{T}^2, j)$ must always be biholomorphically equivalent to $(\mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z}), i)$ for some $\lambda \in \mathbb{H}$. Using the diffeomorphism

$$\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z}) : [a + ib] \mapsto [a + \lambda b], \qquad a, b \in \mathbb{R},$$

we can then identify $(\mathbb{C}/(\mathbb{Z} + \lambda\mathbb{Z}), i)$ with $(\mathbb{T}^2, j_\lambda)$ for a translation-invariant complex structure $j_\lambda \in \mathcal{J}(\mathbb{T}^2)$ determined by $\lambda \in \mathbb{H}$, thus defining a natural map

$$\mathbb{H} \to \mathcal{T}(\mathbb{T}^2) := \mathcal{T}(\mathbb{T}^2, \varnothing) : \lambda \mapsto [j_\lambda]$$

whose projection to $\mathcal{M}_{1,0} \cong \mathcal{T}(\mathbb{T}^2)/M(\mathbb{T}^2)$ is surjective. Show that the map $\mathbb{H} \to \mathcal{T}(\mathbb{T}^2)$ is in fact a homeomorphism, hence the family $\{j_\lambda\}_{\lambda \in \mathbb{H}}$ can be regarded as a *global* Teichmüller slice through any of its elements, and it is also invariant under their automorphisms.
Hint 1: It will help to have an explicit picture of the mapping class group of $\mathbb{T}^2$—classifical results imply that every isomorphism of $H_1(\mathbb{T}^2) = \mathbb{Z}^2$ to itself is induced by a unique mapping class on $\mathbb{T}^2$, thus giving an isomorphism $M(\mathbb{T}^2) \cong \mathrm{SL}(2, \mathbb{Z})$.
Hint 2: Every element of $\varphi \in \mathrm{Diff}_0(\mathbb{T}^2) := \mathrm{Diff}_0(\mathbb{T}^2, \varnothing)$ can be lifted to a diffeomorphism of $\mathbb{C}$ that (after composing with a translation) fixes the lattice $\mathbb{Z} + i\mathbb{Z}$.

Exercise 10.5. For the case $\chi(\Sigma\backslash\zeta) < 0$, compare the formula we computed in lecture for $\dim \mathcal{T}(\Sigma, \zeta)$ with the number of geodesics involved in an arbitrary pair-of-pants decomposition of (Σ, j, ζ). Then google the term "Fenchel-Nielsen coordinates".

Exercise 10.6. Prove that the notion of Fredholm regularity for an element $[(\Sigma, j, \zeta, u)] \in \mathcal{M}_{g,m}(J, A)$ does not depend on the various choices involved in the definition, notably the Sobolev parameters k, p and the Teichmüller slice $\mathcal{T} \subset \mathcal{J}(\Sigma)$ through j.
Hint: By definition, $T_j\mathcal{T}$ is complementary in $L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ to the image of the operator $\mathbf{D}_{\Sigma,\zeta} : W^{1,p}_\zeta(T\Sigma) \to L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$. Show that the image of the operator $D\bar{\partial}_J(j, u)(y, \eta) = \mathbf{D}_u\eta + J \circ Tu \circ y$ does not change if y is allowed to take arbitrary values in $L^p(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma))$ rather than just in the subspace $T_j\mathcal{T}$. This has something to do with reparametrizations of the map $u : \Sigma \to M$.

11. Week 11

Lecture 19 (10.01.2023): Simple curves and multiple covers.

- The automorphism group of an element $[(\Sigma, j, \zeta, u)] \in \mathcal{M}_{g,m}(J, A)$:
$$\mathrm{Aut}(u) := \left\{ \varphi \in \mathrm{Aut}(\Sigma, j, \zeta) \;\middle|\; u \circ \varphi = u \right\}.$$
- $\mathrm{Aut}(u)$ is finite if and only if $[(\Sigma, j, \zeta, u)]$ is stable; in particular, $\mathrm{Aut}(u)$ is finite whenever u is nonconstant. (follows from the factorization theorem below)
- Holomorphic maps $\varphi : (\Sigma, j) \to (\Sigma', j')$ of degree $\deg(\varphi) =: d \geqslant 0$ between closed Riemann surfaces of genera g and g' respectively:
  - $d = 0 \Leftrightarrow \varphi$ is constant
  - $d = 1 \Leftrightarrow \varphi$ is a biholomorphic map
  - $d \geqslant 2 \Leftrightarrow \varphi$ is a branched covering, and thus becomes a covering map of degree d after removing finitely many points from Σ and Σ': in particular, $|\mathrm{Aut}(\varphi)| \leqslant d$.

- $\varphi : (\Sigma, j) \to (\Sigma', j')$ has a branch point of branching order $k \geqslant 2$ at $\zeta \in \Sigma \Leftrightarrow d\varphi \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, \varphi^* T\Sigma'))$ has a zero of order $k-1$ at ζ; we define the algebraic count of branch points
$$Z(d\varphi) := \#(d\varphi)^{-1}(0) := \sum_{\zeta \in (d\varphi)^{-1}(0)} \mathrm{ord}(d\varphi; \zeta) \geqslant 0,$$
where equality holds if and only if φ is an honest covering map.
- Riemann-Hurwitz formula: $-\chi(\Sigma) + d\chi(\Sigma') = Z(d\varphi) \geqslant 0$, with equality iff there are no branch points.
Proof: Compute $c_1(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, \varphi^* T\Sigma'))$.
- For $[(\Sigma, j, \varnothing, \varphi)] \in \mathcal{M}_{g,0}(j', d[\Sigma'])$,
$$\text{vir-dim}\, \mathcal{M}_{g,0}(j', d[\Sigma']) = (1-3)\chi(\Sigma) + 2c_1(\varphi^* T\Sigma') = 2[-\chi(\Sigma) + d\chi(\Sigma')] = 2Z(d\varphi).$$
Interpretation (can be proved via classical methods): the values of the branch points of φ define local coordinates for $\mathcal{M}_{g,0}(j', d[\Sigma'])$
- Proposition: Every $[(\Sigma, j, \varnothing, \varphi)] \in \mathcal{M}_{g,0}(j', d[\Sigma'])$ is Fredholm regular, hence $\mathcal{M}_{g,0}(j', d[\Sigma'])$ is naturally a smooth orbifold of dimension $2[2g - 2 + d(2 - 2g')]$.
(Note: by the Riemann-Hurwitz formula, the space is empty if this integer is negative.)
Proof in case $Z(d\varphi) = 0$: Use the isomorphism $d\varphi : T\Sigma \to \varphi^* T\Sigma'$ to identify
$$D\bar{\partial}_{j'}(j, \varphi) : T_j\mathcal{T} \oplus W^{k,p}(\varphi^* T\Sigma') \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, \varphi^* T\Sigma') : (y, \eta) \mapsto \mathbf{D}_\varphi \eta + j' \circ T\varphi \circ y$$
with the operator
$$T_j\mathcal{T} \oplus W^{k,p}(T\Sigma) \to W^{k-1,p}(\overline{\mathrm{End}}_{\mathbb{C}}(T\Sigma)) : (y, X) \mapsto \mathbf{D}_\Sigma X + jy,$$
which is surjective by the definition of a Teichmüller slice.
- Definition: $u : (\Sigma, j) \to (M, J)$ is a d-fold **multiple cover** of $v : (\Sigma', j') \to (M, J)$ if $u = v \circ \varphi$ for some holomorphic branched cover $\varphi : (\Sigma, j) \to (\Sigma', j')$ of degree $d \geqslant 2$. We call u **simple** if it is not a multiple cover of any other curve.
- Factorization theorem: Every closed nonconstant J-holomorphic curve $u : (\Sigma, j) \to (M, J)$ factors as $u = v \circ \varphi$ for a closed simple curve $v : (\Sigma', j') \to (M, J)$ that is embedded after removing finitely many points $\Gamma \subset \Sigma'$ from its domain, and a nonconstant holomorphic map $\varphi : (\Sigma, j) \to (\Sigma', j')$. In particular, u is either simple (if $\deg(\varphi) = 1$) or it is a multiple cover $v \circ \varphi$ with **covering multiplicity**
$$\mathrm{cov}(u) = d := \deg(\varphi) \geqslant 2,$$
and $\mathrm{Aut}(u) = \mathrm{Aut}(\varphi)$ has order at most d.
- Local lemma 1 (intersections): Given an almost complex structure J on $\mathbb{C}^n$ and two nonconstant J-holomorphic maps $u, v : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ with $u(0) = v(0)$, there exist neighborhoods $\mathcal{U}, \mathcal{V} \subset \mathbb{D}$ of 0 such that either $u(\mathcal{U}) = v(\mathcal{V})$ or $u(\mathcal{U}\backslash\{0\}) \cap v(\mathcal{V}) = u(\mathcal{U}) \cap v(\mathcal{V}\backslash\{0\}) = \varnothing$.
Proof: Apply the similarity principle after choosing local coordinates very cleverly...
- Local lemma 2 (branching): Given an almost complex structure J on $\mathbb{C}^n$ and a nonconstant J-holomorphic map $u : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ with $u(0) = 0$ and $du(0) = 0$, one can biholomorphically reparametrize a neighborhood of 0 in $\mathbb{D}$ such that $u(z) = v(z^k)$ for some $k \in \mathbb{N}$ and an injective J-holomorphic map $v : (\mathbb{D}, i) \to (\mathbb{C}^n, J)$ with $dv(z) \neq 0$ for all $z \neq 0$.
Proof: Similarity principle again...
- Proof of the factorization theorem: the two lemmas imply that the sets $C := \{z \in \Sigma \mid du(z) = 0\}$ and
$$\Delta := \{z \in \Sigma \mid u(z) = u(w) \text{ for some } w \neq z \text{ and the intersection is isolated}\}$$

are both finite, and $\dot{\Sigma}' := u(\Sigma \backslash (C \cup \Delta))$ is then a smooth submanifold of M on which $J|_{T\dot{\Sigma}'}$ defines a complex structure j', and $(\dot{\Sigma}', j')$ is biholomorphically equivalent to the complement of a finite subset $\Gamma \subset \Sigma'$ in a closed Riemann surface (Σ', j'). Define $v : (\Sigma', j') \to (M, J)$ by extending the inclusion $\dot{\Sigma}' \hookrightarrow M$ over Γ, and define $\varphi : (\Sigma, j) \to (\Sigma', j')$ by extending $u : \Sigma \backslash (C \cup \Delta) \to \dot{\Sigma}'$.

- Transversality: $D\bar{\partial}_J(j, u)$ is surjective if and only if the intersection of $\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \to \mathcal{E}$ with the zero-section of $\mathcal{E} \to \mathcal{T} \times \mathcal{B}$ at (j, u) is transverse.
  Question: Can this intersection be made transverse by perturbing J?
- Bad news: $\bar{\partial}_J$ is defined to be equivariant under the action of some group $\mathrm{Aut}(\Sigma, j_0, \zeta)$, and this will remain true no matter how J is perturbed, i.e. the class of available perturbations of $\bar{\partial}_J$ is rather restrictive. This is a danger especially near points with nontrivial isotropy.
- The Calabi-Yau example: suppose $\dim M = 6$ and $c_1(A) = 0$, so $\text{vir-dim}\, \mathcal{M}_{g,0}(J, dA) = (n-3)(2-2g) + 2c_1(dA) = 0$ for every $g, d \geqslant 0$. For any element of $\mathcal{M}_{g',0}(J, A)$ parametrized by a simple curve $v : (\Sigma', j') \to (M, J)$ and any $d \geqslant 2$, the space $\mathcal{M}_{g,0}(J, dA)$ then contains the set
$$\left\{ u = v \circ \varphi \;\middle|\; \varphi \in \mathcal{M}_{g,0}(j', d[\Sigma']) \right\},$$
  which for sufficiently large $g > 0$ is a nonempty orbifold of dimension $2[2g-2+d(2-2g')] > 0$. So in this situation, $\mathcal{M}_{g,0}(J, dA)$ can *never* be an orbifold of dimension equal to its virtual dimension, and the multiple covers with branch points can never be regular.

Lecture 20 (11.01.2023): Generic transversality for simple curves.

- Main theorem: for generic J in $\mathcal{J}(M, \omega)$ or $\mathcal{J}_\tau(M, \omega)$ on a closed symplectic manifold (M, ω), all simple J-holomorphic curves are Fredholm regular.
- Terminology: for X a complete metric space, a subset $Y \subset X$ is **comeager** if it contains a countable intersection of open and dense subsets.
  (Baire category theorem: comeager implies dense. But density on its own is not good enough, because two dense subsets can easily have an empty intersection, whereas a countable intersection of comeager subsets is again comeager!)
  A statement depending on a choice of $x \in X$ is said to be true **for generic** $x \in X$ if there exists a comeager subset $Y \subset X$ such that it holds for all $x \in Y$.
- Sard-Smale theorem: Suppose X and Y are separable Banach manifolds of class C^k for some $k \geqslant 1$ and $f : X \to Y$ is a map of class C^k such that at every point $x \in X$, the tangent map $T_x f : T_x X \to T_{f(x)} Y$ is a Fredholm operator with $k \geqslant \mathrm{ind}(T_x f) + 1$. Then generic $y \in Y$ are regular values of f, meaning $T_x f$ is surjective for every $x \in f^{-1}(y)$.
- Definition: For a C^1-map $u : \Sigma \to M$, $z \in \Sigma$ is called an **injective point** of u if $T_z u : T_z \Sigma \to T_{u(z)} M$ is injective and $u^{-1}(u(z)) = \{z\}$. We call u **somewhere injective** if it admits an injective point.
  Yesterday's theorem implies: for a closed J-holomorphic curve, somewhere injective $\Leftrightarrow$ simple $\Leftrightarrow$ not multiply covered $\Leftrightarrow$ the set of injective points is open and dense.
- Technical version of the main theorem: fix $J^{\mathrm{fix}} \in \mathcal{J}(M, \omega)$ and an open subset $\mathcal{U} \subset M$ with compact closure (called the "perturbation domain"), define the complete metrizable space
$$\mathcal{J}^{\mathcal{U}} := \left\{ J \in \mathcal{J}(M, \omega) \;\middle|\; J = J^{\mathrm{fix}} \text{ on } M \backslash \mathcal{U} \right\},$$
  and for each $J \in \mathcal{J}^{\mathcal{U}}$ let
$$\mathcal{M}^{\mathcal{U}}(J) = \mathcal{M}^{\mathcal{U}}_{g,m}(J, A) \subset \mathcal{M}_{g,m}(J, A)$$
  denote the open set of curves $[(\Sigma, j, \zeta, u)] \in \mathcal{M}_{g,m}(J, A)$ such that $u : \Sigma \to M$ has an injective point $z \in \Sigma$ with $u(z) \in \mathcal{U}$. Then for generic $J \in \mathcal{J}^{\mathcal{U}}$, every curve in $\mathcal{M}^{\mathcal{U}}(J)$

is Fredholm regular, and $\mathcal{M}^{\mathcal{U}}(J)$ is therefore a smooth manifold[12] with dimension equal to its virtual dimension. Similar statements hold with $\mathcal{J}(M,\omega)$ replaced by $\mathcal{J}_\tau(M,\omega)$ or $\mathcal{J}(M)$.

- Proof "modulo technical hassles":
  - Step 1: Pretend that $\mathcal{J}^{\mathcal{U}}$ is a smooth Banach manifold with

$$T_J\mathcal{J}^{\mathcal{U}} = \left\{ Y \in \Gamma(\overline{\mathrm{End}}_{\mathbb{C}}(TM,J)) \;\middle|\; Y = 0 \text{ on } M\backslash\mathcal{U} \text{ and } \omega(Yv,w) + \omega(v,Yw) = 0 \text{ for all } v,w \right\}.$$

  The latter condition is the linearization of $\omega(Jv,Jw) = \omega(v,w)$, needed for ω-compatibility; we would remove it if we only need tameness or no symplectic condition. But of course, $\mathcal{J}^{\mathcal{U}}$ is *not* a Banach manifold in any natural way, as the space of smooth sections of a bundle with support in a fixed compact subset is at best a Fréchet space, not a Banach space.[13] We will ignore this issue for now and rectify it next week.
  - Step 2: Define the "universal" moduli space

$$\mathcal{M}^{\mathcal{U}} := \left\{ (u,J) \;\middle|\; J \in \mathcal{J}^{\mathcal{U}} \text{ and } u \in \mathcal{M}^{\mathcal{U}}(J) \right\}.$$

  - Step 3 (the main one): Prove that $\mathcal{M}^{\mathcal{U}}$ is a smooth (and separable) Banach manifold. This uses the implicit function theorem in roughly the same functional-analytic setup that we used for studying the space of Fredholm regular J-holomorphic curves for a fixed J. The main task is to show that for every $[(\Sigma,j,\zeta,u)] \in \mathcal{M}^{\mathcal{U}}(J)$, the linear operator

$$\begin{aligned}\mathbf{L} := D\bar{\partial}(j,u,J) : T_j\mathcal{T} \oplus W^{k,p}(u^*TM) \oplus T_J\mathcal{J}^{\mathcal{U}} &\to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)) \\ (y,\eta,Y) &\mapsto \mathbf{D}_u\eta + Y \circ Tu \circ j + J \circ Tu \circ y\end{aligned}$$

  is surjective. In fact, this holds even after restricting $\mathbf{L}$ to the set of triples (y,η,Y) where $y = 0$ and η vanishes at the marked points. Here the case $k \geqslant 2$ follows from the case $k = 1$ via elliptic regularity. For $k = 1$, one argues by contradiction using the Hahn-Banach theorem: if the operator is not surjective, then for $\frac{1}{p} + \frac{1}{q} = 1$ there is a nontrivial section $\alpha \in L^q(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ that is L^2-orthogonal to every $\mathbf{L}(0,\eta,Y)$, which means the two conditions

$$\begin{aligned}\langle \mathbf{D}_u\eta, \alpha\rangle_{L^2} &= 0 \qquad \text{for all } \eta \in W^{1,p}_\zeta(u^*TM), \\ \langle Y \circ Tu \circ j, \alpha\rangle_{L^2} &= 0 \qquad \text{for all } Y \in T_J\mathcal{J}^{\mathcal{U}}.\end{aligned}$$

  The first implies that α is a weak solution of class L^q_{loc} to $\mathbf{D}_u^*\alpha = 0$ on $\Sigma\backslash\zeta$, thus it is smooth on this region and (by the similarity principle) has only isolated zeroes. Choosing an injective point $z_0 \in \Sigma\backslash\zeta$ of u with $u(z_0) \in \mathcal{U}$ and $\alpha(z_0) \neq 0$, one can then find $Y \in T_j\mathcal{J}^{\mathcal{U}}$ with support near $u(z_0)$ such that the second condition is violated.
  Note: This last part is the only detail that depends on our restriction to ω-compatible almost complex structures, as $Y \in T_J\mathcal{J}^{\mathcal{U}}$ needs to satisfy an extra condition to ensure compatibility. If we don't care about compatibility, this condition is dropped and step 3 becomes slightly easier.
  - Step 4: The projection $\pi : \mathcal{M}^{\mathcal{U}} \to \mathcal{J}^{\mathcal{U}} : (u,J) \mapsto J$ is a smooth map, and its derivative at each point (u,J) is Fredholm, and surjective if and only if u is a Fredholm regular curve. This follows from an algebraic exercise: suppose $D : X \to Z$ and

[12] We are saying "manifold" instead of "orbifold" here because all curves in $\mathcal{M}^{\mathcal{U}}(J)$ are simple and thus have trivial automorphism groups. To put it another way, the implicit function theorem in this situation identifies $\mathcal{M}^{\mathcal{U}}(J)$ locally with the quotient of $\bar{\partial}_J^{-1}(0)$ by an action that is both proper and free.

[13] Fréchet spaces are nice objects that arise very naturally in applications, but their usefulness suffers from the fact that there is no Banach fixed point theorem, and thus no inverse or implicit function theorem.

$A : Y \to Z$ are linear maps between (possibly infinite-dimensional) vector spaces and the map $L : X \oplus Y \to Z : (x, y) \mapsto Dx + Ay$ is surjective. Then the projection $\ker L \to Y : (x, y) \mapsto y$ has kernel and cokernel naturally isomorphic to those of D.

  – Step 5: The Sard-Smale theorem implies that generic $J \in \mathcal{J}^{\mathcal{U}}$ are regular values of the projection $\pi : \mathcal{M}^{\mathcal{U}} \to \mathcal{J}^{\mathcal{U}}$, and by step 4, all $u \in \mathcal{M}^{\mathcal{U}}(J)$ for these J are Fredholm regular.

- Enhancement: Consider the evaluation map $\mathrm{ev} : \mathcal{M}_{g,m}(J, A) \to M^{\times m}$ and **forgetful map** $\Phi : \mathcal{M}_{g,m}(J, A) \to \mathcal{M}_{g,m} : [(\Sigma, j, \zeta, u)] \mapsto [(\Sigma, j, \zeta)]$, both of which can also be defined on the universal moduli space $\mathcal{M}^{\mathcal{U}}$. In the local picture of this space as a zero-set $\bar\partial^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}^{\mathcal{U}}$, the map $(\mathrm{ev}, \Phi) : \mathcal{M}^{\mathcal{U}} \to M^{\times m} \times \mathcal{M}_{g,m}$ then looks like
$$\mathcal{T} \times \mathcal{B} \times \mathcal{J}^{\mathcal{U}} \supset \bar\partial^{-1}(0) \to M^{\times m} \times \mathcal{T} : (j, u, J) \mapsto (u(\zeta_1), \ldots, u(\zeta_m), j),$$
with derivative
$$\ker D\bar\partial(j, u, J) \to T_{u(\zeta_1)}M \times \ldots \times T_{u(\zeta_m)}M \times T_j\mathcal{T} : (y, \eta, Y) \mapsto (\eta(\zeta_1), \ldots, \eta(\zeta_m), y).$$
The latter is surjective since we showed that $D\bar\partial(j, u, J)$ is surjective on triples with $y = 0$ and η vanishing at the marked points, implying that $(\mathrm{ev}, \Phi) : \mathcal{M}^{\mathcal{U}} \to M^{\times m} \times \mathcal{M}_{g,m}$ is a submersion. Given any submanifold $Z \subset M^{\times m} \times \mathcal{M}_{g,m}$, we can now replace $\mathcal{M}^{\mathcal{U}}$ in the main argument above with the finite-codimensional submanifold $(\mathrm{ev}, \Phi)^{-1}(Z) \subset \mathcal{M}^{\mathcal{U}}$ and use it to prove (again modulo technical hassles to be dealt with next time):
- Corollary: For any submanifold $Z \subset M^{\times m} \times \mathcal{M}_{g,m}$, there exists a comeager subset $\mathcal{J}^{\mathcal{U}}_Z \subset \mathcal{J}^{\mathcal{U}}$ such that for all $J \in \mathcal{J}^{\mathcal{U}}_Z$, all $u \in \mathcal{M}^{\mathcal{U}}(J)$ are Fredholm regular and the map $(\mathrm{ev}, \Phi) : \mathcal{M}^{\mathcal{U}}(J) \to M^{\times m} \times \mathcal{M}_{g,m}$ is transverse to Z, so in particular, the constrained moduli space
$$\mathcal{M}^{\mathcal{U}}(J; Z) := \left\{ u \in \mathcal{M}^{\mathcal{U}}(J) \;\middle|\; (\mathrm{ev}(u), \Phi(u)) \in Z \right\}$$
is a smooth submanifold of $\mathcal{M}^{\mathcal{U}}(J)$ with codimension equal to the codimension of $Z \subset M^{\times m} \times \mathcal{M}_{g,m}$.
Caution: In this statement, the space $\mathcal{J}^{\mathcal{U}}_Z$ depends on the choice of submanifold Z, and we cannot find a single comeager set of J's that achieves transversality of $(\mathrm{ev}, \Phi) : \mathcal{M}^{\mathcal{U}}(J) \to M^{\times M} \times \mathcal{M}_{g,m}$ to all submanifolds Z.

Suggested reading. The factorization theorem on simple and multiply covered curves is proved in [Wena, §2.15]; a nearly identical proof is also in [MS12]. The main technical work underlying this result consists of the "local lemmas" 1 and 2, and for these there are at least two approaches one can take: McDuff-Salamon base their exposition on a deep local formula due to Micallef and White [MW95] for the structure of a nonconstant J-holomorphic curve (or more generally a minimal surface) in the neighborhood of a point where its derivative vanishes. Appendix E (written with Laurent Lazzarini) of [MS12] contains an exposition of this formula. Alternatively, there is an "approximate" version of the Micallef-White formula that suffices for our applications and has a technically easier proof; this is the approach taken in [Wena], and the local results appear in §2.14, but with the minor problem that the proof given there is not correct. (I will get around to fixing this someday.) A fully correct version does appear however in Appendix B of the book [Wen20b], which is also available for free on the arXiv.

The general version of the theorem that all holomorphic branched covers of Riemann surfaces are Fredholm regular is proved in [Wen10, Example 3.16], where it is derived from a more general setup that was developed for proving transversality results for punctured J-holomorphic curves in dimension four. The main technical steps in the argument are really Lemma 3.14 and 3.15 in that paper, but I don't really recommend reading this now unless you are intensely curious about it, as it will take a while to understand the setup.

The generic transversality argument for simple J-holomorphic curves is quite standard, and you will find very similar treatments of it in [MS12], [Wena, §4.4.1] and (in a more general context) [Wenc, Lecture 8]. Appendix A.5 of [MS12] also includes a concise proof of the Sard-Smale theorem (reducing it to the finite-dimensional Sard's theorem), which was originally proved in [Sma65].

Exercises (for the Übung on 18.01.2023).

Exercise 11.1. For a nodal J-holomorphic curve $\mathbf{u} = [(S, j, \zeta, u, \Delta)]$ of arithmetic genus $g \geqslant 0$ with $m \geqslant 0$ marked points, the automorphism group $\mathrm{Aut}(\mathbf{u})$ is defined to be the group of all self-equivalences, in other words, biholomorphic maps $\varphi : (S, j) \to (S, j)$ that fix each marked point, map nodes to nodes (not necessarily preserving any order but preserving their grouping into pairs) and satisfy $u \circ \varphi = u$. (Note that S may in general be disconnected and φ is not required to preserve any connected components that don't have marked points!) Show that $\mathrm{Aut}(\mathbf{u})$ is finite if and only if $\mathbf{u}$ is stable.

Exercise 11.2. The moduli space $\mathcal{M}_{g,0}(j', d[\Sigma'])$ of degree $d \geqslant 0$ holomorphic maps $\varphi : (\Sigma, j) \to (\Sigma', j')$ (up to parametrization) between two Riemann surfaces of genera g and g' respectively sounds like a wonderful object when you hear that its elements are always Fredholm regular, so that it is always a smooth orbifold of the correct dimension. However, $\mathcal{M}_{g,0}(j', d[\Sigma'])$ seems less wonderful when you look at its compactification $\overline{\mathcal{M}}_{g,0}(j', d[\Sigma'])$. Show that for $g > g'$ and $d = 1$, $\mathcal{M}_{g,0}(j', [\Sigma'])$ is an *empty* moduli space with a positive virtual dimension, and its compactification $\overline{\mathcal{M}}_{g,0}(j', [\Sigma'])$ is nonempty. This contradicts the tempting intuition that $\mathcal{M}_{g,m}(J, A)$ should always be an open and dense subset of $\overline{\mathcal{M}}_{g,m}(J, A)$.

Exercise 11.3. Assume (M, J) is a $2n$-dimensional almost complex manifold, $A \in H_2(M)$, and $v : (\Sigma, j) \to (M, J)$ represents a Fredholm regular element of the moduli space $\mathcal{M}_{g,0}(J, A)$. Prove:

(a) For any sequence $J_k \in \mathcal{J}(M)$ converging in the C^∞-topology to J, there exists for sufficiently large k a sequence of J_k-holomorphic curves $v_k : (\Sigma, j_k) \to (M, J_k)$ such that $j_k \to j$ and $v_k \to v$ in C^∞.
Hint: Use the implicit function theorem in infinite dimensions.

(b) If $g = 0$, $n \geqslant 4$ and $\text{vir-dim}\, \mathcal{M}_{0,0}(J, A) = 0$, then for all $J' \in \mathcal{J}(M)$ sufficiently C^∞-close to J, there exist elements of $\mathcal{M}_{0,0}(J', dA)$ for $d > 1$ that are not Fredholm regular.

Exercise 11.4. Assume (M, ω) is a closed symplectic manifold.

(a) Prove that if $\dim M \geqslant 6$, then for generic $J \in \mathcal{J}(M, \omega)$, generic elements $[(\Sigma, j, \zeta, u)]$ of the moduli space of all simple J-holomorphic curves have the property that the map $u : \Sigma \to M$ is injective. Show in particular that for generic J, all simple curves lying in moduli spaces of virtual dimension 0 are injective.
Hint: For each $g, m \geqslant 0$ and $A \in H_2(M)$, consider elements $u \in \mathcal{M}_{g,m+2}(J, A)$ satisfying the constraint that the last two marked points evaluate to the same point in M. Estimate the dimension of this set, and compare it with $\text{vir-dim}\, \mathcal{M}_{g,m}(J, A)$.

(b) Prove similarly that if $\dim M \geqslant 6$, then for generic $J \in \mathcal{J}(M, \omega)$, generic pairs of elements of the moduli space of simple J-holomorphic curves have disjoint images.

(c) Do you think the statement in part (b) is likely to be true for $\dim M = 4$? Consider for example $M = \mathbb{CP}^2$.

(d) Show that if $\dim M \geqslant 4$, then for generic $J \in \mathcal{J}(M, \omega)$, generic elements $[(\Sigma, j, \zeta, u)]$ of the moduli space of all simple J-holomorphic curves have the property that the map $u : \Sigma \to M$ has no *triple points*, meaning self-intersections $u(z_1) = u(z_2) = u(z_3)$ such that the points $z_1, z_2, z_3 \in \Sigma$ are all distinct.

(e) Show that the statement in part (d) is false for $\dim M = 2$.

Exercise 11.5. Fix a closed Riemann surface (Σ, j) of genus $g \geqslant 2$, an almost complex manifold (M, J) of dimension $2n$, and a homology class $A \in H_2(M)$ that is *primitive*, meaning it is not dB for any integer $d \geqslant 2$ and $B \in H_2(M)$. Let $\widetilde{\mathcal{M}}(j, J, A)$ denote the moduli space of *parametrized* J-holomorphic maps $u : (\Sigma, j) \to (M, J)$ with $[u] = A$; here the word "parametrized" means that different maps are different elements of $\widetilde{\mathcal{M}}(j, J, A)$ even if they are related to each other by reparametrization.

(a) How is $\widetilde{\mathcal{M}}(j, J, A)$ related to the moduli space $\mathcal{M}_{g,0}(J, A)$ of *unparametrized* J-holomorphic curves with the same genus and homology class (and no marked points)? Express your answer in terms of the forgetful map $\Phi : \mathcal{M}_{g,0}(J, A) \to \mathcal{M}_{g,0}$.
(b) Prove that for generic J, $\widetilde{\mathcal{M}}(j, J, A)$ is a smooth manifold of dimension $n(2-2g)+2c_1(A)$. If the latter is negative, does it follow that $\mathcal{M}_{g,0}(J, A)$ is empty?
(c) Assuming $n \geqslant 3$, find an optimal constant $C(n) \geqslant 0$ such that if $n(2-2g)+2c_1(A) \leqslant C(n)$, then for generic J, every map in $\widetilde{\mathcal{M}}(j, J, A)$ is injective. You may use as a black box the following fact (which can be proved via an intelligent construction of Teichmüller slices): for every $g, m \geqslant 0$ with $2g + m \geqslant 3$, the projection $\mathcal{M}_{g,m+1} \to \mathcal{M}_{g,m}$ defined by forgetting the final marked point is a submersion.

12. Week 12

Lecture 21 (17.01.2023): Technicalities on transversality.

- Two remedies for the fact that $\mathcal{J}^{\mathcal{U}}$ (with the C^∞-topology) is not a Banach manifold:
  (1) Use almost complex structures of class C^k for some $k < \infty$ large (e.g. [MS12] does this, but then $\bar\partial_J$ is no longer smooth, thus neither are the moduli spaces, and one must carefully keep track of how many derivatives exist)
  (2) Use the "Floer C_ε-space"
- Definition: For $\varepsilon \in S := \{\text{sequences } \{\varepsilon_m > 0\}_{m=0}^\infty \text{ with } \varepsilon_m \to 0\}$ and a vector bundle $E \to M$ over a compact manifold, define the separable Banach space
$$C_\varepsilon(E) := \left\{ \eta \in \Gamma(E) \;\middle|\; \|\eta\|_{C_\varepsilon} := \sum_{m=0}^\infty \varepsilon_m \|\eta\|_{C^m} < \infty \right\}.$$
  Note: All norms in this discussion depend on various choices, and while the C^m-topologies are independent of those choices, the C_ε-topology (and the space $C_\varepsilon(E)$ itself) is not. We just need to accept that.
- Properties: Define a partial order on S by saying $\varepsilon \leqslant \varepsilon'$ if and only if there is a constant $C > 0$ such that $\varepsilon_m \leqslant C\varepsilon'_m$ for all $m \in \mathbb{N}$.
  (1) There are continuous inclusions $C_\varepsilon(E) \hookrightarrow \Gamma(E)$ (the latter with the C^∞-topology) for every $\varepsilon \in S$
  (2) There is a continuous inclusion $C_{\varepsilon'}(E) \hookrightarrow C_\varepsilon(E)$ whenever $\varepsilon \leqslant \varepsilon'$
  (3) $\bigcup_{\varepsilon \in S} C_\varepsilon(E) = \Gamma(E)$
  (4) Every countable subset of S has a lower bound in S.
  Corollary: Any countable subset of $\Gamma(E)$ is contained in $C_\varepsilon(E)$ for some $\varepsilon \in S$.
- Rigorous proof of the main transversality theorem from last time:
  – Given any $J^{\text{ref}} \in \mathcal{J}^{\mathcal{U}}$, define a smooth Banach manifold (with one chart) consisting of C_ε-perturbations of J^{ref}:
$$\mathcal{J}^{\mathcal{U}}_\varepsilon := \left\{ \left(\mathbb{1} + \frac{1}{2} J^{\text{ref}} Y\right) J^{\text{ref}} \left(\mathbb{1} + \frac{1}{2} J^{\text{ref}} Y\right)^{-1} \;\middle|\; Y \in T_{J^{\text{ref}}} \mathcal{J}^{\mathcal{U}}_\varepsilon \text{ with } \|Y\|_{C^0} < \delta \right\}$$

for suitably small $\delta > 0$, where

$$T_{J^{\mathrm{ref}}}\mathcal{J}^{\mathcal{U}}_\varepsilon := \left\{ Y \in T_{J^{\mathrm{ref}}}\mathcal{J}^{\mathcal{U}} \;\middle|\; \|Y\|_{C_\varepsilon} < \infty \right\}.$$

Now there is a continuous inclusion $\mathcal{J}^{\mathcal{U}}_\varepsilon \hookrightarrow \mathcal{J}^{\mathcal{U}}$, and a smooth section $\bar\partial : \mathcal{T}\times\mathcal{B}\times\mathcal{J}^{\mathcal{U}}_\varepsilon \to \mathcal{E} : (j,u,J) \mapsto \bar\partial_J(j,u)$.

- Call $[(\Sigma, j, \zeta, u)] \in \mathcal{M}^{\mathcal{U}}(J)$ ε**-regular** if $D\bar\partial(j,u,J)$ is surjective on $T_j\mathcal{T}\oplus T_u\mathcal{B}\oplus T_J\mathcal{J}^{\mathcal{U}}_\varepsilon$. Since every smooth section is of class C_ε for some $\varepsilon \in S$, what we proved last time implies: Given $u \in \mathcal{M}^{\mathcal{U}}(J^{\mathrm{ref}})$, there exists $\varepsilon_0 \in S$ such that u is ε-regular for every $\varepsilon \leqslant \varepsilon_0$.
- Corollary (using the properties of C_ε listed above and the fact that $\mathcal{M}^{\mathcal{U}}(J^{\mathrm{ref}})$ is second countable): One can choose $\varepsilon \in S$ such that every $u \in \mathcal{M}^{\mathcal{U}}(J^{\mathrm{ref}})$ is ε-regular.
- Define the universal moduli space of ε-regular curves:

$$\mathcal{M}^{\mathcal{U}}_\varepsilon := \left\{ (u,J) \;\middle|\; J \in \mathcal{J}^{\mathcal{U}}_\varepsilon,\ u \in \mathcal{M}^{\mathcal{U}}(J), \text{ and } u \text{ is } \varepsilon\text{-regular} \right\}.$$

The implicit function theorem implies this is a smooth Banach manifold, so the projection $\mathcal{M}^{\mathcal{U}}_\varepsilon \to \mathcal{J}^{\mathcal{U}}_\varepsilon$ can be fed into the Sard-Smale theorem.
- Taubes trick: Prove that transversality can be achieved on a sequence of compact subsets exhausting $\mathcal{M}^{\mathcal{U}}(J)$. Choose compact subsets

$$\mathcal{J}_1(\Sigma) \subset \mathcal{J}_2(\Sigma) \subset \mathcal{J}_3(\Sigma) \subset \ldots \subset \mathcal{J}(\Sigma)$$

whose union projects onto $\mathcal{M}_{g,m}$ (possible because Teichmüller slices are finite-dimensional and thus locally compact), then define $\mathcal{M}^{\mathcal{U}}_N(J) \subset \mathcal{M}^{\mathcal{U}}(J)$ for each $N \in \mathbb{N}$ to consist of curves $[(\Sigma, j, \zeta, u)]$ satisfying closed conditions that prevent degenerations and define quantitative versions of the conditions defining $\mathcal{M}^{\mathcal{U}}(J)$:
  1. $j \in \mathcal{J}_N(\Sigma)$
  2. $\|du\|_{C^0} \leqslant N$
  3. There exists a point $z_0 \in \Sigma$ at which

$$\mathrm{dist}(u(z_0), M\backslash\mathcal{U}) \geqslant \frac{1}{N}, \quad |du(z_0)| \geqslant \frac{1}{N} \quad \text{and} \quad \inf_{z\in\Sigma\backslash\{z_0\}} \frac{\mathrm{dist}(u(z_0), u(z))}{\mathrm{dist}(z_0, z)} \geqslant \frac{1}{N}.$$

Then $\mathcal{M}^{\mathcal{U}}(J) = \bigcup_{N\in\mathbb{N}} \mathcal{M}^{\mathcal{U}}_N(J)$, and for any C^∞-convergent sequence $J_k \to J$, sequences $u_k \in \mathcal{M}^{\mathcal{U}}_N(J_k)$ have subsequences converging to elements of $\mathcal{M}^{\mathcal{U}}_N(J)$. Define

$$\mathcal{J}^{\mathcal{U}}_N := \left\{ J \in \mathcal{J}^{\mathcal{U}} \;\middle|\; \text{all } u \in \mathcal{M}^{\mathcal{U}}_N(J) \text{ are Fredholm regular} \right\} \subset \mathcal{J}^{\mathcal{U}}$$

and claim: $\mathcal{J}^{\mathcal{U}}_N$ is open and dense (in the C^∞-topology). Openness follows from the compactness statement above, and for density, it suffices to choose $J^{\mathrm{ref}} \in \mathcal{J}^{\mathcal{U}}_N$ arbitrarily and show that for any sequence $J_k \in \mathcal{J}^{\mathcal{U}}_\varepsilon$ converging to J^{ref} and consisting of regular values of the projection $\mathcal{M}^{\mathcal{U}}_\varepsilon \to \mathcal{J}^{\mathcal{U}}_\varepsilon$, every $u_k \in \mathcal{M}^{\mathcal{U}}(J_k)$ for $k \gg 1$ is Fredholm regular. This holds because a subsequence of u_k converges to something in $\mathcal{M}^{\mathcal{U}}(J^{\mathrm{ref}})$, which is ε-regular, implying that u_k is also ε-regular and therefore (as a regular point of the projection $\mathcal{M}^{\mathcal{U}}_\varepsilon \to \mathcal{J}^{\mathcal{U}}_\varepsilon$) Fredholm regular. Now

$$\bigcap_{N\in\mathbb{N}} \mathcal{J}^{\mathcal{U}}_N \subset \mathcal{J}^{\mathcal{U}}$$

is the comeager set we want.

- How to achieve $\bar\partial_J \pitchfork 0$ for all (not just somewhere injective) curves: inhomogeneous perturbations! Given $J \in \mathcal{J}(M)$, define the vector bundle $P \to \mathcal{T}\times\Sigma\times M$ with fibers $P_{(j,z,p)} = \overline{\mathrm{Hom}}_{\mathbb{C}}((T_z\Sigma, j), (T_pM, J))$, so any $K \in \Gamma(E)$ determines a section $\nu : \mathcal{T}\times\mathcal{B} \to \mathcal{E}$

by $\nu(j,u)(z) := K(j,z,u(z))$. We then consider the **inhomogeneous nonlinear Cauchy-Riemann equation**

$$\bar\partial_J(j,u) = \nu(j,u), \qquad \text{i.e.} \qquad \bar\partial_{J,\nu}(j,u) := \bar\partial_J(j,u) - \nu(j,u) = 0.$$

- The linearization of $\bar\partial_{J,\nu}$ at $(j,u) \in \bar\partial_{J,\nu}^{-1}(0)$ with respect to $\eta \in T_u\mathcal{B}$ is another linear Cauchy-Riemann type operator $\mathbf{D}_u^\nu$ on u^*TM.
- Proposition: For generic $K \in \Gamma(P)$, $\bar\partial_{J,\nu}$ is transverse to the zero-section everywhere. Proof: Same idea as before, but easier because the perturbation $K(j,z,u(z))$ depends explicitly on z and not just $u(z)$.

Lecture 22 (18.01.2023): Gluing.

- Goal: Describe a neighborhood of $U_0 := [(S, j_0, \zeta, \Delta, u_0)]$ in $\overline{\mathcal{M}}_{g,m}(J,A)$. In particular, must this neighborhood contain a sequence of smooth curves in $\mathcal{M}_{g,m}(J,A)$ degenerating to U_0? (Exercise 11.2 shows that this is not always true.)
- Choose a Teichmüller slice $\mathcal{T} \subset \mathcal{J}(S)$ through j_0 (i.e. a product of Teichmüller slices for each connected component of S) with $\zeta \cup \Delta$ regarded as the set of marked points, and let

  $$W^{k,p}_\Delta(u_0^*TM) := \left\{\eta \in W^{k,p}(u_0^*TM) \;\middle|\; \eta(z^+) = \eta(z^-) \text{ for each node } \{z^+, z^-\} \in \Delta\right\}.$$

  Definition: The nodal curve U_0 is **Fredholm regular** if the restriction of the linearization $D\bar\partial_J(j_0,u_0)$ to the domain $T_{j_0}\mathcal{T} \oplus W^{k,p}_\Delta(u_0^*TM)$ is surjective onto $W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(TS, u_0^*TM))$.
- Meaning: Suppose U_0 has N nodes, denoted by $\Delta = \left\{\{z_1^+, z_1^-\}, \ldots, \{z_N^+, z_N^-\}\right\}$, and let $\mathcal{B}$ denote the Banach manifold $W^{k,p}(S,M)$. Fredholm regularity of U_0 then means that all component curves in U_0 are Fredholm regular (hence $\bar\partial_J^{-1}(0) \subset \mathcal{T} \times \mathcal{B}$ is a smooth finite-dimensional manifold near (j_0,u_0)) and, additionally, the map

  $$\mathrm{ev}_\Delta : \bar\partial_J^{-1}(0) \to M^{\times 2N} : (j,u) \mapsto (u(z_1^+), u(z_1^-), \ldots, u(z_N^+), u(z_N^-))$$

  is transverse at (j_0,u_0) to the submanifold

  $$\mathrm{Diag} := \left\{(p_1, p_1, \ldots, p_N, p_N) \;\middle|\; p_1, \ldots, p_N \in M\right\} \subset M^{\times 2N}.$$

  The set $\widetilde{\mathcal{M}}_\Delta := \mathrm{ev}_\Delta^{-1}(\mathrm{Diag}) \subset \bar\partial_J^{-1}(0)$ parametrizes the set of all other nodal curves in $\overline{\mathcal{M}}_{g,m}(J,A)$ near U_0 that also have N nodes, and it is then a manifold of dimension

  $$\dim \widetilde{\mathcal{M}}_\Delta = \mathrm{ind}\, D\bar\partial_J(j_0,u_0) - 2nN$$

- Dimensional comparison: assume for simplicity that all domains in this discussion are stable, hence their automorphism groups are finite. One then computes:

  $$\dim \widetilde{\mathcal{M}}_\Delta = \text{vir-dim}\, \mathcal{M}_{g,m}(J,A) - 2N \qquad \text{and} \qquad \dim \mathcal{T} = \dim \mathcal{M}_{g,m} - 2N.$$

  Upshot: If we want to see all smooth curves near $U_0 \in \overline{\mathcal{M}}_{g,m}(J,A)$ or all smooth Riemann surfaces close to $[(S, j_0, \zeta, \Delta)] \in \overline{\mathcal{M}}_{g,m}$, we need to add 2 extra "gluing parameters" for each node.
- Define the space of **gluing parameters**

  $$\Gamma := ([0,\infty) \times S^1)^{\times N} \ni (R_1, \theta_1, \ldots, R_N, \theta_N) =: \gamma$$

  and its compactification $\overline{\Gamma} := \left(\overline{[0,\infty) \times S^1}\right)^{\times N}$ where $\overline{[0,\infty) \times S^1} := \left([0,\infty] \times S^1\right) / \left(\{\infty\} \times S^1\right)$ is topologically a 2-disk, its center identified with the quotient of $\{\infty\} \times S^1$. Denote the point with $R_1 = \ldots = R_N = \infty$ by $\overline{\Gamma}^\infty \in \overline{\Gamma}$.

- Pre-gluing of Riemann surfaces: Assuming all $j \in \mathcal{T}$ are identical near Δ, associate to each $(j, \gamma) \in \mathcal{T} \times \Gamma$ a smooth Riemann surface $(\Sigma_\gamma, j_\gamma)$ of genus g, defined by cutting out of (S, j) small disk-like neighborhoods of the nodal points $z_i^\pm$, then connecting the boundaries of neighborhoods of z_i^+ and z_i^- by necks $[-R_i, R_i] \times S^1$ and "twisting" each neck by $\theta_i \in S^1$. (In other words: after choosing holomorphic cylindrical coordinates near $z_i^\pm$, θ_i represents the S^1-freedom in how to glue $[-R_i, R_i] \times S^1$ biholomorphically to these neighborhoods.) The resulting map
$$\mathcal{T} \times \Gamma \to \mathcal{M}_{g,m} : (j, \gamma) \mapsto [(\Sigma_\gamma, j_\gamma, \zeta)]$$
has a natural continuous extension to $\mathcal{T} \times \overline{\Gamma} \to \overline{\mathcal{M}}_{g,m}$ whose image is onto a neighborhood of $[(S, j_0, \zeta, \Delta)] \in \overline{\mathcal{M}}_{g,m}$. (For this extension, the node $\{z_i^+, z_i^-\} \in \Delta$ is replaced by a neck of length $2R_i$ whenever $R_i < \infty$, but the node is left intact when $R_i = \infty$.)
- Pre-gluing of J-holomorphic curves: Associate to each $(j, u, \gamma) \in \widetilde{\mathcal{M}}_\Delta \times \Gamma$ a smooth and *approximately* J-holomorphic map $u_\gamma : (\Sigma_\gamma, j_\gamma) \to (M, J)$, defined such that $u_\gamma = u$ outside the disk-like neighborhoods of nodal points, and on each neck $[-R_i, R_i] \times S^1$, u_γ interpolates via a smooth cutoff function between $u|_{\mathrm{nbhd}(z_i^+)}$ and $u|_{\mathrm{nbhd}(z_i^-)}$. There is also a natural extension to $\gamma \in \overline{\Gamma}$ that leaves the node $\{z_i^+, z_i^-\}$ intact whenever $R_i = \infty$.
- Main technical lemma ("gluing estimates"): One can choose a family of norms (depending on the gluing parameters) such that:
  (1) $\|\bar\partial_J(j_\gamma, u_\gamma)\| \to 0$ as $\gamma \to \overline{\Gamma}^\infty$
  (2) If U_0 is Fredholm regular, then the linearization[14] $\nabla\bar\partial_J(j_\gamma, u_\gamma)$ is surjective for all γ close enough to $\overline{\Gamma}^\infty$ and has a right inverse that is bounded uniformly as $\gamma \to \overline{\Gamma}^\infty$.

  Idea of the proof: For an intelligent choice of norms, one views the operators $\nabla\bar\partial_J(j_\gamma, u_\gamma)$ for $\gamma \to \overline{\Gamma}^\infty$ as converging (in some generalized sense) to the restriction of $D\bar\partial_J(j_0, u_0)$ that was required to be surjective in the definition of Fredholm regularity.
- Main gluing theorem: If U_0 is Fredholm regular,[15] then the pre-glued family of approximately J-holomorphic curves $u_\gamma : (\Sigma_\gamma, j_\gamma) \to (M, J)$ can be modified for $\gamma \in \overline{\Gamma}$ near $\overline{\Gamma}^\infty$ to a family of *exactly* J-holomorphic curves
$$u'_\gamma : (\Sigma_{\gamma'}, j'_{\gamma'}) \to (M, J),$$
where $\gamma \in \overline{\Gamma}$, $j' \in \mathcal{T}$ and u'_γ are all small perturbations of γ, j, u_γ respectively, with the size of the perturbation becoming arbitrarily small as $\gamma \to \overline{\Gamma}^\infty$. Moreover, the image of the resulting map
$$\widetilde{\mathcal{M}}_\Delta \times \overline{\Gamma} \to \overline{\mathcal{M}}_{g,m}(J, A)$$
contains a neighborhood of U_0.

  Idea of the proof: Restricting $\nabla\bar\partial_J(j_\gamma, u_\gamma)$ to a subspace complementary to its kernel gives (for γ close to $\overline{\Gamma}^\infty$) an isomorphism Q satisfying an injectivity estimate of the form $\|Qv\| \geqslant c\|v\|$ for a constant $c > 0$ that is (thanks to the gluing estimate) independent of (j, u, γ). Since $\bar\partial_J(j_\gamma, u_\gamma)$ is small, a quantitative version of the inverse function theorem can then be used to show that (j_γ, u_γ) admits a unique perturbation in the direction of this subspace that hits the zero-set of $\bar\partial_J$.
- Recall: For $[(\Sigma, j, \zeta)] \in \mathcal{M}_{g,m}$, each $\varphi \in \mathrm{Aut}(\Sigma, j, \zeta)$ represents a Fredholm regular element of the moduli space $\mathcal{M}_{g,m}(j, [\Sigma])$, which has virtual dimension $2m$, and this element is also a transverse intersection of $\mathrm{ev} : \mathcal{M}_{g,m}(j, [\Sigma]) \to \Sigma^{\times m}$ with the one-point submanifold

[14] Here we are writing $\nabla\bar\partial_J(j_\gamma, u_\gamma)$ instead of $D\bar\partial_J(j_\gamma, u_\gamma)$ because in general $(j_\gamma, u_\gamma) \notin \bar\partial_J^{-1}(0)$, thus the linearization depends on a choice of connection.

[15] In the lecture I neglected to include this hypothesis, but it really is quite important.

$\{(\zeta_1, \ldots, \zeta_m)\}$. It follows (via the implicit function theorem) that for any choice of Teichmüller slice $\mathcal{T} \subset \mathcal{J}(\Sigma)$ parametrizing $\mathcal{T}(\Sigma, \zeta)$ near j, every $j' \in \mathcal{J}(\Sigma)$ near j determines a unique $j'' \in \mathcal{T}$ near j and $\varphi' \in \mathrm{Diff}(\Sigma, \zeta)$ near φ for which $\varphi' : (\Sigma, j'') \to (\Sigma, j')$ is biholomorphic. (Note that if $\mathcal{T}$ is chosen to be $\mathrm{Aut}(\Sigma, j, \zeta)$-invariant, the uniqueness in this statement implies that $\varphi' = \varphi$ whenever $j' \in \mathcal{T}$.)

One can similarly exploit the Fredholm regularity of automorphisms and plug them into a variation on the gluing construction above, giving:

- Automorphism gluing theorem:[16] Assume the Teichmüller slice $\mathcal{T} \subset \mathcal{J}(S)$ is chosen to be $\mathrm{Aut}(S, j_0, \zeta, \Delta)$-invariant. Then every $\varphi \in \mathrm{Aut}(S, j_0, \zeta, \Delta)$ uniquely determines a family of biholomorphic maps
$$\varphi_{j,\gamma} : (\Sigma_{\gamma'}, j'_{\gamma'}) \to (\Sigma_\gamma, j_\gamma), \qquad (j, \gamma) \in \mathcal{T} \times \Gamma$$
fixing the marked points ζ, such that the map $\varphi_{j,\gamma}$ and the parameters $\gamma' \in \Gamma$ and $j' \in \mathcal{T}$ each depend smoothly on (j, γ). Moreover, there is a natural continuous extension of this family allowing $\gamma \in \overline{\Gamma}$, which defines a continuous family of equivalences between marked nodal Riemann surfaces, and any sequence φ_k of such equivalences converging to φ as $k \to \infty$ can be realized via this construction for $k \gg 1$ and a unique sequence of parameters $(j_k, \gamma_k) \to (j_0, \overline{\Gamma}^\infty) \in \mathcal{T} \times \overline{\Gamma}$.
- Corollary 1: The natural action of $\mathrm{Aut}(S, j_0, \zeta, \Delta)$ on $\widetilde{\mathcal{M}}_\Delta$ by $\varphi \cdot (j, u) := (\varphi^* j, u \circ \varphi)$ extends to an action on a neighborhood of $\widetilde{\mathcal{M}}_\Delta \times \{\overline{\Gamma}^\infty\}$ in $\widetilde{\mathcal{M}}_\Delta \times \overline{\Gamma}$ such that if U_0 is Fredholm regular, the gluing map $\widetilde{\mathcal{M}}_\Delta \times \overline{\Gamma} \to \overline{\mathcal{M}}_{g,m}(J, A)$ descends to the quotient as a homeomorphism
$$\left(\widetilde{\mathcal{M}}_\Delta \times \overline{\Gamma}\right) \Big/ \mathrm{Aut}(S, j_0, \zeta, \Delta) \to \overline{\mathcal{M}}_{g,m}(J, A)$$
from a neighborhood of $[(j_0, u_0, \overline{\Gamma}^\infty)]$ to a neighborhood of U_0. In particular, the open set of Fredholm regular stable nodal J-holomorphic curves in $\overline{\mathcal{M}}_{g,m}(J, A)$ is naturally a topological orbifold[17] with isotropy group $\mathrm{Aut}(U_0)$ at U_0.
- Corollary 2: For $2g + m \geqslant 3$, the Deligne-Mumford space $\overline{\mathcal{M}}_{g,m}$ is a compact topological orbifold with isotropy group $\mathrm{Aut}(S, j, \zeta, \Delta)$ at $[(S, j, \zeta, \Delta)] \in \overline{\mathcal{M}}_{g,m}$.
(Remark: You can view this either as a corollary of Corollary 1 above, or as a direct consequence of the automorphism gluing theorem, as the latter presents a neighborhood of $[(S, j_0, \zeta, \Delta)]$ in $\overline{\mathcal{M}}_{g,m}$ as the quotient of $\mathcal{T} \times \overline{\Gamma}$ by $\mathrm{Aut}(S, j_0, \zeta, \Delta)$. While it does not follow immediately from our gluing construction, one can show in fact that $\overline{\mathcal{M}}_{g,m}$ has a natural smooth structure compatible with the smooth orbifold structure we already had on the open and dense subset $\mathcal{M}_{g,m} \subset \overline{\mathcal{M}}_{g,m}$. It also has a natural complex structure and is thus a complex orbifold, whose tangent spaces all have natural identifications with the cokernels of certain complex-linear Cauchy-Riemann operators.)
- Nice corollary of Corollary 2 (via Deligne-Mumford compactness): For each $g, m \geqslant 0$ with $2g + m \geqslant 3$, there exists a universal bound on the orders of the automorphism groups of genus g Riemann surfaces with m marked points.

[16] I'm no longer sure, but I suspect that I slightly misstated this theorem and Corollary 1 when I presented them in lecture. I'm trying very hard to produce correct statements in this writeup, even if I leave the proofs somewhat to your imagination.

[17] I am not making any claims about the smoothness of this orbifold, because there is no obvious way to control the smoothness of transition maps as gluing parameters go to infinity. This issue has occasionally been a source of controversy among symplectic topologists.

Suggested reading. For the basic properties of the Floer C_ε-space (e.g. why it is separable), see [Wenc, Appendix B]. The method by which I used it in lecture to fix the proof of generic transversality is described in the blog post [Wen]; unfortunately I did not have the correct understanding of this method until relatively recently, and thus have not gotten around to updating any of my books-in-progress accordingly.

If you want to see how generic transversality is proven using finitely-differentiable almost complex structures, see [MS12, §3.2]. The so-called "Taubes trick" also appears in their argument, but for a different purpose: it is used in order to turn a statement about generic C^k-smooth almost complex structures into one in which everything is C^∞.

The inhomogeneous nonlinear Cauchy-Riemann equation doesn't appear in any of my lecture notes, but is discussed in [MS12, Chapter 8].

I wish I could give you a good reference for the general gluing theorems we sketched in lecture, but I really can't; in this form, they are essentially folk theorems. Various similar but more specific and technical gluing theorems can be found (with full gory details) in various places: e.g. the construction of quantum cohomology requires a theorem about the gluing of two J-holomorphic spheres with a domain-dependent almost complex structure, so this is proved in [MS12, Chapter 10]. If you are at all familiar with Floer homology, then it is also worth looking at the gluing theorem for two rigid Floer cylinders explained in [AD14], since this result is somewhat simpler to state (if not to prove). In every case, complete proofs require quite careful definitions of parametrized families of norms and a fairly long sequence of estimates; getting all the details right is a pain in the neck, though the main idea is not so hard to understand.

Exercises (for the Übung on 25.01.2023). *Update (27.01.2023): I have added written solutions to these exercises to make up for the cancellation of the problem session on 25.01.*

Exercise 12.1. The following trick is used instead of inhomogeneous perturbations for the construction in [MS12] of the genus 0 Gromov-Witten invariants of semipositive symplectic manifolds. Given a closed Riemann surface (Σ, j), a **domain-dependent almost complex structure** on a manifold M is a smooth function J on $\Sigma \times M$ whose value at each point $(z,p) \in \Sigma \times M$ is a complex structure $J(z,p) : T_pM \to T_pM$; equivalently, J is a family of almost complex structures $J(z,\cdot) \in \mathcal{J}(M)$ smoothly parametrized by $z \in \Sigma$. If (M,ω) is a symplectic $2n$-manifold, we can define $\mathcal{J}^\Sigma_\tau(M,\omega)$ to be the space of smooth domain-dependent almost complex structures such that $J(z,\cdot)$ is ω-tame for every $z \in \Sigma$; it is easily shown that this space is contractible. For $J \in \mathcal{J}^\Sigma_\tau(M,\omega)$, a smooth map $u : \Sigma \to M$ is then called J-holomorphic if its derivative at every point $z \in \Sigma$ is a complex linear map from $(T_z\Sigma, j)$ to $(T_{u(z)}M, J(z,u(z)))$, so in local holomorphic coordinates (s,t) on some region in Σ, the nonlinear Cauchy-Riemann equation now becomes

$$\partial_s u(s,t) + J(s,t,u(z))\, \partial_t u(s,t) = 0.$$

For $J \in \mathcal{J}^\Sigma_\tau(M,\omega)$ and $A \in H_2(M)$, let

$$\widetilde{\mathcal{M}}(j,J,A) \subset C^\infty(\Sigma, M)$$

denote the space of J-holomorphic maps $u : (\Sigma, j) \to (M,J)$ that satisfy $[u] := u_*[\Sigma] = A$. Notice that for most choices of $J \in \mathcal{J}^\Sigma_\tau(M,\omega)$, there is no meaningful notion of multiply-covered curves in $\widetilde{\mathcal{M}}(j,J,A)$: the composition of a J-holomorphic curve with a holomorphic branched cover of Riemann surfaces will not still be J-holomorphic for a typical domain-dependent J. For the same reason, there is no natural action of $\mathrm{Aut}(\Sigma, j)$ on $\widetilde{\mathcal{M}}(j,J,A)$ and no meaningful equivalence relation defined via biholomorphic reparametrization when J is domain-dependent.

Prove:

(a) For any $J \in \mathcal{J}^\Sigma_\tau(M,\omega)$, the set of J-holomorphic maps $u : (\Sigma, j) \to (M,J)$ satisfying $[u] = 0 \in H_2(M)$ is precisely the set of constant maps $\Sigma \to M$.

Solution:
For the same reasons as in the case of a domain-independent ω-tame almost complex structure, the 2-form $u^*\omega$ on Σ is everywhere nonnegative, and strictly positive wherever the derivative of u does not vanish. If $[u] = 0$, the computation $\int_\Sigma u^*\omega = \langle [\omega], [u] \rangle = 0$ thus implies that u is constant.

(b) For generic $J \in \mathcal{J}_\tau^\Sigma(M, \omega)$ and every $A \neq 0 \in H_2(M)$, $\widetilde{\mathcal{M}}(j, J, A)$ is a smooth manifold of dimension $n\chi(\Sigma) + 2c_1(A)$.

Solution:
One can set this up in the same way that we analyzed the local structure of $\mathcal{M}_{g,m}(J, A)$, but there is no need to worry about Teichmüller slices since j is fixed. The nonlinear Cauchy-Riemann operator is thus defined as a smooth section $\bar\partial_J : \mathcal{B} \to \mathcal{E}$ with $\mathcal{B} = W^{k,p}(\Sigma, M)$ and $\mathcal{E}_u = W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$, where in the case of domain-dependent J, one can take the complex structure on the bundle $u^*TM \to \Sigma$ to be given by $J(z, u(z))$ at each point $z \in \Sigma$. The linearization at $u \in \bar\partial_J^{-1}(0)$ is then a Cauchy-Riemann type operator $\mathbf{D}_u = D\bar\partial_J(u) : T_u\mathcal{B} \to \mathcal{E}_u$ on u^*TM, and thus has Fredholm index $n\chi(\Sigma) + 2c_1(u^*TM) = n\chi(\Sigma) + 2c_1(A)$. The only difference that the domain-dependence of J makes to this discussion is that if we choose a symmetric connection ∇ on M in order to write down the explicit formula

$$\mathbf{D}_u\eta(z) = \nabla\eta(z) + J(z, u(z)) \circ \nabla\eta(z) \circ j + \nabla_\eta J(z, u(z)) \circ Tu \circ j,$$

then the terms J and $\nabla_\eta J$ both depend explicit on $z \in \Sigma$ and not just on $u(z) \in M$.

Here is a sketch of the generic transversality proof "modulo technical hassles", i.e. pretending that certain Fréchet manifolds are actually Banach manifolds, a discrepancy that can as usual be repaired using C_ε-spaces. Let us pretend in particular that $\mathcal{J}_\tau^\Sigma(M, \omega)$ is a Banach manifold, and use it to define a universal moduli space

$$\mathcal{M}(\mathcal{J}) := \left\{ (u, J) \;\middle|\; J \in \mathcal{J}_\tau^\Sigma(M, \omega) \text{ and } u \in \widetilde{\mathcal{M}}(j, J, A) \right\}.$$

The main step is to show that $\mathcal{M}(\mathcal{J})$ is a smooth Banach manifold, and the rest of the argument then proceeds as usual via the Sard-Smale theorem and the Taubes trick. To prove smoothness of $\mathcal{M}(\mathcal{J})$, one needs to extend $\bar\partial_J$ in the obvious way to a section $\bar\partial$ defined on a bundle over $\mathcal{B} \times \mathcal{J}_\tau^\Sigma(M, \omega)$, whose linearization at each $(u, J) \in \bar\partial^{-1}(0)$ will then be an operator $\mathbf{L} : W^{k,p}(u^*TM) \oplus T_J\mathcal{J}_\tau^\Sigma(M, \omega) \to W^{k-1,p}(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ of the form

$$\mathbf{L}(\eta, Y) = \mathbf{D}_u\eta + Y \circ Tu \circ j.$$

The crucial difference between this and the situation we already considered in lecture is that Y is now allowed to depend explicitly on both $z \in \Sigma$ and $u(z) \in M$, i.e. Y can be any smooth function on $\Sigma \times M$ whose value at (z, p) is a linear map $Y(z, p) : T_pM \to T_pM$ that anticommutes with $J(z, p)$ and satisfies the additional condition $\omega(Yv, Jw) + \omega(Jv, Yw) = 0$ for all tangent vectors v, w; the latter is the result of linearizing the ω-compatibility condition $\omega(Jv, Jw) = \omega(v, w)$. Consider the case $k = 1$, from which the rest will as usual follow via elliptic regularity. We know that $\mathbf{L}$ has closed image since $\mathbf{D}_u$ is Fredholm, so if $\mathbf{L}$ is not surjective, there exists a nontrivial section $\alpha \in L^q(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$ for $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\begin{aligned} \langle \mathbf{D}_u\eta, \alpha \rangle_{L^2} &= 0 \qquad \text{for all } \eta \in W^{1,p}(u^*TM), \\ \langle Y \circ Tu \circ j, \alpha \rangle_{L^2} &= 0 \qquad \text{for all } Y \in T_J\mathcal{J}_\tau^\Sigma(M, \omega). \end{aligned}$$

The first condition implies via elliptic regularity that α is a smooth solution to $\mathbf{D}_u^*\alpha = 0$, so by the similarity principle, it has only isolated zeroes. The next step is where we needed to assume in lecture that $u : \Sigma \to M$ has an injective point, but the domain-dependence of Y makes that assumption unnecessary here. Instead, it suffices to be able to choose a point $z_0 \in \Sigma$ at which $\alpha(z_0) \neq 0$ and $T_{z_0}u \neq 0$, the existence of which is guaranteed if $A \neq 0$ since, by part (a), u is not constant. Indeed, we can then choose $Y \in T_J\mathcal{J}_\tau^\Sigma(M,\omega)$ such that $Y(z_0, u(z_0)) \circ T_{z_0}u \circ j$ has a positive inner product with $\alpha(z_0)$ and then multiply by a cutoff function depending only on $z \in \Sigma$ to make the integrand of $\langle Y \circ Tu \circ j, \alpha \rangle_{L^2}$ vanish outside an arbitrarily small neighborhood of z_0. This ensures $\langle Y \circ Tu \circ j, \alpha \rangle_{L^2} > 0$ and thus brings about a contradiction, proving that $\mathbf{L}$ is surjective.

(c) The statement about $\widetilde{\mathcal{M}}(j, J, A)$ in part (b) also holds for $A = 0 \in H_2(M)$ and all *time-independent* $J \in \mathcal{J}_\tau(M,\omega)$ if Σ has genus 0, but it does not hold for any $J \in \mathcal{J}_\tau^\Sigma(M,\omega)$ if Σ has positive genus.[18]

Solution:
If $J \in \mathcal{J}_\tau^\Sigma(M,\omega)$ and $u : \Sigma \to M$ is a constant map with value $p \in M$, then the complex vector bundle $u^*TM \to \Sigma$ has fiber $(T_pM, J(z,p))$ at each point $z \in \Sigma$, so it is a trivial bundle, and the linearized Cauchy-Riemann operator $\mathbf{D}_u$ takes the form

$$\mathbf{D}_u\eta(z) = d\eta(z) + J(z,p) \circ d\eta(z) \circ j.$$

Here we have written ordinary differentials instead of covariant derivatives since η is just a function $\Sigma \to T_pM$ with values in a fixed vector space. Since the bundle is trivial, the operator $\mathbf{D}_u$ has index $n\chi(\Sigma)$, which is nonpositive if Σ has genus $g > 0$, even though there clearly always exists a $2n$-dimensional family of constant J-holomorphic maps $\Sigma \to M$, implying that $\mathbf{D}_u$ cannot be surjective. On the other hand, if $g = 0$ and J is domain-independent, then choosing a complex basis of $(T_pM, J(p))$ identifies $\mathbf{D}_u$ with the standard Cauchy-Riemann operator $\bar\partial$ on the trivial bundle $S^2 \times \mathbb{C}^n \to S^2$. The kernel of this operator consists of the holomorphic functions $S^2 \to \mathbb{C}^n$, which are all constant since S^2 is compact, thus $\dim\ker\mathbf{D}_u = 2n = \operatorname{ind}\mathbf{D}_u$ and it follows that $\mathbf{D}_u$ is surjective.

Comment: The trouble with higher-genus curves in the homology class $0 \in H_2(M)$ is a reason to prefer inhomogeneous perturbations; cf. the next exercise.

Exercise 12.2. Fix a closed Riemann surface (Σ, j) of genus $g \geqslant 0$ with a point $z_0 \in \Sigma$, and assume (M,ω) is a closed symplectic $2n$-manifold with a fixed tame (and possibly domain-dependent) almost complex structure $J \in \mathcal{J}_\tau^\Sigma(M,\omega)$. Let $P \to \Sigma \times M$ denote the vector bundle whose fiber at $(z,p) \in \Sigma \times M$ is the space of complex-antilinear maps from $(T_z\Sigma, j)$ to $(T_pM, J(z,p))$. Any section $K \in \Gamma(P)$ then determines an inhomogeneous nonlinear Cauchy-Riemann equation for maps $u : \Sigma \to M$ in the form

$$\bar\partial_J u = \nu(u) \qquad \text{on } \Sigma,$$

where at $z \in \Sigma$, $\bar\partial_J u(z) := T_zu + J(z, u(z)) \circ T_zu \circ j \in P_{(z,u(z))}$ and $\nu(u)(z) := K(z, u(z))$. For $A \in H_2(M)$, let

$$\widetilde{\mathcal{M}}(j, J, K, A) \subset C^\infty(\Sigma, M)$$

denote the space of solutions $u : \Sigma \to M$ to this equation that satisfy $[u] = A$. Prove:

(a) For generic $K \in \Gamma(P)$ and every $A \in H_2(M)$, $\widetilde{\mathcal{M}}(j, J, K, A)$ is a smooth finite-dimensional manifold. What is its dimension?

[18] I have modified this statement slightly from the original version, which claimed that transversality also holds in the genus zero case for all domain-dependent J. On closer inspection, I don't think that's true.

Solution:
I'll skip to the crucial detail: the operator that needs to be surjective for each $u \in \widetilde{\mathcal{M}}(j, J, K, A)$ in order for the relevant universal moduli space to be smooth takes the form

$$\begin{aligned}\mathbf{L} : W^{1,p}(u^*TM) \oplus \Gamma(P) &\to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))\\ \mathbf{L}(\eta, Q)(z) &= \left(\mathbf{D}_u^\nu \eta\right)(z) - Q(z, u(z)),\end{aligned}$$

where $\mathbf{D}_u^\nu$ is a linear Cauchy-Riemann type operator on u^*TM. If $\mathbf{L}$ is not surjective, then there is a nontrivial section α of class L^q for $\frac{1}{p} + \frac{1}{q} = 1$ satisfying

$$\begin{aligned}\langle \mathbf{D}_u^\nu \eta, \alpha\rangle_{L^2} &= 0 \qquad \text{for all } \eta \in W^{1,p}(u^*TM),\\ \langle Q(\cdot, u), \alpha\rangle_{L^2} &= 0 \qquad \text{for all } Q \in \Gamma(P).\end{aligned}$$

As usual, the first condition implies via elliptic regularity and the similarity principle that α is smooth and has only isolated zeroes. One can then easily violate the second condition by a suitable choice of $Q(z, p)$ with support for z close to some point z_1 where $\alpha(z_1) \neq 0$; this does not require any assumptions at all about the map $u : \Sigma \to M$. By the usual Sard-Smale/C_ε-space/Taubes trick argument, it follows that $\widetilde{\mathcal{M}}(j, J, K, A)$ is a smooth manifold for generic $K \in \Gamma(P)$, and its dimension is the index of $\mathbf{D}_u^\nu$, which is $n\chi(\Sigma) + 2c_1(A)$.

(b) If $g = 0$ and $J \in \mathcal{J}_\tau(M, \omega)$ is domain-independent,[19] then there exists a neighborhood $\mathcal{U} \subset \Gamma(P)$ of 0 such that for every $p \in M$ and every $K \in \mathcal{U}$, $\widetilde{\mathcal{M}}(j, J, K, 0)$ contains a unique solution $u : \Sigma \to M$ with $u(z_0) = p$. In what circumstances will this solution be constant?

Solution:
We will deduce the result from the implicit function theorem after observing that for each $p \in M$, if $g = 0$, $K = 0$ and J is domain-independent, then the moduli space

$$\widetilde{\mathcal{M}}(j, J, K, 0\,;\, p) := \left\{ u \in \widetilde{\mathcal{M}}(j, J, K, 0) \;\middle|\; u(z_0) = p \right\}$$

is "cut out transversely," meaning the following. Since $K = 0$, $\widetilde{\mathcal{M}}(j, J, K, 0)$ contains only constant maps, and as we saw in Exercise 12.1, the linearized Cauchy-Riemann operator $\mathbf{D}_u$ at each of these maps can be identified with the standard operator $\bar\partial$ on a trivial bundle, which is surjective and has kernel consisting of all constant functions. Now pick $m \in \mathbb{N}$ large and consider the section

$$\bar\partial_{J,\nu} : \mathcal{B} \times C^m(P) \to \mathcal{E} : (u, K) \mapsto \bar\partial_J u - K(\cdot, u),$$

where $\mathcal{B} := W^{1,p}(\Sigma, M)$ and $\mathcal{E}_{(u,K)} := L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM))$. Since there is a continuous composition pairing $(F, u) \mapsto F \circ u \in W^{1,p}$ for $F \in C^1$ and $u \in W^{1,p}$, the section $\bar\partial_{J,\nu}$ is of class C^{m-1} due to the term $K(\cdot, u)$, so we need to assume at least $m \geqslant 2$ in order to apply the inverse and implicit function theorems. At $(u, K) \in \bar\partial_{J,\nu}^{-1}(0)$, its linearization takes the form

$$D\bar\partial_{J,\nu}(u, K) : W^{1,p}(u^*TM) \oplus C^m(P) \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TM)) : (\eta, Q) \mapsto \mathbf{D}_u^\nu \eta - Q(\cdot, u),$$

where $\mathbf{D}_u^\nu$ is a linear Cauchy-Riemann type operator on u^*TM. Whenever $K = 0$ and u is a constant map, $\mathbf{D}_u^\nu = \mathbf{D}_u$ has the aforementioned identification with the standard

[19] As with Exercise 12.1(c), I have modified the statement here to add the assumption that J is domain-independent in the genus zero case.

operator $\bar\partial$ and is thus surjective, implying that $D\bar\partial_{J,\nu}(u,0)$ is also surjective, hence $\bar\partial_{J,\nu}^{-1}(0)$ is a C^{m-1}-smooth Banach manifold in some open neighborhood of the subset

$$\widetilde{\mathcal{M}}(j,J,0,0) = \left\{(u,0) \in \bar\partial_{J,\nu}^{-1}(0)\right\},$$

which is compact since it consists only of constant maps. We claim moreover that the map

$$\bar\partial_{J,\nu}^{-1}(0) \to C^m(P) \times M : (u,K) \mapsto (K, u(z_0))$$

is a local C^{m-1}-diffeomorphism on some neighborhood of $\widetilde{\mathcal{M}}(j,J,0,0)$. Indeed, the derivative of this map at $(u,0) \in \bar\partial_{J,\nu}^{-1}(0)$ with $u \equiv p \in M$ is

$$\ker D\bar\partial_{J,\nu}(u,0) \to C^m(P) \oplus T_pM : (\eta, Q) \mapsto (Q, \eta(z_0)). \tag{12.1}$$

If $(Q,\eta(z_0)) = 0$ here, then η is an element of $\ker \mathbf{D}_u$ with $\eta(z_0) = 0$, implying $\eta \equiv 0$ since $\ker \mathbf{D}_u$ contains only constant functions, so we've proved the derivative is injective. For surjectivity, suppose $(Q,X) \in C^m(P) \oplus T_pM$ is given. Since $\mathbf{D}_u$ is surjective, we can then find $\eta \in W^{1,p}(u^*TM)$ with $\mathbf{D}_u\eta = Q(\cdot,u)$, and then add to this a constant function to achieve $\eta(z_0) = X$. We then have $D\bar\partial_{J,\nu}(u,0)(\eta,Q) = \mathbf{D}_u\eta - Q(\cdot,u) = 0$, and have thus proven that the map (12.1) is surjective. Using the inverse function theorem and appealing to the compactness of $\widetilde{\mathcal{M}}(j,J,0,0)$ and the uniqueness of maps in this moduli space through any given point $p \in M$, we now find a neighborhood $\mathcal{U}^m \subset C^m(P)$ of 0 such that the map

$$\left\{(u,K) \;\middle|\; K \in \mathcal{U}^m \text{ and } u \in \widetilde{\mathcal{M}}(j,J,K,0)\right\} \to \mathcal{U}^m \times M : (u,K) \mapsto (K, u(z_0))$$

is a C^{m-1}-smooth diffeomorphism. Take $\mathcal{U} \subset \Gamma(P)$ to be the set of all smooth elements in $\mathcal{U}^m$.

Given $p \in M$ the map $u \in \widetilde{\mathcal{M}}(j,J,K,0)$ with $u(z_0) = p$ will be constant if and only if $K(z,p) = 0$ for all $z \in \Sigma$. This implies in particular that for generic choices of K, *none* of these maps are constant.

(c) If $g = 1$ and $H \subset M$ is a smooth hypersurface, then for generic $K \in \Gamma(P)$, $\widetilde{\mathcal{M}}(j,J,K,0)$ contains no solution $u : \Sigma \to M$ with $u(z_0) \in H$. Find also an explicit $K \in \Gamma(P)$ that is not generic in this sense.

Solution:
We saw in part (a) that for generic K, $\widetilde{\mathcal{M}}(j,J,K,0)$ in the genus 1 case is a smooth manifold of dimension $n\chi(\Sigma) + 2c_1(A) = 0$, i.e. a discrete set. One can extend this as follows to a result involving the transversality of the "evaluation" map

$$\mathrm{ev} : \widetilde{\mathcal{M}}(j,J,K,A) \to M : u \mapsto u(z_0).$$

The idea is first to show that the obvious extension of ev to the universal moduli space[20]

$$\widetilde{\mathcal{M}} := \left\{(u,K) \;\middle|\; K \in \Gamma(P) \text{ and } u \in \widetilde{\mathcal{M}}(j,J,K,A)\right\}$$

is a submersion. The key point here is that the operator $\mathbf{L}$ that we needed to prove is surjective in part (a) remains surjective if $\eta \in W^{1,p}(u^*TM)$ is restricted to the space of sections satisfying $\eta(z_0) = 0$. This extra condition makes no difference to the argument in part (a); one only needs to observe that α is still smooth on $\Sigma\backslash\{z_0\}$ and has isolated zeroes, so one can pick a point $z_1 \in \Sigma\backslash\{z_0\}$ at which $\alpha(z_1) \neq 0$ and choose $Q(z,p)$ to have support

[20] Here we are again pretending that $\Gamma(P)$ is a Banach manifold, and as usual, the discrepancy can be remedied using C_ε-spaces.

near $z = z_1$ and not intersecting $z = z_0$. With this understood, any smooth hypersurface $H \subset M$ now determines a smooth codimension 1 Banach submanifold

$$\mathrm{ev}^{-1}(H) \subset \widetilde{\mathcal{M}},$$

and by applying the Sard-Smale theorem to the projection $\mathrm{ev}^{-1}(H) \to \Gamma(P) : (u, K) \mapsto K$, one finds a comeager subset $\Gamma^{\mathrm{reg}}(P) \subset \Gamma(P)$ depending on H such that for all $K \in \Gamma^{\mathrm{reg}}(P)$, $\widetilde{\mathcal{M}}(j, J, K, 0)$ is a smooth 0-manifold containing $\mathrm{ev}^{-1}(H) \subset \widetilde{\mathcal{M}}(j, J, K, 0)$ as a codimension 1 submanifold. By definition, a codimension 1 submanifold of a 0-manifold is the empty set.

We observe however that if $K = 0$, then $\widetilde{\mathcal{M}}(j, J, K, 0)$ contains all constant maps $\Sigma \to M$ and thus contains a map through every hypersurface of M, implying $0 \notin \Gamma^{\mathrm{reg}}(P)$.

(d) If $g \geqslant 2$, then for generic $K \in \Gamma(P)$, $\widetilde{\mathcal{M}}(j, J, K, 0) = \varnothing$. Find also an explicit $K \in \Gamma(P)$ that is not generic in this sense.

Solution:
The point here is that $\dim \widetilde{\mathcal{M}}(j, J, K, 0) = n\chi(\Sigma)$ is negative if $g \geqslant 2$. Once again $K = 0$ is clearly not generic since $\widetilde{\mathcal{M}}(j, J, 0, 0)$ contains all constant maps $\Sigma \to M$ and is thus not empty.

Exercise 12.3. Fix (Σ, j) and (M, ω) with a domain-dependent almost complex structure $J \in \mathcal{J}_\tau^\Sigma(M, \omega)$ and inhomogeneous perturbation $K \in \Gamma(P)$ as in the previous two exercises.

(a) Find a (domain-independent) almost complex structure $\widehat{J}$ on $\Sigma \times M$ such that the natural projection $(\Sigma \times M, \widehat{J}) \to (\Sigma, j)$ is pseudoholomorphic and the map $\hat{u} : \Sigma \to \Sigma \times M : z \mapsto (z, u(z))$ is $\widehat{J}$-holomorphic if and only if $u : \Sigma \to M$ satisfies the inhomogeneous nonlinear Cauchy-Riemann equation $\bar{\partial}_J u = \nu(u)$ of Exercise 12.2.
Hint: If you regard $\Sigma \times M \to \Sigma$ as a trivial fiber bundle with almost complex fibers, you can construct a connection on this bundle for which $\bar{\partial}_J u = \nu(u)$ is equivalent to the condition that the covariant derivative of the section $\hat{u}$ is everywhere complex-linear.
Remark: This exercise has the convenient consequence that almost everything one needs to know about the moduli space $\widetilde{\mathcal{M}}(j, J, K, A)$ follows from things we have previously proved about the usual moduli space of J-holomorphic curves with domain-independent J.

Solution:
Following the hint, let $\pi : E \to \Sigma$ denote the trivial fiber bundle $E = \Sigma \times M$, with fibers $E_z = \{z\} \times M$, and write $VE \subset TE$ for the vertical subbundle over E, whose fiber $V_{(z,p)}E$ at $(z, p) \in E$ is $T_{(z,p)}E_z \cong T_pM$. We can use $J \in \mathcal{J}_\tau^\Sigma(M, \omega)$ to endow each fiber E_z with the almost complex structure $J_z := J(z, \cdot) \in \mathcal{J}_\tau(M, \omega)$, which can also be viewed as a complex structure on the vector bundle $VE \to E$. A connection on $E \to \Sigma$ defines a horizontal subbundle $HE \subset TE$ complementary to VE, thus giving a splitting $TE = HE \oplus VE$ such that for each $(z, p) \in E$, $\pi_* : TE \to T\Sigma$ restricts to an isomorphism $H_{(z,p)}E \to T_z\Sigma$. In light of the canonical isomorphism $V_{(z,p)}E = T_pM$, the connection thus determines an isomorphism

$$T_{(z,p)}E = H_{(z,p)}E \oplus V_{(z,p)}E \cong T_z\Sigma \oplus T_pM, \tag{12.2}$$

where we should stress that unless we have chosen the *trivial* connection, this isomorphism will not be the obvious one arising from the fact that E is a product $\Sigma \times M$. Let $K : TE \to VE$ denote the fiberwise linear projection along HE. The covariant derivative of a section $s : \Sigma \to E$ in the direction $X \in T_z\Sigma$ at a point $z \in \Sigma$ is then given by

$$\nabla_X s = K(Ts(X)) \in V_{s(z)}E.$$

To be more explicit about this, we can write the projection $K_{(z,p)} : T_{(z,p)}(\Sigma \times M) \to V_{(z,p)}E = T_pM$ in block form with respect to the "obvious" splitting $T_{(z,p)}(\Sigma \times M) = T_z\Sigma \oplus T_pM$ as

$$K_{(z,p)} = \begin{pmatrix} \nu(z,p) & \mathbb{1} \end{pmatrix} : T_z\Sigma \oplus T_pM \to T_pM$$

for some linear map $\nu(z,p) : T_z\Sigma \to T_pM$ that depends smoothly on $(z,p) \in \Sigma \times M$. It will turn out to be convenient in the following if $\nu(z,p)$ defines a complex-antilinear map $(T_z\Sigma, j(z)) \to (T_pM, J(z,p))$, so let us assume this henceforth. A section $\hat{u} : \Sigma \to E = \Sigma \times M$ takes the form $\hat{u}(z) = (z, u(z))$ for some smooth map $u : \Sigma \to M$, and its covariant derivative with respect to $X \in T_z\Sigma$ is then

$$\nabla_X \hat{u} = K_{(z,u(z))} \begin{pmatrix} X \\ T_zu(X) \end{pmatrix} = \nu(z,u(z))X + T_zu(X) \in T_{u(z)}M. \tag{12.3}$$

Let us call $\hat{u}$ a J-∇-holomorphic section if its covariant derivative defines a complex-linear map $(T_z\Sigma, j(z)) \to (T_{u(z)}M, J(z,u(z)))$ at every point $z \in \Sigma$; in light of (12.3), this condition means

$$T_zu \circ j(z) + \nu(z,u(z)) \circ j(z) = J(z,u(z)) \circ T_zu + J(z,u(z)) \circ \nu(z,u(z)).$$

Since ν is complex antilinear, the two terms containing ν can now be combined into one, and the equation rewritten as

$$\bar{\partial}_J u(z) := T_zu + J(z,u(z)) \circ T_zu \circ j(z) = -2\nu_{(z,u(z))}.$$

Up to a factor of -2, this is the usual inhomogeneous nonlinear Cauchy-Riemann equation (with domain-dependent J) for the map $u : \Sigma \to M$, and we see that once J has been chosen, there is a canonical bijective correspondence between choices of inhomogeneous perturbation and choices of connection for which the term ν in the vertical projection is complex-antilinear. Finally, we construct a domain-independent almost complex structure $\hat{J}$ on the total space $E = \Sigma \times M$ such that $\hat{u}$ is a J-∇-holomorphic section if and only if it is a $\hat{J}$-holomorphic map $\Sigma \to E$. It will be most convenient to write $\hat{J}(z,p) : T_{(z,p)}E \to T_{(z,p)}E$ in block form with respect to the splitting (12.2), and the obvious formula to try is then

$$\hat{J}(z,p) = \begin{pmatrix} j(z) & 0 \\ 0 & J(z,p) \end{pmatrix} : H_{(z,p)}E \oplus V_{(z,p)}E \to H_{(z,p)}E \oplus V_{(z,p)}E,$$

i.e. $\hat{J}$ is the unique almost complex structure that matches J on the vertical subbundle and makes each horizontal subspace a complex subspace whose projection to the tangent space of the base is complex linear. In the same splitting, the tangent map $T_z\hat{u} : T_z\Sigma \to T_{\hat{u}(z)}E$ takes the form

$$T_z\hat{u} = \begin{pmatrix} \mathbb{1} \\ \nabla\hat{u}(z) \end{pmatrix} : T_z\Sigma \to H_{\hat{u}(z)}E \oplus V_{\hat{u}(z)}E,$$

thus the equation $T_z\hat{u} \circ j(z) = \hat{J}(\hat{u}(z)) \circ T_z\hat{u}$ becomes

$$\begin{pmatrix} j(z) \\ \nabla\hat{u}(z) \circ j(z) \end{pmatrix} = \begin{pmatrix} j(z) & 0 \\ 0 & J(z,u(z)) \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ \nabla\hat{u}(z) \end{pmatrix} = \begin{pmatrix} j(z) \\ J(z,u(z)) \circ \nabla\hat{u}(z) \end{pmatrix},$$

which holds if and only if $\nabla\hat{u}(z)$ is complex linear.

(b) Describe a natural compactification for the moduli space $\widetilde{\mathcal{M}}(j, J, K, A)$. Think up two ways to deduce this result: (1) By a direct bubbling argument, and (2) Using the construction in part (a) to derive it from Gromov's compactness theorem.

Solution: Let's first think in terms of the trivial fiber bundle $\pi : E = \Sigma \times M \to \Sigma$ and almost complex structure $\hat{J} \in \mathcal{J}(E)$ from part (a). Given $A \in H_2(M)$, let $\hat{A} \in H_2(E)$ denote the homology class represented by a section of the form $\hat{u}(z) = (z, u(z))$ if $u : \Sigma \to M$ represents A. Not every $\hat{J}$-holomorphic map $\hat{u} : (\Sigma, j') \to (E, \hat{J})$ homologous to $\hat{A}$ need be of this form, but we observe that since $\pi : (E, \hat{J}) \to (\Sigma, j)$ is pseudoholomorphic, every $\hat{J}$-holomorphic map $\hat{u} : (\Sigma, j') \to (E, \hat{J})$ has the property that $\pi \circ \hat{u} : (\Sigma, j') \to (\Sigma, j)$ is a holomorphic map between Riemann surfaces. If $[\hat{u}] = \hat{A}$, then since $\pi_* \hat{A} = [\Sigma]$, the map $\pi \circ \hat{u} : \Sigma \to \Sigma$ must have degree 1 and is therefore a *biholomorphic* map. It follows that $\hat{u}$ has a unique biholomorphic reparametrization making $\pi \circ \hat{u}$ the identity map, so up to reparametrization, we can indeed assume without loss of generality that $j' = j$ and $\hat{u}$ is a section of the bundle $\pi : E \to \Sigma$, giving rise to a bijective correspondence

$$\begin{aligned} \widetilde{\mathcal{M}}(j, J, K, A) &\xrightarrow{\cong} \mathcal{M}_{g,0}(\hat{J}, \hat{A}) \\ u &\mapsto [(\Sigma, j, \varnothing, \hat{u})]. \end{aligned}$$

This makes it seem natural to call $\overline{\mathcal{M}}_{g,0}(\hat{J}, \hat{A})$ the compatification of $\widetilde{\mathcal{M}}(j, J, K, A)$, but we should think a bit about what this means in practice. Nodal curves $[(S, j', \varnothing, \Delta, \hat{v})] \in \overline{\mathcal{M}}_{g,0}(\hat{J}, \hat{A})$ may in general have several components $S = S_1 \sqcup \ldots \sqcup S_r$, and writing $\hat{v}_i := \hat{v}|_{S_i}$ for each $i = 1, \ldots, r$, the map $\hat{v}_i : (S_i, j') \to (E, \hat{J})$ will also have the property that $\pi \circ \hat{v}_i : (S_i, j') \to (\Sigma, j)$ is a holomorphic map between closed Riemann surfaces, with some degree $d_i \geqslant 0$. Since $\pi_* \hat{A} = [\Sigma]$, there are not many possibilities for these degrees: they must all add up to 1, which means that exactly one of them (say for $i = 1$) is 1 and the rest vanish. This means that $\hat{v}_1$ can be reparametrized biholomorphically to define a section $\hat{v}_1 : (\Sigma, j) \to (E, \hat{J}) : z \mapsto (z, v_1(z))$ for some $v_1 \in \widetilde{\mathcal{M}}(j, J, K, A_1)$, where $A_1 \in H_2(M)$ need not match A, whereas for $i = 2, \ldots, r$, $\hat{v}_i$ is a map of the form $\hat{v}_i(z) = (z_i, v_i(z))$ with $z_i \in \Sigma$ constant. Maps of this form are $\hat{J}$-holomorphic if and only if $v_i : (S_i, j') \to (M, J(z_i, \cdot))$ is an honest pseudoholomorphic curve, i.e. it satisfies the nonlinear Cauchy-Riemann equation with no inhomogeneous term and a domain-independent almost complex structure. The homology classes $A_i := [v_i] \in H_2(M)$ of these curves must satisfy

$$A = \sum_{i=1}^{r} A_i,$$

and since S_1 has the same genus as Σ, the only way to attach these components together without increasing the arithmetic genus is if the following holds:

- None of the nodes $\{z^+, z^-\} \in \Delta$ have both nodal points in S_1;
- $S_i \cong S^2$ for $i = 2, \ldots, r$;
- The spheres S_i for $i = 2, \ldots, r$ are arranged into a finite collection of "bubble trees", each consisting of finitely many components attached to each other by nodes, and with exactly one node attaching one of them to S_1, each individual tree being attached to S_1 at a separate point.

In particular, there is a finite set of distinct points $z_1, \ldots, z_b \in \Sigma$ such that each z_i corresponds to a specific bubble tree in which the bubbles are all $J(z_i, \cdot)$-holomorphic spheres.

Once you've seen the picture of finitely-many bubble trees attached to an element $v_1 \in \widetilde{\mathcal{M}}(j, J, K, A_1)$, it is not hard to imagine how you might deduce this picture directly without passing through Gromov's compactness theorem. By elliptic regularity, a sequence $u_k \in \widetilde{\mathcal{M}}(j, J, K, A)$ will have a C^∞-convergent subsequence if it satisfies a uniform C^1-bound, and energy quantization implies that such a bound will hold away from finitely many points $\Gamma \subset \Sigma$, so that u_k at least has a convergent subsequence on $\Sigma \backslash \Gamma$. The correspondence with $\widehat{J}$-holomorphic curves allows us to apply Gromov's removable singularity theorem to the limit of this sequence and extend it over Γ; the resulting map $v_1 : \Sigma \to M$ then satisfies the inhomogeneous nonlinear Cauchy-Riemann equation but might represent a different homology class than A. One can then use the usual rescaling trick to analyze the formation of bubbles along sequences converging to Γ, and the key observation here is that reparametrizing these maps to zoom in on small disks around points causes both the inhomogeneous term and the domain-dependence of J to disappear from the Cauchy-Riemann equation in the limit. As a consequence, the bubbles that form are honest pseudoholomorphic curves with no inhomogeneous term or domain dependence.

For the development of the Gromov-Witten invariants, the following point is important to understand: the bubbles that arise in the compactification of $\widetilde{\mathcal{M}}(j, J, K, A)$ may be multiply covered, and might therefore live in moduli spaces that fail to be smooth or have the wrong dimension, even if both J and K are generic. This is the reason why inhomogeneous perturbations, useful as they are, do not actually suffice to solve *all* of the transversality problems that arise in Gromov-Witten theory. We will find at least that this problem can be circumvented if the symplectic manifold (M, ω) satisfies certain technical assumptions that always hold up to dimension six.

Final remark: the following more general scenario also arises in the definition of the Gromov-Witten invariants. Instead of a sequence u_k in the fixed moduli space $\widetilde{\mathcal{M}}(j, J, K, A)$, suppose we have $u_k \in \widetilde{\mathcal{M}}(j_k, J, K_k, A)$ where j_k is a sequence of complex structures such that for some fixed set of marked points ζ in Σ, the sequence $[(\Sigma, j_k, \zeta)] \in \mathcal{M}_{g,m}$ degenerates to a nodal marked Riemann surface in Deligne-Mumford space $\overline{\mathcal{M}}_{g,m}$. Here we assume also that K_k is a corresponding sequence of inhomogeneous perturbations which depend smoothly on the position of $[(\Sigma, j_k, \zeta)]$ in Deligne-Mumford space. Concretely, assume that j_k converges to a smooth limit j_∞ outside of a finite collection of disjoint circles $C \subset \Sigma \backslash \zeta$ whose lengths with respect to the Poincaré metric are collapsing to 0. The singular limit $(\Sigma, j_\infty, \zeta)$ then corresponds to a stable nodal Riemann surface representing an element of $\overline{\mathcal{M}}_{g,m}$. If the maps u_k satisfy a uniform C^1-bound, they will then have a subsequence that converges on each component of $\Sigma \backslash C$ to a smooth map satisfying an inhomogeneous nonlinear Cauchy-Riemann equation, so the limit can be understood as a *nodal* solution to $\bar{\partial}_J u = \nu(u)$ whose domain is a stable nodal Riemann surface in $\overline{\mathcal{M}}_{g,m}$. However, uniform C^1-bounds may as usual fail at finitely many points, producing additional bubbles that are pseudoholomorphic spheres with domain-independent almost complex structures. As a consequence of these bubbles, the nodal Riemann surface that serves as the domain of our limiting object will not in general be stable, because it can have spherical components with no marked points and fewer than three nodal points, on which the limiting map may again be a multiply covered holomorphic sphere. If not for these multiply covered spheres, we could use inhomogeneous perturbations to solve all transversality problems, and the Gromov-Witten invariants would be much more straightforward to define than they actually are.

Exercise 12.4. Assume (M, ω) is a closed symplectic manifold.

(a) What conditions on a stable nodal J-holomorphic curve $U = [(S, j, \zeta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(J, A)$ suffice to ensure that U is Fredholm regular if $J \in \mathcal{J}(M, \omega)$ is generic?
Hint: One could for instance require $u|_{S_i} : S_i \to M$ to be a somewhere injective map for each connected component of $S_i \subset S$. This is almost but not quite enough.

Solution:
One can analyze the universal moduli space of (possibly disconnected) holomorphic curves near $u : (S, j) \to (M, J)$ as the zero-set of a section $\bar\partial : \mathcal{T} \times \mathcal{B} \times \mathcal{J}(M, \omega) \to \mathcal{E} : (j', u', J') \mapsto \bar\partial_{J'}(j', u')$, where $\mathcal{B} = W^{1,p}(S, M)$ and $\mathcal{T} \subset \mathcal{J}(S)$ is a suitable Teichmüller slice, formed as a product of Teichmüller slices for the connected components of S, with everything in $\zeta \cup \Delta$ regarded as a marked point. The key step is to prove that the linearization of this section at each $(j, u, J) \in \bar\partial^{-1}(0)$ is surjective, but we need a bit more: we would like to show not just that $\bar\partial_J^{-1}(0) \subset \mathcal{T} \times \mathcal{B}$ is smooth for generic J but also that the map $\mathrm{ev}_\Delta : \bar\partial_J^{-1}(0) \to M^{\times 2N}$ defined by evaluating u at its nodal points $\Delta = \{\{z_1^+, z_1^-\}, \dots, \{z_N^+, z_N^-\}\}$ is transverse to the submanifold $D := \{(p_1, p_1, \dots, p_N, p_N)\} \subset M^{\times 2N}$. To make this possible, we restrict the domain of $D\bar\partial(j, u, J)$ to sections $\eta \in W^{1,p}_\Delta(u^*TM) \subset W^{1,p}(u^*TM)$ that vanish at Δ; if the resulting operator is surjective, it will follow that ev_Δ is a submersion on the universal moduli space, and thus transverse to everything. As usual, the Teichmüller slice will not play any role in this argument, so let's ignore it and consider the operator

$$\mathbf{L} : W^{1,p}_\Delta(u^*TM) \oplus T_J\mathcal{J}(M, \omega) \to L^p(\overline{\mathrm{Hom}}_{\mathbb{C}}(TS, u^*TM)) : (\eta, Y) \mapsto \mathbf{D}_u\eta + Y \circ Tu \circ j.$$

If it is not surjective, then there is a nontrivial section $\alpha \in L^q(\overline{\mathrm{Hom}}_{\mathbb{C}}(TS, u^*TM))$ for $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\begin{aligned} \langle \mathbf{D}_u\eta, \alpha\rangle_{L^2} &= 0 \qquad \text{for all } \eta \in W^{1,p}_\Delta(u^*TM), \\ \langle Y \circ Tu \circ j, \alpha\rangle_{L^2} &= 0 \qquad \text{for all } Y \in T_J\mathcal{J}(M, \omega). \end{aligned}$$

From the first condition, we deduce via elliptic regularity and the similarity principle that α is smooth on $S \backslash \Delta$ and, on each connected component of $S \backslash \Delta$, either vanishes identically or has at most isolated zeroes. By assumption there is a connected component $S_1 \subset S$ on which α does not vanish identically, and if we can choose a point $z_0 \in S_1$ at which $\alpha(z_0) \neq 0$ such that z_0 is also an injective point of the map $u : S \to M$, then the rest of the argument will work exactly the same as for the genericity result that we proved in lecture. We just need to notice that since S may be disconnected, the existence of an injective point of $u : S \to M$ on $S_1 \subset S$ requires more than just the assumption that $u|_{S_1} : S_1 \to M$ is not a multiply covered curve: first, we need to require this for *all* components of S since it cannot be predicted on which components α will be nontrivial, and second, we also need to know that no two of these components are reparametrizations of each other, so that their intersections with each other will be isolated.

First conclusion: *Let $\overline{\mathcal{M}}^*_{g,m}(J, A) \subset \overline{\mathcal{M}}_{g,m}(J, A)$ denote the open set of nodal curves whose connected components are all simple curves and have pairwise nonidentical images. Then for generic $J \in \mathcal{J}(M, \omega)$, every $U \in \overline{\mathcal{M}}^*_{g,m}(J, A)$ is Fredholm regular.*

Actually, one can do somewhat better. The statement above excludes from $\overline{\mathcal{M}}^*_{g,m}(J, A)$ any elements that have so-called *ghost* components $S_i \subset S$, meaning components on which $u|_{S_i}$ is constant. A ghost component that has genus zero is often also called a **ghost bubble**, and these can arise quite naturally. For example, if $U = (S, j, \zeta, \Delta)$ is a nodal curve in the set $\overline{\mathcal{M}}^*_{g,m}(J, A)$ defined above and $m \geqslant 1$, then moving one of the marked points $\zeta_1 \in S$ until it coincides with a nodal point $z_1^+ \in \Delta$ will produce a new element of

$\overline{\mathcal{M}}_{g,m}(J,A)$ that must have a ghost bubble: indeed, the way to make Δ and ζ disjoint in this situation is to add to S an extra copy of S^2 on which u is defined to be a constant map with value $u(z_1^+) = u(z_1^-)$, place the marked point ζ_1 on S^2 and replace the original node $\{z_1^+, z_1^-\}$ with two nodes, one connecting z_1^+ to the new ghost bubble and another connecting that ghost bubble to z_1^-. Notice that in this construction, the number of marked plus nodal points on the ghost bubble is exactly three, so there is no ambiguity about the complex structure on S^2 and no freedom of reparametrization.

If $U = [(S, j, \zeta, \Delta, u)] \in \overline{\mathcal{M}}_{g,m}(J,A)$ has a ghost component $S_i \subset S$ with image at a point $p \in M$, then the restriction of the operator $D\bar{\partial}_J(j,u)(y,\eta) = \mathbf{D}_u\eta + J \circ Tu \circ y$ to S_i is easy to understand: the term involving y disappears since Tu vanishes, and any choice of complex basis for T_pM trivializes the bundle $u^*TM|_{S_i}$ so that $\mathbf{D}_u$ gets identified with the standard $\bar{\partial}$-operator on the trivial bundle $S_i \times \mathbb{C}^n \to S_i$. Writing $\mathcal{T}_i \subset \mathcal{J}(S_i)$ for the restriction of our chosen Teichmüller slice to S_i, the index of this restricted operator is now $\dim \mathcal{T}_i + n\chi(S_i)$, and its kernel consists of all (y,η) where $y \in \mathcal{T}_i$ and $\eta : S_i \to T_pM$ is a constant function; the kernel thus has dimension $\dim \mathcal{T}_i + 2n$, which matches the index (meaning that $D\bar{\partial}_J(j,u)$ is surjective) if and only if S_i has genus zero. This is a good reason to exclude ghost components with positive genus, but there still seems to be hope of establishing transversality in the presence of ghost bubbles. Here is what can be proved:

Better conclusion: *For generic $J \in \mathcal{J}(M,\omega)$, all elements of the open set $\overline{\mathcal{M}}^*_{g,m}(J,A) \subset \overline{\mathcal{M}}_{g,m}(J,A)$ are Fredholm regular, where $\overline{\mathcal{M}}^*_{g,m}(J,A)$ consists of all stable nodal curves $U = (S, j, \zeta, \Delta, u)$ with the following properties:*

- *All nonconstant connected components of U are simple curves with pairwise distinct images;*
- *Every connected nodal curve that can be formed from U by taking a union of ghost components together with all nodes that connect them to each other has arithmetic genus zero.*

The second condition is a generalization of the condition that ghost components of positive genus must be excluded, and you can infer its necessity from the following thought-experiment: suppose $U \in \overline{\mathcal{M}}_{g,m}(J,0)$ is a stable nodal curve in which *all* components are ghost bubbles, but the total arithmetic genus is positive. (It is not hard to draw a picture of such a curve; just arrange the nodes on a collection of spheres so that cycles are formed.) If this curve were Fredholm regular, then it would be possible to glue it to a smooth curve in $\mathcal{M}_{g,m}(J,0)$, which would necessarily be constant since its homology class is 0, but would have positive genus and be Fredholm regular due to the regularity of U; this is a contradiction since, for the reasons cited above, constant curves of positive genus are never Fredholm regular.[21]

The proof of the improved result requires only one or two ingredients beyond what we have already discussed. One of these is the fact that on the universal moduli space $\bar{\partial}^{-1}(0) \subset \mathcal{T} \times \mathcal{B} \times \mathcal{J}(M,\omega)$ of disconnected curves with $2N$ nodal points, the map $\mathrm{ev}_\Delta : \bar{\partial}^{-1}(0) \to M^{\times 2N}$ is not just transverse to the particular submanifold $\{(p_1, p_1, \dots, p_N, p_N)\} \subset M^{\times 2N}$, but to *every* submanifold. This is relevant if we have ghost bubbles because, for instance, if U contains exactly one ghost bubble which has $k \geqslant 3$ nodal points, then there are also k other nodal points $z_1, \dots, z_k$ on nonconstant components at which the map $u : S \to M$

[21]Any in any case, the smooth curve in $\mathcal{M}_{g,m}(J,0)$ in this thought-experiment would not belong to $\overline{\mathcal{M}}^*_{g,m}(J,0)$, implying that $\overline{\mathcal{M}}^*_{g,m}(J,A)$ could not generally be an *open* subset of $\overline{\mathcal{M}}_{g,m}(J,A)$ if it were allowed to contain such objects.

and all nearby maps will necessarily satisfy the incidence relation

$$(u(z_1), \dots, u(z_k)) \in \{(p, \dots, p) \mid p \in M\} \subset M^{\times k}.$$

For generic J, we have the freedom to require that the evaluation map on nodal points should also be transverse to submanifolds like this one. With this understood, the result can be proved in the following way if there is only one ghost bubble and it has k nodal points. Write $S = S^0 \sqcup S'$ where S^0 is the connected component on which u is constant. The space $\bar{\partial}_J^{-1}(0) \subset \mathcal{T} \times \mathcal{B}$ of J-holomorphic maps $S \to M$ near (j, u) can be written as a product of two pieces $\mathcal{M}^0 \times \mathcal{M}'$, where $\mathcal{M}'$ consists of nonconstant J-holomorphic curves near (j, u) restricted to S', and $\mathcal{M}^0$ is the space of constant curves defined on S^0, which may also have varying complex structures. By assumption there are k nodes $\{\{z_1^0, z_1'\}, \dots, \{z_k^0, z_k'\}\}$ such that for each $i = 1, \dots, k$, $z_i^0 \in S^0$ and $z_i' \in S'$, and we have a pair of evaluation maps

$$\mathrm{ev}_\Delta^0 : \mathcal{M}^0 \to M^{\times k}, \qquad \mathrm{ev}_\Delta' : \mathcal{M}' \to M^{\times k}$$

defined by evaluating curves at these nodal points. Our goal is to show that if J is generic, then the combined evaluation map

$$\mathrm{ev}_\Delta = \mathrm{ev}_\Delta^0 \times \mathrm{ev}_\Delta' : \mathcal{M}^0 \times \mathcal{M}' \to M^{\times k} \times M^{\times k}$$

is transverse to the diagonal submanifold in $M^{\times k} \times M^{\times k}$. What we know already about the two maps ev_Δ^0 and ev_Δ' is the following: first, ev_Δ' is transverse to the "thin" diagonal $\{(p, \dots, p)\} \subset M^{\times k}$. Second, since $\mathcal{M}^0$ is a space of constant maps, the image of ev^0 is this same thin diagonal in $M^{\times k}$, and it is a submersion onto that submanifold. From here it is a straightforward linear algebra exercise to prove that ev_Δ is transverse to the diagonal in $M^{\times k} \times M^{\times k}$. This was a special case, but the general case follows from a similar argument, just with more notation and bookkeeping.

(b) Suppose $\dim M = 4$ and the nodal curve U has exactly two connected components, both of which are embedded Fredholm regular J-holomorphic curves living in moduli spaces with virtual dimension 0. What additional condition then ensures that the nodal curve U is Fredholm regular?

Solution:
Let's denote the two components of U by $u_+ : (S_+, j_+) \to (M, J)$ and $u_- : (S_-, j_-) \to (M, J)$ and write $z_\pm \in S_\pm$ for the two nodal points, where $u_+(z_+) = u_-(z_-)$. By assumption, both curves represent isolated elements of their respective 0-dimensional moduli spaces—call them $\mathcal{M}_{g_\pm,0}(J, A_\pm)$. The moduli spaces $\mathcal{M}_{g_\pm,1}(J, A_\pm)$ are therefore 2-dimensional, and the maps

$$S_\pm \to \mathcal{M}_{g_\pm,1}(J, A_\pm) : z \mapsto [(S_\pm, j_\pm, z)]$$

define diffeomorphisms onto connected components of these 2-dimensional moduli spaces. Identifying the relevant components of $\mathcal{M}_{g_\pm,1}(J, A_\pm)$ with $S_\pm$ in this way, the map

$$\mathrm{ev}_\Delta : \mathcal{M}_{g_+,1}(J, A_+) \times \mathcal{M}_{g_-,1}(J, A_-) \to M \times M : (v_+, v_-) \mapsto (\mathrm{ev}(v_+), \mathrm{ev}(v_-))$$

gets identified with the map

$$S_+ \times S_- \to M \times M : (z_1, z_2) \mapsto (u_+(z_1), u_-(z_2)),$$

and the nodal curve U is therefore Fredholm regular if and only if the latter map intersects the diagonal in $M \times M$ transversely at the point (z_+, z_-). This is simply the condition that the intersection of u_+ with u_- at the node is transverse.

Caution: You should guard against letting this simple example inform too much of your *intuition about what Fredholm regularity of a nodal curve means in higher dimensions. An intersection* $u_+(z_+) = u_-(z_-)$ *of two J-holomorphic curves in a manifold of dimension* $\dim M \geqslant 6$ *can never be transverse—transversality of the evaluation map to the diagonal thus means something a bit harder to visualize in higher dimensions.*

13. Week 13

The lecture and Übung on Wednesday of this week were cancelled, so there was only the Tuesday lecture. A make-up lecture will take place next week in place of the usual Übung.

Lecture 23 (24.01.2023): The fairyland definition of $\mathrm{GW}_{g,m,A}$**.**

- Theorem: All smooth moduli spaces that arise in this course have canonical orientations.
  Proof sketch: Ingredient 1: Cauchy-Riemann type operators on bundles over closed Riemann surfaces can always be deformed through a family of Fredholm operators to make them complex linear.
  Ingredient 2: On the space of Fredholm operators between two Banach spaces, there exists a continuous real line bundle Det (the **determinant line bundle**) such that $\mathrm{Det}_T = \Lambda^{\max}(\ker T)$ whenever T is surjective. It follows that on the universal moduli space $\mathcal{M} = \{(u, J)\}$ of J-holomorphic curves, there is a continuous and canonically oriented line bundle $\mathrm{Det} \to \mathcal{M}$ whose fiber over (u, J) matches $\Lambda^{\max} T_u \mathcal{M}_{g,m}(J, A)$ whenever u is a Fredholm regular curve.
- Throughout this lecture, (M, ω) is a closed symplectic $2n$-manifold and $J \in \mathcal{J}_\tau(M, \omega)$ will be assumed generic whenever convenient. For an initial "fairyland" definition of the Gromov-Witten invariants, we pretend that the following are true:
  (1) Transversality is always possible.
  (2) Automorphism groups are always trivial.
  Various spaces that are either smooth orbifolds or not smooth at all in the real world will therefore be manifolds in fairyland: in particular, $\overline{\mathcal{M}}_{g,m}(J, A)$ will now be a closed oriented topological manifold[22] with dimension $D := (n-3)(2-2g) + 2c_1(A) + 2m$, and therefore has a natural fundamental class in singular homology $H_D(\overline{\mathcal{M}}_{g,m}(J, A))$. We will similarly be pretending whenever convenient that $\overline{\mathcal{M}}_{g,m}$ is a topological manifold, which is true at least in the case $g = 0$, though we know that more generally it is an orbifold. We'll discuss next time how to lift these unrealistic assumptions.
- Definition: For integers $g, m \geqslant 0$ with $2g + m \geqslant 3$ and a homology class $A \in H_2(M)$, recall the evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_m) : \overline{\mathcal{M}}_{g,m}(J, A) \to M^{\times m}$ and forgetful map $\Phi : \overline{\mathcal{M}}_{g,m}(J, A) \to \overline{\mathcal{M}}_{g,m}$. Under some dimensional conditions to be specified below, we define the multilinear map

$$\mathrm{GW}_{g,m,A} : (H^*(M; \mathbb{Q}))^{\times m} \times H_*(\overline{\mathcal{M}}_{g,m}; \mathbb{Q}) \to \mathbb{Q}$$

[22] I am saying "topological" manifold in order to avoid making any overly ambitious assumptions about the smoothness of transition maps that arise from gluing when gluing parameters go to infinity. In fairyland, $\mathcal{M}_{g,m}(J, A)$ has a natural smooth structure, but $\overline{\mathcal{M}}_{g,m}(J, A)$ might have only continuous transition maps.

in any of the following equivalent ways (see below for explanations of the notation):

$$\begin{aligned}
\mathrm{GW}_{g,m,A}(\alpha_1,\dots,\alpha_m,\beta) &:= \left\langle \mathrm{ev}_1^*\alpha_1 \cup \dots \cup \mathrm{ev}_m^*\alpha_m \cup \Phi^*(\mathrm{PD}^{-1}(\beta)), [\overline{\mathcal{M}}_{g,m}(J,A)]\right\rangle \\
&= \int_{\overline{\mathcal{M}}_{g,m}(J,A)} \mathrm{ev}_1^*\alpha_1 \wedge \dots \wedge \mathrm{ev}_m^*\alpha_m \wedge \Phi^*(\mathrm{PD}^{-1}(\beta)) \\
&= (\mathrm{ev},\Phi)_*[\overline{\mathcal{M}}_{g,m}(J,A)] \bullet (\mathrm{PD}(\alpha_1)\times\dots\times\mathrm{PD}(\alpha_m)\times\beta) \\
&= \#(\mathrm{ev},\Phi)^{-1}\left(\bar{\alpha}_1\times\dots\times\bar{\alpha}_m\times\bar{\beta}\right) \\
&= \#\left\{u\in\overline{\mathcal{M}}_{g,m}(J,A) \;\middle|\; \mathrm{ev}_i(u)\in\bar{\alpha}_i \text{ for } i=1,\dots,m \text{ and } \Phi(u)\in\bar{\beta}\right\}.
\end{aligned}$$

Here PD denotes the Poincaré duality isomorphism from cohomology to homology, either on M or on $\overline{\mathcal{M}}_{g,m}$.[23] The integral in the second line implicitly assumes a choice of differential forms to represent the cohomology classes α_i and $\mathrm{PD}^{-1}(\beta)$; this version of the formula is favored by physicists, since Witten's original presentation of these invariants derived them from a computation of a Feynman path-integral in some quantum field theory. In the third line, "$\bullet$" denotes the homological intersection number in $M^{\times m}\times\overline{\mathcal{M}}_{g,m}$ and "$\times$" is the homological cross product. This intersection number is computed in the last two lines by choosing closed smooth oriented submanifolds $\bar{\alpha}_i\subset M$ and $\bar{\beta}\subset\overline{\mathcal{M}}_{g,m}$ to represent the homology classes $\mathrm{PD}(\alpha_i)$ and β respectively, and making generic perturbations so that $(\mathrm{ev},\Phi)\pitchfork(\bar{\alpha}_1\times\dots\times\bar{\alpha}_m\times\bar{\beta})$. In each case "#" should be understood as a count of intersections with signs. (It will also need to include rational weights when we leave fairyland and worry about orbifold singularities.) This computation of intersection numbers makes sense whenever vir-dim $\mathcal{M}_{g,m}(J,A)$ matches the codimension of $\bar{\alpha}_1\times\dots\times\bar{\alpha}_m\times\bar{\beta}$ in $M^{\times m}\times\overline{\mathcal{M}}_{g,m}$, which means

$$\sum_{i=1}^m |\alpha_i| - |\beta| = n(2-2g) + 2c_1(A).$$

Whenever this condition is not satisfied, we define $\mathrm{GW}_{g,m,A}(\alpha_1,\dots,\alpha_m,\beta) := 0$.

- Remark: A theorem of Thom from [Tho54] guarantees that in a smooth manifold, every integral homology class has an integer multiple that can be represented by a closed smooth oriented submanifold, hence the submanifolds $\bar{\alpha}_i$ (with rational factors) suffice for representing everything in $H_*(M;\mathbb{Q})$. This is one of the reasons to use rational coefficients and define $\mathrm{GW}_{g,m,A}$ having values in $\mathbb{Q}$ instead of $\mathbb{Z}$, though we will see later that there are more compelling reasons, and the definition *can* in some situations be reformulated to take integer values.
- Invariance theorem: $\mathrm{GW}_{g,m,A}$ depends only on (g,m,A) and the symplectic deformation class of ω; in particular, it does not depend on the choice of tame almost complex structure. Fairyland proof: Given a symplectic deformation $\{\omega_s\in\Omega^2(M)\}_{s\in[0,1]}$ and $J_i\in\mathcal{J}_\tau(M,\omega_i)$ for $i=0,1$, one can extend J_i to a smooth family $\{J_s\in\mathcal{J}_\tau(M,\omega_s)\}_{s\in[0,1]}$ and then define the **parametric moduli space**

$$\overline{\mathcal{M}}_{g,m}(\{J_s\},A) := \left\{(u,s) \;\middle|\; s\in[0,1] \text{ and } u\in\overline{\mathcal{M}}_{g,m}(J_s,A)\right\}.$$

One can analyze the local structure of this space as the zero-set of a smooth section $\bar{\partial}_{\{J_s\}} : \mathcal{T}\times\mathcal{B}\times[0,1]\to\mathcal{E} : (j,u,s)\mapsto\bar{\partial}_{J_s}(j,u)$, and call an element of $\mathcal{M}_{g,m}(\{J_s\},A)$ **parametrically regular** if it corresponds to some $(j,u,s)\in\bar{\partial}_{\{J_s\}}^{-1}(0)$ at which the linearization $D\bar{\partial}_{\{J_s\}}(j,u,s)$ is surjective. (This is obviously true whenever u is Fredholm

[23] Closed oriented orbifolds also have Poincaré duality, though in general only with rational coefficients. The main property this isomorphism should be assumed to have is that evaluating a cohomology class α on a homology class β is equivalent to computing the intersection number of $\mathrm{PD}(\alpha)$ with β.

regular, but the converse is false.) A variation on the usual Sard-Smale argument proves: for generic families $\{J_s \in \mathcal{J}_\tau(M,\omega_s)\}_{s\in[0,1]}$, every $(u,s) \in \mathcal{M}_{g,m}(\{J_s\},A)$ with u somewhere injective is parametrically regular. In fairlyland, we pretend *everything* in $\overline{\mathcal{M}}_{g,m}(\{J_s\},A)$ is parametrically regular, which makes it a compact oriented topological manifold with boundary

$$\partial\overline{\mathcal{M}}_{g,m}(\{J_s\},A) = \overline{\mathcal{M}}_{g,m}(J_1,A) \sqcup \left(-\overline{\mathcal{M}}_{g,m}(J_0,A)\right),$$

i.e. it defines an oriented cobordism between $\overline{\mathcal{M}}_{g,m}(J_0,A)$ and $\overline{\mathcal{M}}_{g,m}(J_1,A)$ on which (ev,Φ) has an obvious continuous extension, implying

$$(\mathrm{ev},\Phi)_*[\overline{\mathcal{M}}_{g,m}(J_0,A)] = (\mathrm{ev},\Phi)_*[\overline{\mathcal{M}}_{g,m}(J_1,A)] \in H_*(M^{\times m}\times\overline{\mathcal{M}}_{g,m}).$$

Equivalently, if $(\mathrm{ev},\Phi)\pitchfork(\bar\alpha_1\times\ldots\times\bar\alpha_m\times\bar\beta)$, then

$$(\mathrm{ev},\Phi)^{-1}(\bar\alpha_1\times\ldots\times\bar\alpha_m) \subset \overline{\mathcal{M}}_{g,m}(\{J_s\},A)$$

is now a compact oriented 1-manifold whose boundary therefore contains zero points when counted with signs, and this count is the difference between the two versions of $\mathrm{GW}_{g,m,A}(\alpha_1,\ldots,\alpha_m,\beta)$ computed with J_0 and J_1. (If you've never seen this type of argument before, spend the weekend reading [Mil97].)

- Observation (moving from fairyland back toward the real world): the intersection count $\#(\mathrm{ev},\Phi)^{-1}\left(\bar\alpha_1\times\ldots\times\bar\alpha_m\times\bar\beta\right)$ can be computed without assuming that $\overline{\mathcal{M}}_{g,m}(J,A)$ is globally a topological manifold with a well-defined fundamental class. Key fact (proved last week): on the set of nodal curves with $N\geqslant 1$ nodes, (ev,Φ) factors through a map defined on a space with virtual dimension $\text{vir-dim}\,\mathcal{M}_{g,m}(J,A)-2N$, so this stratum should never hit $\bar\alpha_1\times\ldots\times\bar\alpha_m\times\bar\beta$ (even under deformations $\{J_s\}$) if everything is transverse.
- Definition: a d**-dimensional pseudocycle** (V,f) in a smooth orbifold M consists of a smooth and oriented (but not necessarily compact) d-manifold V and a smooth map $f: V\to M$ such that the set

$$\Omega_f := \bigcap_{K\subset V\text{ compact}} \overline{f(V\backslash K)} \subset M$$

"has dimension at most $d-2$," meaning that it is contained in a countable union of images of smooth maps defined on manifolds of dimension at most $d-2$. We call two pseudocycles (V_0,f_0) and (V_1,f_1) **bordant** if there exists a smooth oriented $(d+1)$-manifold V and smooth map $f: V\to M$ such that $\partial V = V_1\sqcup(-V_0)$, $f|_{V_i} = f_i$ for $i=0,1$, and Ω_f has dimension at most $d-1$.
- Theorem:
  (1) For any two pseudocycles (V,f) and (W,g) in M with $\dim V+\dim W = \dim M$, after a generic perturbation of f, there are only finitely many intersections of f with g, all transverse, and their signed count

  $$f\bullet g := \#\left\{(x,y)\in V\times W \;\middle|\; f(x)=g(y)\right\}\in\mathbb{Z}$$

  depends on (V,f) and (W,g) only up to bordism.
  (2) For the purpose of computing intersection numbers, any singular homology class $A\in H_d(M;\mathbb{Z})$ can be represented by a d-dimensional pseudocycle in M.

Pseudocycle representation of $A\in H_d(M;\mathbb{Z})$: write $A=[\sum_i\epsilon_i\sigma_i]$ for a finite collection of singular simplices $\sigma_i:\Delta^d\to M$ and signs $\epsilon_i=\pm1$. Since $\partial\sum_i\epsilon_i\sigma_i=\sum_i\epsilon_i\,\partial\sigma_i=0$, the $(d-1)$-boundary faces of these singular simplices must cancel in pairs. Define V by gluing together all the domains of the σ_i along the cancelling $(d-1)$-dimensional boundary faces, then deleting all boundary faces of dimension less than $d-1$, and define $f: V\to M$ as a smoothing of the obvious continuous map determined by the σ_i.

- Remark: Zinger [Zin08] has shown that in a smooth manifold M, the correspondence described above gives an isomorphism from singular homology $H_d(M;\mathbb{Z})$ to the abelian group of d-dimensional pseudocycles up to bordism, with addition defined via disjoint unions. I've never checked whether this is also true when M is an orbifold, but in practice one does not really need to know this: the message is that for the purposes of defining homological intersection theory, pseudocycles are natural objects to work with, in some sense even more natural than singular homology.

Suggested reading. The determinant line bundle over the space of Fredholm operators is described in [MS12, Appendix A.2]. The same book is also the standard reference for pseudocycles (see §6.5), which were introduced specifically for the purpose of defining Gromov-Witten invariants, though as Zinger's paper [Zin08] demonstrates, they are in fact quite natural objects which ought to be better known in algebraic topology. The papers [RT95, RT97] of Ruan and Tian which rigorously defined Gromov-Witten invariants concurrently with [MS94] used the slightly different but closely related notion of *pseudo-manifolds*, which are expressed in terms of simplicial complexes.

Exercises. Next week there will be a make-up lecture instead of an Übung, so I have not thought up any exercises.

14. Week 14

Lecture 24 (31.01.2023): The GW-invariants in semipositive symplectic manifolds.

- Idea (due to Ruan-Tian [RT95, RT97]) for making $(\mathrm{ev}, \Phi) : \mathcal{M}_{g,m}(J,A) \to M^{\times m} \times \overline{\mathcal{M}}_{g,m}$ a pseudocycle: replace $\bar\partial_J(j,u) = 0$ by $\bar\partial_J(j,u) = \nu(j,u)$ for $\nu(j,u)(z) := K(j,z,u(z))$ and a generic inhomogeneous term K depending on $j \in \mathcal{T}$, $z \in \Sigma$ and $u(z) \in M$. But we need a way to set up the moduli space *globally* without choosing Teichmüller slices.
- Definition: the **universal curve** over $\overline{\mathcal{M}}_{g,m}$ is the map
$$\pi : \overline{\mathcal{M}}_{g,m+1} \to \overline{\mathcal{M}}_{g,m} : [(S,j,(\zeta_1,\dots,\zeta_{m+1}),\Delta)] \mapsto \mathrm{st}\left([(S,j,(\zeta_1,\dots,\zeta_m),\Delta)]\right),$$
where the *stabilization* operation st turns general marked nodal Riemann surfaces into stable ones by removing spherical components with fewer than three marked or nodal points and putting any orphaned marked points in the obvious places on adjacent components.
- Observation: Given $x = [(S,j,\zeta,\Delta)] \in \overline{\mathcal{M}}_{g,m}$, let Σ_x denote the singular Riemann surface obtained from (S,j) by identifying the two nodal points in each node. Then there is a natural surjective map
$$i_x : \Sigma_x \to \pi^{-1}(x) \subset \overline{\mathcal{M}}_{g,m+1}$$
that adds $z \in \Sigma_x$ to ζ as the $(m+1)$-st marked point whenever $z \notin (\zeta \cup \Delta)$, and otherwise adds an extra spherical component containing the $(m+1)$-st marked point (plus one other if $z \in \zeta$).
Easy exercise: i_x descends to a bijection $\Sigma_x/\mathrm{Aut}(x) \to \pi^{-1}(x)$, so the fiber over $x \in \overline{\mathcal{M}}_{g,m}$ of the universal curve is an explicit (singular) Riemann surface representing x (modulo its automorphisms).
- Case $g = 0$ and $m \geqslant 3$: Here $\mathrm{Aut}(x)$ is always trivial, so $\overline{\mathcal{M}}_{0,m}$ is a manifold and $\pi^{-1}(x) = \Sigma_x$ is a faithful representative of each $x \in \overline{\mathcal{M}}_{0,m}$. Given $J \in \mathcal{J}_\tau(M,\omega)$, we define the **space of inhomogeneous perturbations**
$$\mathcal{K} \subset \left\{\text{smooth functions on } \overline{\mathcal{M}}_{0,m+1} \times M\right\}$$
to consist of functions such that for $p \in M$, $x \in \overline{\mathcal{M}}_{0,m}$ and $z \in \Sigma_x = \pi^{-1}(x) \subset \overline{\mathcal{M}}_{0,m+1}$, $K(z,p)$ vanishes whenever z lies in some fixed neighborhood of the nodes on Σ_x (this way

we don't need to worry about what "smooth" means at the nodes), and otherwise takes its value in $\overline{\mathrm{Hom}}_{\mathbb{C}}(T_z\Sigma_x, T_pM)$. For $A \in H_2(M)$, we then define a moduli space

$$\begin{aligned}\mathcal{M}_{0,m}(J, K, A) := \big\{(x, u) \;\big|\; &x \in \mathcal{M}_{0,m},\ u : \Sigma_x \to M \text{ satisfies } \bar{\partial}_J u = \nu(u) \\ &\text{for } \nu(u)(z) := K(z, u(z)),\ \text{and } [u] := u_*[\Sigma_x] = A\big\}.\end{aligned}$$

Remark 1: The inhomogeneous term $\nu(u)(z)$ depends implicitly on the complex structure of the domain, because z is not just a point in a fixed Riemann surface, but rather a point in (some fiber of) the universal curve.
Remark 2: Since $\mathrm{Aut}(x)$ is always trivial, there is no freedom of biholomorphic reparametrization, i.e. using the universal curve gives us a *canonical* parametrization of each J-holomorphic curve. (You should take a moment to check: if $K = 0$, this space is equivalent to our usual $\mathcal{M}_{0,m}(J, A)$.)

- Theorem (via the usual Sard-Smale argument): For generic $K \in \mathcal{K}$, $\mathcal{M}_{0,m}(J, K, A)$ is a smooth manifold of dimension equal to $\text{vir-dim}\,\mathcal{M}_{0,m}(J, A)$, and $(\mathrm{ev}, \Phi) : \mathcal{M}_{0,m}(J, K, A) \to M^{\times m} \times \overline{\mathcal{M}}_{0,m}$ can also be made transverse to any given submanifold.
- Remaining to check: Does the image of (ev, Φ) on $\overline{\mathcal{M}}_{0,m}(J, K, A)\backslash\mathcal{M}_{0,m}(J, K, A)$ have codimension at least 2?
- $\overline{\mathcal{M}}_{0,m}(J, K, A)$ was discussed in Exercise 12.3(b): it consists of pairs (x, u) where $x \in \overline{\mathcal{M}}_{g,m}$ and $u : \Sigma \to M$ is a so-called **stable map** on a (not necessarily stable) singular Riemann surface with $\mathrm{st}(\Sigma) = x$, satisfying $\bar{\partial}_J u = \nu(u)$ for $\nu(u)(z) = K(\mathrm{st}(z), u(z))$, where the map $\mathrm{st} : \Sigma \to \Sigma_x$ is constant on non-stable spherical components of Σ. The latter can arise in limits of sequences where bubbling occurs.
- Good news: If there is no bubbling, so Σ is stable, and K is generic, then the inhomogeneous tern looks generic on every component, so all components of u live in smooth moduli spaces of the correct dimension, and the strata of nodal curves therefore always have codimension at least 2.
- Bad news: If bubbling occurs and Σ has non-stable spherical components $S^2 \subset \Sigma$, then $K(\mathrm{st}(z), u(z)) = 0$ on these, so $u|_{S^2}$ is just an ordinary J-holomorphic sphere, which may be multiply covered. Assume J is also generic—then:
  - If $u|_{S^2}$ is simple, it lives in a smooth moduli space of the correct dimension and we have no problem.
  - If $u|_{S^2}$ is a d-fold cover of a simple curve v, then v is also a sphere (spheres can only cover other spheres since everything else has trivial π_2) and belongs to a smooth moduli space $\mathcal{M}_{0,k}(J, B)$ for some $B \in H_2(M)$, where k is the number of marked points on $u|_{S^2}$ (we can project them down to the simple curve), and we have

    $$\dim \mathcal{M}_{0,k}(J, B) = 2(n - 3) + 2c_1(B) + 2k,$$

    whereas $u|_{S^2}$ lives in the (possibly non-smooth) moduli space $\mathcal{M}_{0,k}(J, dB)$ with virtual dimension $2(n-3)+2dc_1(B)+2k$. If the former is no larger than the latter, then we can make (ev, Φ) a pseudocycle by replacing multiply covered components of nodal curves with their underlying simple curves, so this will work if and only if $c_1(B) \geqslant 0$. We need to know that this is true for all classes $B \in H_2(M)$ that can be represented by simple J-holomorphic spheres: these all satisfy $\omega(B) := \langle[\omega], B\rangle > 0$ since the spheres must have positive energy, and also $\text{vir-dim}\,\mathcal{M}_{0,0}(J, B) = 2(n-3) + 2c_1(B) \geqslant 0$ since J is generic.
- Definition: A $2n$-dimensional symplectic manifold (M, ω) is **semipositive** if for all $A \in H_2(M)$ in the image of the Hurewicz map $\pi_2(M) \to H_2(M)$,

$$\omega(A) > 0 \text{ and } c_1(A) \geqslant 3 - n \qquad \Rightarrow \qquad c_1(A) \geqslant 0.$$

One should not try to interpret any deep geometric meaning behind this condition; it is just a hypothesis that makes the proof of our theorem work, and has the advantage that it holds in many situations of interest, e.g. it is always true if $\dim M \leqslant 6$, and also in higher dimensions for certain important classes of projective varieties such as Fano manifolds. We've proved:

- Theorem: If (M, ω) is a closed semipositive symplectic manifold, then for generic J in $\mathcal{J}_\tau(M, \omega)$ or $\mathcal{J}(M, \omega)$ and generic $K \in \mathcal{K}$, $(\mathrm{ev}, \Phi) : \mathcal{M}_{0,m}(J, K, A) \to M^{\times m} \times \overline{\mathcal{M}}_{0,m}$ is a pseudocycle for each $m \geqslant 3$ and $A \in H_2(M)$, whose bordism class depends only on the symplectic deformation class of ω, but not on the choice of J and K. It gives rise to the **rational Gromov-Witten invariants**[24]
$$\mathrm{GW}_{0,m,A} : H^*(M; \mathbb{Z}) \times \ldots \times H^*(M, \mathbb{Z}) \times H_*(\overline{\mathcal{M}}_{0,m}; \mathbb{Z}) \to \mathbb{Z},$$
$$\mathrm{GW}_{0,m,A}(\alpha_1, \ldots, \alpha_m, \beta) := (\mathrm{ev}, \Phi) \bullet (\mathrm{PD}(\alpha_1) \times \ldots \mathrm{PD}(\alpha_m) \times \beta).$$
Note: In this definition, we can use integer coefficients and get integer values because $\overline{\mathcal{M}}_{0,m}$ is a manifold (not an orbifold) and all integral singular homology classes can be represented by pseudocycles. More general definitions that apply when $g > 0$, $m < 3$ or (M, ω) is not semipositive will typically give rational-valued invariants.
- Case $g > 0$ and $2g + m \geqslant 3$: the trouble here is that fibers $\pi^{-1}(x)$ of the universal curve for $x \in \overline{\mathcal{M}}_{g,m}$ are no longer faithful representatives of x, but instead look like $\Sigma_x / \mathrm{Aut}(x)$ where $\mathrm{Aut}(x)$ may act on Σ_x nontrivially. If we then try to define the moduli space as we did above, the inhomogeneous term in $\bar{\partial}_J(u) = \nu(u)$ will be an $\mathrm{Aut}(x)$-invariant function on Σ_x, thus not generic, and we cannot achieve transversality this way.
- Remedy (due to Looijenga [Loo94]): For each $N \geqslant 2$, we can functorially associate to each stable marked Riemann surface $x = (\Sigma, j, \zeta)$ a finite set
$$P_x = \{\text{“Prym level-}N\text{ structures on }x\text{”}\}$$
and to each biholomorphic equivalence $x \xrightarrow{\varphi} y$ of stable marked Riemann surfaces a bijection $\varphi_* : P_x \to P_y$ such that if $N \geqslant 6$ and N is even, then:
  (1) The action of $\mathrm{Aut}(x)$ on P_x is free for every x;
  (2) Writing $(x, p) \sim (y, \varphi_* p)$ for $p \in P_x$ and equivalences $\varphi : x \to y$, the resulting finite cover
$$\mathcal{M}^P_{g,m} := \frac{\{(x,p) \mid p \in P_x\}}{\sim} \xrightarrow{P} \mathcal{M}_{g,m} : [(x,p)] \mapsto [x]$$
has a natural extension to a finite cover $P : \overline{\mathcal{M}}^P_{g,m} \to \overline{\mathcal{M}}_{g,m}$, where $\overline{\mathcal{M}}^P_{g,m}$ is a compact manifold.[25]
- Covering the universal curve: for a Prym cover $P : \overline{\mathcal{M}}^P_{g,m} \to \overline{\mathcal{M}}_{g,m}$ as described above, there is now a natural lift $\widetilde{\pi} : \overline{\mathcal{M}}^P_{g,m+1} \to \overline{\mathcal{M}}^P_{g,m}$ of the universal curve $\pi : \overline{\mathcal{M}}_{g,m+1} \to \overline{\mathcal{M}}_{g,m}$ such that for each $x \in \overline{\mathcal{M}}^P_{g,m}$, $\widetilde{\pi}^{-1}(x) =: \Sigma_x$ is a singular Riemann surface representing $P(x) \in \overline{\mathcal{M}}_{g,m}$. One thus finds multiple explicit (and faithful) representatives of each $x \in \overline{\mathcal{M}}_{g,m}$ among the fibers of the lifted universal curve.

[24] In this setting the word “rational” refers to the fact that we are restricting to genus zero curves; it has nothing to do with the choice of coefficients.

[25] Be aware that in the world of orbifolds, the standard definition of the term “covering map” does not match the usual topological definition of this term. You will notice for instance of you inspect $P : \mathcal{M}^P_{g,m} \to \mathcal{M}_{g,m}$ closely that it is not a local homeomorphism; locally near a point $x \in \mathcal{M}_{g,m}$ and $y \in P^{-1}(x)$, it looks rather like the composition of a homeomorphism with the quotient projection for an action by $\mathrm{Aut}(x)$. As a consequence, the number of points in $P^{-1}(x)$ depends on the order of $\mathrm{Aut}(x)$, and matches the number of elements in P_x (which we define to be the **degree** of the cover) if and only if $\mathrm{Aut}(x)$ is trivial.

- Definition of $\mathrm{GW}_{g,m,A}$: Choose J in $\mathcal{J}_\tau(M,\omega)$ or $\mathcal{J}(M,\omega)$, modify the previous definition of $\mathcal{K}$ so that $K \in \mathcal{K}$ is now a function on $\overline{\mathcal{M}}^P_{g,m+1} \times M$, and for generic $K \in \mathcal{K}$, define the moduli space $\mathcal{M}^P_{g,m}(J,K,A)$ to consist of pairs (x,u) where $x \in \mathcal{M}^P_{g,m}$ and $u : \Sigma_x \to M$ satisfies the inhomogeneous nonlinear Cauchy-Riemann equation determined by J and K. If J and K are generic, this will be a smooth manifold of the correct dimension and (ev,Φ) will be a pseudocycle, but its intersection with cycles in $M^{\times m} \times \overline{\mathcal{M}}_{g,m}$ overcounts due to the existence of multiple representatives of each domain in the lifted universal curve. We therefore define
$$\mathrm{GW}_{g,m,A}(\alpha_1,\ldots,\alpha_m,\beta) := \frac{1}{\deg(P)}(\mathrm{ev},\Phi) \bullet (\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m) \times \beta) \in \mathbb{Q},$$
where the α_i and β can be cohomology/homology classes with rational coefficients; in other words, there is no advantage to taking integer coefficients since the values of the invariant may be rational anyway.[26] The so-called *rational pseudocycle* $\frac{1}{\deg(P)}(\mathrm{ev},\Phi)$ defined on $\mathcal{M}^P_{g,m}(J,K,A)$ defines a rational bordism class independent of the choices of J and K, as well as the choice of Prym cover $P : \overline{\mathcal{M}}^P_{g,m} \to \overline{\mathcal{M}}_{g,m}$.

Lecture 25 (1.02.2023): Some computations in genus zero. Throughout this lecture (and in fact for the remainder of the course), we assume without always saying so that (M,ω) is closed and semipositive whenever a rigorous definition of the Gromov-Witten invariants is required. Many definitions (e.g. "symplectically uniruled") make sense without assuming semipositivity, but whatever we prove about them will then be dependent on some more general definition of the invariants that we haven't given unless semipositivity is assumed.

- Digression on counting in orbifolds: suppose $f : M \to N$ is a smooth map between closed oriented orbifolds and $\Sigma \subset N$ is a closed oriented smooth suborbifold with $\dim M + \dim \Sigma = \dim N$. Generalizing the standard homological intersection theory for manifolds, one obtains a rational-valued intersection number $f \bullet \Sigma \in \mathbb{Q}$, which depends only on the homology classes $[f] := f_*[M]$ and $[\Sigma]$ in $H_*(N;\mathbb{Q})$, such that:
  1. If $M = \widetilde{M}/G$, $N = \widetilde{N}/H$ and $\Sigma = \widetilde{\Sigma}/H$ for smooth manifolds $\widetilde{M}$, $\widetilde{N}$ and $\widetilde{\Sigma}$ that are acted upon by finite groups G and H, where $\widetilde{\Sigma} \subset \widetilde{N}$ is an H-invariant submanifold, and $f : M \to N$ is induced by an equivariant smooth map $\widetilde{f} : \widetilde{M} \to \widetilde{N}$, then
$$f \bullet \Sigma = \frac{1}{|G|}\widetilde{f} \bullet \widetilde{\Sigma},$$
where $\bullet$ on the right hand side means the usual (integer-valued) homological intersection product for manifolds.
  2. If $f \pitchfork \Sigma$, then
$$f \bullet \Sigma = \sum_{p \in f^{-1}(\Sigma)} \frac{\epsilon(p)}{|\operatorname{Aut}(p)|},$$
where $\epsilon(p) = \pm 1$ is a sign determined in the usual manner by orientations, and $\operatorname{Aut}(p)$ denotes the isotropy group of the orbifold M at p.

  You should be able to convince yourself that if $\bullet$ is required to match the usual intersection number of submanifolds in the absence of isotropy (so e.g. when G and H act on $\widetilde{M}$ and $\widetilde{N}$ freely), then both of these properties must hold, so in particular, one cannot define $f \bullet \Sigma$ to be integer-valued in general.

[26] And in any case, the homology class represented by a closed oriented suborbifold of $\overline{\mathcal{M}}_{g,m}$ will sometimes have rational instead of integer coefficients—see Exercise 14.2.

- By the same token, the **Euler number** $e(E) \in \mathbb{Z}$ of an oriented vector bundle $E \to B$ over a closed oriented manifold B of the same dimension can be defined as the self-intersection number of the zero-section, so it counts (with signs) the zeroes of a generic section. If the base B is an orbifold, then one also considers **orbibundles** $E \to B$, which locally look like the quotient of a vector bundle by a finite group acting by linear bundle maps, hence sections look locally like functions that are equivariant with respect to finite group actions. For an orbibundle, $e(E)$ is also defined as the self-intersection number of the zero-section, which is therefore generally a rational number, and for any section $\eta \in \Gamma(E)$ that is transverse to the zero-section,[27] one can compute it as
$$e(E) = \sum_{p \in \eta^{-1}(0)} \frac{\epsilon(p)}{|\operatorname{Aut}(p)|} \in \mathbb{Q}$$
for signs $\epsilon(p) = \pm 1$ determined in the usual way. There is no way in general to define an *integer*-valued count of zeroes that does not depend on the choice of section.
- Theorem (computing $\mathrm{GW}_{g,m,A}$ in the nicest possible scenario):
Suppose we have $J \in \mathcal{J}_\tau(M, \omega)$, submanifolds $\bar{\alpha}_i \subset M$ representing $\mathrm{PD}(\alpha_i) \in H_*(M)$ for each $i = 1, \dots, m$, and a suborbifold $\bar{\beta} \subset \overline{\mathcal{M}}_{g,m}$ representing $\beta \in H_*(\overline{\mathcal{M}}_{g,m})$, such that
$$\mathcal{M}(\alpha, \beta) := (\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m \times \bar{\beta}) \subset \overline{\mathcal{M}}_{g,m}(J, A)$$
has only finitely many elements, all of them *smooth* (i.e. non-nodal) Fredholm regular curves, and the resulting intersections of (ev, Φ) with $\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m \times \bar{\beta}$ are all transverse. Then
$$\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m, \beta) = \sum_{u \in \mathcal{M}(\alpha,\beta)} \frac{\epsilon(u)}{|\operatorname{Aut}(u)|},$$
where the sign $\epsilon(u) = \pm 1$ is positive whenever the associated linearized Cauchy-Riemann operator $\mathbf{D}_u$ is complex-linear.
Proof: Choose a Prym cover $P : \overline{\mathcal{M}}_{g,m}^P \to \overline{\mathcal{M}}_{g,m}$ with degree $k \in \mathbb{N}$. For each $u \in \mathcal{M}(\alpha, \beta)$, let $x := \Phi(u) \in \mathcal{M}_{g,m}$. Then $P^{-1}(x) \subset \mathcal{M}_{g,m}^P$ contains $k/|\operatorname{Aut}(x)|$ elements, and for each $y \in P^{-1}(x)$, one can parametrize u on the associated fiber $\Sigma_y \subset \overline{\mathcal{M}}_{g,m+1}^P$ of the lifted universal curve and obtain elements $(y, u \circ \varphi) \in \mathcal{M}_{g,m}^P(J, 0, A)$ for each $\varphi \in \operatorname{Aut}(x)$, though only $|\operatorname{Aut}(x)|/|\operatorname{Aut}(u)|$ of these elements are distinct. In total, u thus gives rise to exactly
$$\frac{k}{|\operatorname{Aut}(x)|} \frac{|\operatorname{Aut}(x)|}{|\operatorname{Aut}(u)|} = \frac{k}{|\operatorname{Aut}(u)|}$$
elements of $\mathcal{M}_{g,m}^P(J, 0, A)$, so counting them with the correct sign and dividing by $k = \deg(P)$ gives the contribution $\pm 1/|\operatorname{Aut}(u)|$. The transversality assumptions guarantee via the implicit function theorem that this result will not change if we instead count elements of $\mathcal{M}_{g,m}^P(J', K, A)$ for any generic J' near J and generic small inhomogeneous perturbation K.
- Example (from Gromov's non-squeezing theorem): Suppose $(M, \omega) = (S^2 \times W, d\,\mathrm{vol} \oplus \mu)$ for a positive area form $d\,\mathrm{vol} \in \Omega^2(S^2)$, where (W, μ) is any $(2n-2)$-dimensional closed symplectic manifold with $\pi_2(W) = 0$. Let $A = [S^2 \times \{\mathrm{const}\}] \in H_2(M)$, and choose $\alpha_1 \in H^{2n}(M)$ such that $\mathrm{PD}(\alpha_1) = [\mathrm{pt}] \in H_0(M)$ is the homology class represented by a

[27] This observation comes with the caveat that on an orbibundle, local equivariance can sometimes prevent the existence of any sections that are transverse to the zero-section. Thus in order to formulate a general definition of $e(E) \in \mathbb{Q}$, one must in general either appeal to algebraic topology or formulate a cleverer notion of generic perturbations, e.g. "multivalued" sections.

single point; this cohomology class will come up in several examples, so let's abbreviate it by

$$\mathrm{pt} := \mathrm{PD}^{-1}([\mathrm{pt}]) \in H^{2n}(M).$$

Set $\alpha_2 = \alpha_3 := [\{\mathrm{const}\} \times W] \in H^{2n-2}(M)$ and $\beta := [\overline{\mathcal{M}}_{0,3}] \in H_0(\overline{\mathcal{M}}_{0,3})$; the latter is just the canonical generator of $H_0(\overline{\mathcal{M}}_{0,3}) = \mathbb{Z}$ since $\overline{\mathcal{M}}_{0,3}$ is a one-point space, and we shall therefore omit β from the notation for the rest of this discussion and regard $\mathrm{GW}_{0,3,A}$ as a function of just $\alpha_1, \alpha_2, \alpha_3$. Claim:

$$\mathrm{GW}_{0,3,A}(\alpha_1, \alpha_2, \alpha_3) = 1.$$

Proof: Choose any $J_W \in \mathcal{J}_\tau(W, \mu)$ and set $J := i \oplus J_W \in \mathcal{J}_\tau(M, \omega)$. The curves in $\mathcal{M}_{0,3}(J, A)$ are then all uniquely parametrizable in the form $u(z) = (f(z), p)$ for constants $p \in W$ and maps $f \in \mathrm{PSL}(2, \mathbb{C}) = \mathrm{Aut}(S^2, i)$, with marked points $0, 1, \infty$. Choosing representative submanifolds $\bar{\alpha}_1 = \{(0, w_0)\}$ for some $w_0 \in W$, $\bar{\alpha}_2 = \{1\} \times W$ and $\bar{\alpha}_3 = \{\infty\} \times W$, the condition $\mathrm{ev}(u) \in \bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\alpha}_3$ then implies $f = \mathrm{Id}$ and $p = w_0$, so there is exactly one solution, and Exercise 14.3 below shows that there are no nodal curves in $\overline{\mathcal{M}}_{0,3}(J, A)$ satisfying this condition. For the unique curve u, the bundle u^*TM splits into the direct sum of TS^2 with a trivial bundle over S^2, and $\mathbf{D}_u$ respects the splitting, so that it becomes the direct sum of the canonical Cauchy-Riemann operator on TS^2 with the trivial operator $\bar{\partial}$ on the trivial bundle, both of which are surjective and complex linear. The necessary transversality of ev to $\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\alpha}_3$ can be verified with a picture.

- Definition: (M, ω) is called **symplectically uniruled** if for some $A \neq 0 \in H_2(M)$, some $m \geqslant 3$ and some $\alpha_2, \ldots, \alpha_m \in H^*(M)$ and $\beta \in H_*(\overline{\mathcal{M}}_{0,m})$,
$$\mathrm{GW}_{0,m,A}(\mathrm{pt}, \alpha_2, \ldots, \alpha_m, \beta) \neq 0.$$
- Theorem: If (M, ω) is symplectically uniruled, then for every $p \in M$ and every $J \in \mathcal{J}_\tau(M, \omega)$, there exists a nonconstant J-holomorphic sphere passing through p.
(This is the fact about $S^2 \times \mathbb{T}^{2n-2}$ that is used in Gromov's proof of nonsqueezing.)
Proof: Choosing $\bar{\alpha}_1 = \{p\}$ as the submanifold representing PD(pt), the nonvanishing of $\mathrm{GW}_{0,m,A}(\mathrm{pt}, \alpha_2, \ldots, \alpha_m, \beta)$ guarantees the existence of a sequence $u_k \in \mathcal{M}_{0,m}(J_k, K_k, A)$ with $\mathrm{ev}_1(u_k) = p$ for any sequence of generic pairs (J_k, K_k) with $J_k \to J$ and $K_k \to 0$. It will then have a subsequence convergent to a nodal J-holomorhic curve $u_\infty \in \overline{\mathcal{M}}_{0,m}(J, A)$ with $\mathrm{ev}_1(u_\infty) = p$, and this nodal curve necessarily has a nonconstant component that passes through p.
- Remark: The uniruled manifolds are considered to be an especially nice class within all symplectic manifolds. It is known for instance that in dimension four, there are vanishingly few of them: some classic results of McDuff [McD90, McD92] imply that every one is either $(\mathbb{CP}^2, c\omega_{\mathrm{FS}})$ for a scaling constant $c > 0$, a symplectic S^2-bundle with J-holomorphic fibers, or a symplectic blowup of one of these. (For a fuller discussion, see [Wen18, Chapter 7].)
- Definition: (M, ω) is **symplectically rationally connected** if for some $A \in H_2(M)$, some $m \geqslant 3$, some $\alpha_3, \ldots, \alpha_m \in H^*(M)$ and some $\beta \in H_*(\overline{\mathcal{M}}_{0,m})$,
$$\mathrm{GW}_{0,m,A}(\mathrm{pt}, \mathrm{pt}, \alpha_2, \ldots, \alpha_m, \beta) \neq 0.$$
- Theorem: If (M, ω) is symplectically rationally connected (with respect to a homology class $A \in H_2(M)$), then for every pair of distinct points $p_1, p_2 \in M$ and every $J \in \mathcal{J}_\tau(M)$, there exists a nodal J-holomorphic sphere $u \in \mathcal{M}_{0,0}(J, A)$ that passes through both p_1 and p_2.
Proof: Choose $\bar{\alpha}_1 = \{p_1\}$ and $\bar{\alpha}_2 = \{p_2\}$, then do the same thing as in the uniruled case. We do not claim the existence of a smooth J-holomorphic curve through both p_1 and p_2

because the nodal curve $u_\infty \in \overline{\mathcal{M}}_{0,m}(J, A)$ in the limit may have the first two marked points on separate connected components. However:

- Observation of Shah during the lecture: If we are willing to assume J is generic, then the nodal curve in the previous theorem can be replaced by a smooth curve without loss of generality.
  Proof: One can replace any multiply covered components of u_∞ by their underlying simple curves and thus construct a nodal curve v_∞ that satisfies the hypotheses of Exercise 12.4(a). For generic J we can also assume the map $(\mathrm{ev}_1, \mathrm{ev}_2)$ on the stratum of nodal curves containing v_∞ is transverse to $\{(p_1, p_2)\} \subset M \times M$. It follows that all nodal curves near v_∞ can be glued, and the resulting moduli space of smooth curves must contain some whose first two marked points are mapped to (p_1, p_2).
  Question (to which I do not immediately know the answer): Can you find a counterexample showing that genericity of J really is necessary here?
- Theorem: $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is symplectically rationally connected for every $n \geqslant 2$.
  Proof sketch: for the standard complex structure $i \in \mathcal{J}(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, there is a unique holomorphic line through any two distinct points $p_1, p_2 \in \mathbb{CP}^n$. Choose these so that they do not lie in the hypersurface at infinity $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$, and observe that the generator $[L] \in H_2(\mathbb{CP}^n)$ has intersection number 1 with this hypersurface. Taking $\bar{\alpha}_1 = \{p_1\}$, $\bar{\alpha}_2 = \{p_2\}$ and $\bar{\alpha}_3 = \mathbb{CP}^{n-1}$, the moduli space $\overline{\mathcal{M}}_{0,3}(i, [L])$ now contains exactly one smooth curve u in $\mathrm{ev}^{-1}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\alpha}_3)$, and no nodal curves (see Exercise 14.3). One can check that $\mathbf{D}_u$ is surjective and complex-linear and deduce from the existence of curves through arbitrary pairs of points that $\mathrm{ev} \pitchfork (\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\alpha}_3)$, thus $\mathrm{GW}_{0,3,[L]}(\mathrm{pt}, \mathrm{pt}, \mathrm{PD}^{-1}[\mathbb{CP}^{n-1}]) \neq 0$.
- Corollary: For any $J \in \mathcal{J}_\tau(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ and any two distinct points $p_1, p_2 \in \mathbb{CP}^n$, there exists a smooth "J-holomorphic line" (i.e. a J-holomorphic curve homologous to the generator $[L] \in H_2(\mathbb{CP}^n)$) passing through p_1 and p_2.
  Proof: Rational connectivity guarantees a nodal curve in $\overline{\mathcal{M}}_{0,0}(J, [L])$ through both points, which has energy $\omega([L])$, but every connected component of such a curve is either constant or has energy equal to a positive integer multiple of $\omega([L])$, implying that at most one component can be nonconstant, and it must therefore pass through both points. (In fact, stability implies in this situation that no other components can exist.)
- Nicer corollary ($n = 2$): For any $J \in \mathcal{J}_\tau(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, there exists *exactly one* J-holomorphic line through any two distinct points in $\mathbb{CP}^2$.
  Proof: Since $[L]$ has self-intersection number 1, positivity of intersections implies that any two distinct J-holomorphic lines have exactly one intersection point, and it is always transverse.
  Remark: This result originally appeared in [Gro85] *with a more direct proof that did not require constructing the GW-invariants, and it has been used many times since then for clever applications, e.g. toward obstructing the existence of symplectic embeddings, or proving the existence of periodic orbits of Hamiltonian systems.*

Lecture 26 (1.02.2023): Higher genus and obstruction bundle computations.

- Definition: the m-**point** Gromov-Witten invariants in genus g for $2g + m \geqslant 3$ are the multilinear maps $\mathrm{GW}_{g,m,A} : H^*(M)^{\times m} \to \mathbb{Q}$ defined by

$$\mathrm{GW}_{g,m,A}(\alpha_1, \ldots, \alpha_m) := \mathrm{GW}_{g,m,A}(\alpha_1, \ldots, \alpha_m, [\overline{\mathcal{M}}_{g,m}]),$$

  which means that they count curves satisfying m marked point constraints but no constraints on their domain complex structures. We extend this definition to allow $2g + m < 3$

and $A \neq 0$ by requiring the condition[28]

$$\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m) = \frac{\mathrm{GW}_{g,m+1,A}(\alpha_1, \dots, \alpha_m, \alpha)}{\langle \alpha, A \rangle} \in \mathbb{Q} \tag{14.1}$$

for all $\alpha \in H^2(M)$ with $\langle \alpha, A \rangle \neq 0$.
Justification: if $\mathrm{PD}(\alpha)$ is represented by a codimension 2 submanifold $\bar{\alpha} \subset M$ that intersects a given curve $u : \Sigma \to M$ transversely and positively, then there are exactly $\langle \alpha, A \rangle = [\bar{\alpha}] \bullet A \in \mathbb{N}$ places to put an extra marked point ζ_{m+1} on Σ so that $\mathrm{ev}_{m+1}(u) \in \bar{\alpha}$.
Remark: Even for $g = 0$ and (M, ω) semipositive, the denominator in (14.1) forces $\mathrm{GW}_{0,m,A}$ to take rational values in general when $m < 3$.

- Note: if $m = 0$, we understand a multilinear map $H^*(M)^{\times 0} \to \mathbb{Q}$ to mean simply an element of $\mathbb{Q}$, and $\mathrm{GW}_{g,0,A} \in \mathbb{Q}$ is then understood to be a count of curves in (a suitable perturbation of) $\mathcal{M}_{g,0}(J, A)$ if vir-dim $\mathcal{M}_{g,0}(J, A) = 0$, and otherwise $\mathrm{GW}_{g,0,A} := 0$.
- Example 1: In $(S^2 \times W, d\,\mathrm{vol} \oplus \mu)$ with $\pi_2(W) = 0$ and $A = [S^2 \times \{\mathrm{const}\}] \in H_2(M)$, the computation in the previous lecture shows $\mathrm{GW}_{0,1,A}(\mathrm{pt}) = 1$.
- Example 2: In $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ with the line class $A = [L] \in H_2(\mathbb{CP}^n)$, our previous proof of rational connectedness can now be expressed without the aid of the submanifold $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$: the result is $\mathrm{GW}_{0,2,[L]}(\mathrm{pt}, \mathrm{pt}) = 1$.
- Example 3: In $(\mathbb{T}^2, d\,\mathrm{vol})$, we claim $\mathrm{GW}_{1,0,2[\mathbb{T}^2]} = \frac{3}{2}$.
Proof: Fix $i \in \mathcal{J}(\mathbb{T}^2, d\,\mathrm{vol})$ as the complex structure on the target. The space $\mathcal{M}_{1,0}(i, 2[\mathbb{T}^2])$ has virtual dimension 0 and, by the Riemann-Hurwitz formula, consists of equivalence classes of holomorphic covering maps $\varphi : (\mathbb{T}^2, j) \to (\mathbb{T}^2, i)$ having degree 2, with no branch points. We saw in Lecture 19 that all of these are Fredholm regular, and they have linearized Cauchy-Riemann operators that are complex-linear. Moreover, any smooth covering map $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$ can be made into a holomorphic map $(\mathbb{T}^2, j) \to (\mathbb{T}^2, i)$ by setting $j := \varphi^* i$, so there is a natural bijection between the set of degree 2 smooth covering maps $\mathbb{T}^2 \to \mathbb{T}^2$ up to isomorphism and the moduli space $\mathcal{M}_{1,0}(i, 2[\mathbb{T}^2])$. By the Galois correspondence, the former is equivalent to the set of index 2 subgroups of $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$, and there are exactly three of these, giving rise to the three covering maps[29]
$$\varphi(s, t) = (2s, t) \quad \text{or} \quad (s, 2t) \quad \text{or} \quad (s - t, s + t).$$
By Exercise 14.3, $\overline{\mathcal{M}}_{1,0}(i, 2[\mathbb{T}^2])$ contains no nodal curves, so $\mathrm{GW}_{1,0,2[\mathbb{T}^2]}$ is computed by counting only these three double covers, each of which has $|\operatorname{Aut}(\varphi)| = 2$, thus $\mathrm{GW}_{1,0,2[\mathbb{T}^2]} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$.
- Definition (generalizing the condition $M \pitchfork N$): Two submanifolds $M, N \subset Q$ are said to **intersect cleanly** if $M \cap N$ is also a submanifold and $T_p(M \cap N) = T_p M \cap T_p N$ for every $p \in M \cap N$.
Remark: In general if $M \cap N$ is a submanifold, then $\dim T_p(M \cap N) \leqslant \dim (T_p M \cap T_p N)$ because there is an obvious inclusion $T_p(M \cap N) \subset T_p M \cap T_p N$, thus the intersection is clean if and only if these two dimensions are equal, i.e. $\dim(T_p M \cap T_p N)$ is no larger than it *must* be under the circumstances. Note that this holds automatically if $M \pitchfork N$, but it can also hold without $T_p M$ and $T_p N$ spanning $T_p Q$, in which case the dimension of $M \cap N$ will be strictly larger than in the transverse case.
- Similarly, we say that $\bar{\partial}_J : \mathcal{T} \times \mathcal{B} \to \mathcal{E}$ **intersects the** 0**-section cleanly** if $\bar{\partial}_J^{-1}(0) \subset \mathcal{T} \times \mathcal{B}$ is a smooth finite-dimensional submanifold with $T_{(j,u)} \bar{\partial}_J^{-1}(0) = \ker D\bar{\partial}_J(j, u)$ for every

[28] There is no definition of $\mathrm{GW}_{g,m,A}$ for $2g + m < 3$ and $A = 0$ because elements of $\mathcal{M}_{g,m}(J, A)$ are in this case not stable, hence Gromov's compactness theorem does not apply to them.

[29] In lecture I carelessly misidentified the third covering map as $\varphi(s, t) = (2s, 2t)$, which of course has degree 4, not 2.

$(j,u) \in \bar{\partial}_J^{-1}(0)$.
Remark: This condition follows from the implicit function theorem if $D\bar{\partial}_J(j,u)$ is surjective, but it may still hold without that, in which case $\dim\ker D\bar{\partial}_J(j,u) > \operatorname{ind} D\bar{\partial}_J(j,u)$, so that the moduli space $\bar{\partial}_J^{-1}(0)$ is a manifold of strictly larger dimension than its virtual dimension.

- Example: (M,ω) is called a **symplectic Calabi-Yau 3-fold** if $\dim M = 6$ and $c_1(TM) = 0$, which implies that all moduli spaces without marked points satisfy
$$\text{vir-dim}\,\mathcal{M}_{g,0}(J,A) = (n-3)(2-2g) + 2c_1(A) = 0.$$
As was mentioned in Lecture 19, these moduli spaces cannot always be smooth orbifolds of dimension 0: if $v \in \mathcal{M}_{h,0}(J,A)$ is a simple curve with domain (Σ', j'), then for $d \geqslant 2$ and $g \geqslant h$, $\mathcal{M}_{g,0}(J,dA)$ contains the set $\left\{u = v \circ \varphi \;\middle|\; \varphi \in \mathcal{M}_{g,0}(j', d[\Sigma'])\right\}$ consisting of d-fold branched covers of v with genus g, which is an orbifold of dimension $2[2g-2+d\chi(\Sigma')] =: K_{g,d}$. In particular, $\dim\ker D\bar{\partial}_J(j,u)$ in this situation must always be at least $K_{g,d}$, which is not zero unless the d-fold covers $\varphi : (\Sigma,j) \to (\Sigma',j')$ in question have no branch points.
- Definition: The simple curve v in the above situation is called **super-rigid** if for all of its branched covers $u = v\circ\varphi : (\Sigma,j) \to (M,J)$ of all possible degrees $d \in \mathbb{N}$ and genera $g \geqslant h$, $\dim\ker D\bar{\partial}_J(j,u) = K_{g,d}$. This is a clean intersection condition: it implies via the implicit function theorem that for each simple curve v, the space of d-fold covers of v with genus g is an open and closed subset of $\mathcal{M}_{g,0}(J,dA)$, and $\bar{\partial}_J$ intersects the 0-section cleanly.
- Theorem [Wend]: For generic $J \in \mathcal{J}(M,\omega)$ in a symplectic Calabi-Yau 3-fold, all simple J-holomorphic curves are super-rigid.
Remark: I mention this just to illustrate that clean intersections really are something that occur in nature, and they are sometimes even the generic case in situations where actual transversality is impossible. But we will not use this result in the course, nor discuss its proof.
- Application (toy model): Suppose $\pi : E \to B$ is a vector bundle over a manifold and $s : B \to E$ is a smooth section that intersects the 0-section cleanly but not transversely, such that the linearization $Ds(x) : T_xB \to E_x$ for all $x \in \mathcal{M} := s^{-1}(0)$ is Fredholm with index 0. (If E and B are finite dimensional, the latter just means $\dim B = \operatorname{rank} E$.) The zero-set $\mathcal{M}$ is then a manifold of some dimension $k \in \mathbb{N}$, which we will assume is compact and carries an orientation. Since $\dim\operatorname{coker} Ds(x) = \dim\ker Ds(x) - \operatorname{ind} Ds(x) = k$ is constant along $x \in \mathcal{M}$, there is now a smooth vector bundle
$$\mathrm{Ob} \to \mathcal{M}, \qquad \mathrm{Ob}_x := \operatorname{coker} Ds(x),$$
called the **obstruction bundle**, which we shall also assume carries a natural orientation.
Proposition: Given a neighborhood $\mathcal{U} \subset B$ of $\mathcal{M} = s^{-1}(0)$, there exists a neighborhood $\mathcal{V} \subset \Gamma(E)$ of s such that for any $s_\epsilon \in \mathcal{V}$ that is transverse to the zero-section, its algebraic count of zeroes in $\mathcal{U}$ is the Euler number of the obstruction bundle:
$$\#\left(s_\epsilon^{-1}(0) \cap \mathcal{U}\right) = e(\mathrm{Ob}) \in \mathbb{Z}.$$
Proof: Over $\mathcal{U}$, choose a splitting $E|_{\mathcal{U}} = I \oplus C$ such that $I_x = \operatorname{im} Ds(x)$ for every $x \in \mathcal{M}$, and write the section $s \in \Gamma(E)$ over $\mathcal{U}$ as (f,g) for $f \in \Gamma(I)$ and $g \in \Gamma(C)$. The clean intersection condition implies that $f \in \Gamma(I)$ is transverse to the zero-section, and the same will then hold for any $s_\epsilon = (f_\epsilon, g_\epsilon)$ sufficiently close to s, so that $\mathcal{M}_\epsilon := f_\epsilon^{-1}(0)$ is another k-dimensional manifold diffeomorphic to $\mathcal{M}$ and the subbundle $C|_{\mathcal{M}_\epsilon}$ is isomorphic to $C|_{\mathcal{M}} \cong \mathrm{Ob}$. Zeroes of s_ϵ in $\mathcal{U}$ then correspond to zeroes of g_ϵ along $\mathcal{M}_\epsilon$, which defines a section of $C|_{\mathcal{M}_\epsilon}$.
- Remark: If we add symmetries to the toy model above, then B generally becomes an orbifold and $E \to B$ an orbibundle, whose Euler number is then rational.

- Example: In $(S^2, d\,\mathrm{vol})$, we claim that $\mathrm{GW}_{1,2,[S^2]}(\mathrm{pt}, \mathrm{pt}, [\mathrm{pt}]) = 2$.
  Proof sketch: Fix the standard complex structure $i \in \mathcal{J}(S^2, d\,\mathrm{vol})$ on the target. The strange aspect of this example is that $\mathcal{M}_{1,2}(i, [S^2])$ is empty, as there are no degree 1 holomorphic maps from a torus to S^2; nonetheless, $\overline{\mathcal{M}}_{1,2}(i, [S^2]) \neq \varnothing$, as it contains nodal curves with a constant genus 1 component attached to spherical components, one of which can have degree 1. Pick $\bar{\alpha}_1 = \{0\}$ and $\bar{\alpha}_2 = \{1\} \subset S^2$, and let $\bar{\beta} \subset \overline{\mathcal{M}}_{1,2}$ denote a one-point suborbifold consisting of a nodal Riemann surface with a genus 1 component attached to a genus 0 component such that both marked points are on the spherical component and the nodal point is positioned on the torus component so that there are no nontrivial automorphisms. (For the reason why the latter is important, see Exercise 14.2.) Most elements u of $(\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\beta}) \subset \overline{\mathcal{M}}_{1,2}(i, [S^2])$ can now be described as follows: each is a nodal curve with a constant torus component and a spherical component that is a biholomorphic map $\varphi : (S^2, i) \to (S^2, i)$, whose parametrization is uniquely determined if we call $\zeta_1 := 0$ and $\zeta_2 := 1$ its marked points and $z^+ := \infty$ the nodal point connecting it to the torus. The constraint $\Phi(u) \in \bar{\beta}$ means that the torus component carries a fixed complex structure $j \in \mathcal{J}(\mathbb{T}^2)$ with its nodal point in a fixed position $z^- \in \mathbb{T}^2$ such that $\mathrm{Aut}(\mathbb{T}^2, j, z^-)$ is trivial. Meanwhile, $\mathrm{ev}(u) \in \bar{\alpha}_1 \times \bar{\alpha}_2$ means that $\varphi : S^2 \to S^2$ fixes 0 and 1, so φ is then determined by the value of $\varphi(\infty)$, which determines u also on the torus component, but is not in itself constrained. The only caveat to add here is that in this description, $\varphi(\infty)$ cannot take the values 0 or 1, since that is where the marked points are sent; but one can nonetheless construct exactly two additional elements of $(\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\beta}) \subset \overline{\mathcal{M}}_{1,2}(i, [S^2])$ for which the constant torus component takes these values, by adding an additional ghost bubble and placing one of the marked points on it. The result is that $(\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\beta}) \subset \overline{\mathcal{M}}_{1,2}(i, [S^2])$ has a natural homeomorphism to S^2. On the other hand, its *virtual* dimension is

$$\begin{aligned} \text{vir-dim}(\mathrm{ev}, \Phi)^{-1}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\beta}) &:= \text{vir-dim}\, \mathcal{M}_{1,2}(i, [S^2]) \\ &\quad - \operatorname{codim}(\bar{\alpha}_1 \times \bar{\alpha}_2 \times \bar{\beta} \subset S^2 \times S^2 \times \overline{\mathcal{M}}_{1,2}) = 4 - 4 = 0, \end{aligned}$$

  so the actual dimension of this space is too large by 2, suggesting the existence of a rank 2 obstruction bundle.
  Claim (left as an exercise): The obstruction bundle in this setting is isomorphic to TS^2, whose Euler class computes the GW-invariant as claimed.
- Justifiable question: since $\mathcal{M}_{1,2}(i, [S^2]) = \varnothing$ in the above example, what does the result $\mathrm{GW}_{1,2,[S^2]}(\mathrm{pt}, \mathrm{pt}, [\mathrm{pt}]) \neq 0$ actually mean?
  Answer: If we choose a generic inhomogeneous perturbation, the equation $\bar{\partial}_i \varphi = \nu(\varphi)$ will indeed have solutions that are smooth degree 1 maps $\varphi : \mathbb{T}^2 \to S^2$; they are not required to be branched covers (and thus not diffeomorphisms), since they are not actually holomorphic.

Suggested reading. The main original source for our definition of $\mathrm{GW}_{g,m,A}$ is [RT97]. The Ruan-Tian paper does not use Prym covers, but refers instead to an older construction by Mumford [Mum83] of finite covers of $\overline{\mathcal{M}}_{g,m}$, which are not manifolds, but are normal projective varieties with quotient singularities. The Prym covers introduced in [Loo94] later became the preferred tool for the same purpose. The definition in [Loo94] of Prym level-N structures for smooth Riemann surfaces is not so hard to understand, though proving that they have no automorphisms for even $N \geqslant 6$ takes some nontrivial work, and the definition of the compactification $\overline{\mathcal{M}}_{g,m}^{P}$ uses algebro-geometric methods.

McDuff and Salamon give a slightly different construction of the rational Gromov-Witten invariants in [MS12, Chapter 7], using domain-dependent almost complex structures instead of inhomogeneous perturbations, though it is (on the surface) not quite as versatile because it can accommodate only a very specific class of constraints on the forgetful map. (Actually, one can use some knowledge of $H_*(\overline{\mathcal{M}}_{0,m})$ to recover the general invariant in genus zero from what they define, but it is nontrivial to prove that.) Some more restrictive versions of $\mathrm{GW}_{g,m,A}$ with $g > 0$ are also defined in [MS12, Chapter 8] by reformulating the inhomogeneous nonlinear Cauchy-Riemann equation as an equation for J-holomorphic sections of a fiber bundle (cf. Exercise 12.3).

Most of the computations we discussed this week are probably also covered in some fashion in Chapters 7 and 8 of [MS12], but I have not checked to be sure.

Exercises (for the Übung on 8.02.2023).

Exercise 14.1. A symplectic manifold (M, ω) is called (spherically) **monotone** if there exists a constant $\tau > 0$ such that

$$c_1(A) = \tau\omega(A)$$

for every $A \in H_2(M)$ in the image of the Hurewicz map $\pi_2(M) \to H_2(M)$. Show:

(a) Monotone implies semipositive.
(b) $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is monotone, and so is $(S^2 \times W, d\,\mathrm{vol} \oplus \mu)$ for any symplectic manifold (W, μ) with $\pi_2(W) = 0$.
(c) If (M, ω) is monotone, then for generic J in $\mathcal{J}_\tau(M, \omega)$ or $\mathcal{J}(M, \omega)$, the map

$$\mathrm{ev} : \mathcal{M}^*_{0,m}(J, A) \to M^{\times m}$$

is a pseudocycle for each integer $m \geqslant 0$ and $A \neq 0 \in H_2(M)$, where $\mathcal{M}^*_{0,m}(J, A)$ denotes the open set of simple curves in $\mathcal{M}_{0,m}(J, A)$. The m-point rational Gromov-Witten invariants can thus be computed for any $m \geqslant 0$ and $A \neq 0$ as

$$\mathrm{GW}_{0,m,A}(\alpha_1, \ldots, \alpha_m) = \mathrm{ev} \bullet (\mathrm{PD}(\alpha_1) \times \ldots \times \mathrm{PD}(\alpha_m))$$

with ev restricted to $\mathcal{M}^*_{0,m}(J, A)$; in particular, if the classes $\mathrm{PD}(\alpha_i)$ are represented by submanifolds $\bar{\alpha}_i \subset M$, then for generic J, $\mathrm{GW}_{0,m,A}(\alpha_1, \ldots, \alpha_m)$ is a signed count of simple J-holomorphic curves u satisfying $\mathrm{ev}(u) \in \bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m$, and its value is an integer.
(d) Why do you think that in part (c), I did not suggest similarly computing the more general invariant $\mathrm{GW}_{0,m,A}(\alpha_1, \ldots, \alpha_m, \beta)$ with arbitrary $\beta \neq [\overline{\mathcal{M}}_{0,m}] \in H_*(\overline{\mathcal{M}}_{0,m})$ just by counting simple curves for generic J?

Remark: A similar trick works for computing the m-point invariants $\mathrm{GW}_{g,m,A}(\alpha_1, \ldots, \alpha_m)$ with $A \neq 0$ in any symplectic 4-manifold whenever either $g = 0$ or $m \geqslant 1$, and these invariants are therefore also integers. This trick is explained in [Wen18, §7.2.3].

Exercise 14.2. Let's be a bit more concrete about orbifolds and suborbifolds. Recall that if M is an n-dimensional orbifold, it is covered by an atlas of charts, with each chart consisting of the data $(\mathcal{U}_\alpha, M_\alpha, G_\alpha, \varphi_\alpha)$ where $\mathcal{U}_\alpha \subset M$ is an open set, G_α is a finite group acting smoothly[30] on a smooth n-manifold M_α, and φ_α is a homeomorphism $\mathcal{U}_\alpha \to M_\alpha/G_\alpha$. The chart thus identifies each $p \in \mathcal{U}_\alpha$ with some finite set of G_α-related points in M_α, and the stabilizer of one of these points is a finite subgroup of G_α called the **isotropy** subgroup of p; we shall denote it by $\mathrm{Aut}(p)$. Without going into details, the notion of compatibility of charts is defined so that up to isomorphism, the group $\mathrm{Aut}(p)$ does not depend on the choice of chart. The orbifold is called **effective** if G_α acts

[30] Many sources require M_α to be a G_α-invariant open subset of $\mathbb{R}^n$ on which G_α acts linearly, but this seemingly more rigid condition is actually equivalent, because any smooth finite group action can locally be made linear by a suitable choice of coordinates. Just choose a G_α-invariant Riemannian metric and use the exponential map to identify a neighborhood of a point with a region in its tangent space.

effectively on M_α for every chart $(\mathcal{U}_\alpha, M_\alpha, G_\alpha, \varphi_\alpha)$ in the atlas. Effectivity is not a topological property of M, but is rather a property of its atlas. Indeed, every orbifold can be made effective by a modification of its atlas that does not change the underlying topological space: this only requires modifying each of the charts $(\mathcal{U}_\alpha, M_\alpha, G_\alpha, \varphi_\alpha)$ to replace G_α with its quotient by the subgroup that acts trivially on M_α. On the other hand, we will see below that any suborbifold $\Sigma \subset M$ inherits from M a natural atlas, which might not be effective even if M is, and in this situation, modifying the atlas of Σ would not be the right thing to do.

(a) Show that if M is connected, then there exists a number $N \in \mathbb{N}$ such that $|\operatorname{Aut}(p)| \geqslant N$ for all p in an open and dense subset of M. Conclude that M is effective if and only if $\operatorname{Aut}(p)$ is trivial for almost every $p \in M$.

A subset $\Sigma \subset M$ is a smooth k-dimensional **suborbifold** if for every chart $(\mathcal{U}_\alpha, M_\alpha, G_\alpha, \varphi_\alpha)$, φ_α identifies $\mathcal{U}_\alpha \cap \Sigma$ with the quotient by G_α of a smooth k-dimensional G_α-invariant submanifold $\Sigma_\alpha \subset M_\alpha$. The induced orbifold atlas on Σ then consists of the charts $(\mathcal{U}_\alpha \cap \Sigma, \Sigma_\alpha, G_\alpha, \varphi_\alpha|_\Sigma)$, so in particular, each point $p \in \Sigma$ has the same isotropy group as a point in Σ that it does as a point in M.

The idea of homological intersection theory in oriented orbifolds is predicated on the notion that every closed oriented suborbifold $\Sigma \subset M$ naturally represents a homology class $[\Sigma]$. The point I want to make with this exercise is that $[\Sigma]$ isn't always what you might intuitively expect it to be, and sometimes it belongs to $H_*(M;\mathbb{Q})$ rather than $H_*(M;\mathbb{Z})$. In the following, assume that for any smooth map $f : M \to N$ of closed oriented orbifolds and a closed oriented suborbifold $\Sigma \subset N$ with $\dim M + \dim \Sigma = \dim N$ and $f \pitchfork \Sigma$, the intersection number $f \bullet \Sigma \in \mathbb{Q}$ is defined the way we sketched in lecture as a signed count of the points $p \in f^{-1}(\Sigma)$ divided by $|\operatorname{Aut}(p)|$. It is a bit tricky to say in general what a smooth map of orbifolds is, but in the special case where M is a manifold and N an orbifold, we call $f : M \to N$ smooth if for every chart $(\mathcal{U}_\alpha, M_\alpha, G_\alpha, \varphi_\alpha)$ on N, there exists a smooth map $f_\alpha : f^{-1}(\mathcal{U}_\alpha) \to M_\alpha$ such that $\varphi_\alpha \circ f|_{f^{-1}(\mathcal{U}_\alpha)}$ is the composition of f_α with the quotient projection $M_\alpha \to M_\alpha/G_\alpha$.

(b) Suppose M and N are a closed oriented manifold and effective orbifold respectively, with the same dimension, and $\Sigma = \{p\} \subset N$ is a one-point suborbifold. Show that $f \bullet \Sigma \in \mathbb{Q}$ is *not* independent of the choice of point $p \in N$, but its product with $|\operatorname{Aut}(p)|$ is.

How are we to interpret part (b)? The inevitable conclusion is that if closed oriented suborbifolds represent homology classes and $f \bullet \Sigma$ depends only on those classes, then not all of the 0-dimensional suborbifolds $\{p\}$ represent the same one; the natural convention is in fact to define

$$[p] := \frac{1}{|\operatorname{Aut}(p)|}[\mathrm{pt}] \in H_0(N;\mathbb{Q}),$$

where $[\mathrm{pt}] \in H_0(N)$ is the usual "homology class of a point" (represented by a single singular 0-simplex). The subtlety here is that although N was assumed to be effective, the suborbifold $\{p\} \subset N$ will only be effective if its isotropy group is trivial, so by part (a), almost every point in N represents the obvious class $[\mathrm{pt}] \in H_0(N)$, but not every point does. One can show more generally that every closed, oriented and effective orbifold M of dimension n has a natural fundamental class $[M] \in H_n(M;\mathbb{Z})$, but if M is not effective, then its fundamental class must instead be defined to live in $H_n(M;\mathbb{Q})$.

Exercise 14.3. Most of the computations we carried out in lecture this week (with the major exception of the last one) require showing that there are no *nodal* curves satisfying the relevant constraints. Prove in particular that the following compactified moduli spaces with constraints do not contain any nodal curves:

(a) $\mathrm{ev}^{-1}(p) \subset \overline{\mathcal{M}}_{0,1}(J, A)$ in $(M, \omega) := (S^2 \times W, d\,\mathrm{vol} \oplus \mu)$ with $\pi_2(W) = 0$, where $J = i \oplus J_W \in \mathcal{J}_\tau(M, \omega)$ for some $J_W \in \mathcal{J}_\tau(W, \mu)$, and $p \in M$ is an arbitrary point.

(b) $\mathrm{ev}^{-1}(p_1, p_2) \subset \overline{\mathcal{M}}_{0,2}(i, [L])$ in $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ with the standard almost complex structure $i \in \mathcal{J}(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, where $p_1, p_2 \in \mathbb{CP}^n$ are arbitrary distinct points and $[L] \in H_2(\mathbb{CP}^n)$ is the homology class of a line.
Bonus question: What happens in this example if you allow $p_1 = p_2$?

(c) $\overline{\mathcal{M}}_{1,0}(i, 2[\mathbb{T}^2])$ in $(\mathbb{T}^2, d\,\mathrm{vol})$ with the standard complex structure $i \in \mathcal{J}(\mathbb{T}^2, d\,\mathrm{vol})$.

Exercise 14.4. Outline a direct proof—without relying on the knowledge that Gromov-Witten invariants exist—of the main fact about $(M, \omega) = (S^2 \times W, d\,\mathrm{vol} \oplus \mu)$ with $\pi_2(W) = 0$ that is used in Gromov's proof of the nonsqueezing theorem: namely that for every $J \in \mathcal{J}_\tau(M, \omega)$ and every $p \in M$, there exists a smooth J-holomorphic sphere $u : (S^2, i) \to (M, J)$ homologous to $A := [S^2 \times \{\mathrm{const}\}]$ that passes through p.
Hint: The two essential facts we used for the proof in lecture were that (1) it's true when J is of the form $J_0 := i \oplus J_W$, and (2) the invariants are independent of generic data (J, K). The proof of the latter uses a parametric moduli space.

Exercise 14.5. For $(\mathbb{T}^2, d\,\mathrm{vol})$, compute the 1-point invariant $\mathrm{GW}_{1,1,2[\mathbb{T}^2]}(\mathrm{pt})$ directly, and verify in light of our computation $\mathrm{GW}_{1,0,2[\mathbb{T}^2]} = \frac{3}{2}$ from lecture that the result is consistent with (14.1).

Exercise 14.6. Show that in any (M, ω) and for any $m \geqslant 3$ classes $\alpha_i \in H^*(M)$ with $\sum_{i=1}^m |\alpha_i| = \dim M$ and $\beta := [\mathrm{pt}] \in H_0(\overline{\mathcal{M}}_{0,m})$,

$$\mathrm{GW}_{0,m,0}(\alpha_1, \ldots, \alpha_m, [\mathrm{pt}]) = \langle \alpha_1 \cup \ldots \cup \alpha_m, [M] \rangle.$$

Hint: For generic submanifold representatives $\bar{\alpha}_i \subset M$ of $\mathrm{PD}(\alpha_i)$, the right hand side is a signed count of points in $\bar{\alpha}_1 \cap \ldots \cap \bar{\alpha}_m$.

Exercise 14.7. Here are some calculations that can be carried out using obstruction bundles.

(a) Show that for any (M, ω), taking $\alpha_1 := 1 = \mathrm{PD}^{-1}[M] \in H^0(M)$ and $\beta := [\mathrm{pt}] \in H_0(\overline{\mathcal{M}}_{1,1})$,

$$\mathrm{GW}_{1,1,0}(1, [\mathrm{pt}]) = \chi(M).$$

Hint: For any $J \in \mathcal{J}_\tau(M, \omega)$, you can choose suitable $\bar{\alpha}_1 \subset M$ and $\bar{\beta} \subset \overline{\mathcal{M}}_{1,1}$ so that $\mathrm{ev}^{-1}(\bar{\alpha}_1 \times \bar{\beta}) \subset \overline{\mathcal{M}}_{1,1}(J, 0)$ is a compact smooth family of non-nodal curves $u : (\mathbb{T}^2, j) \to (M, J)$ and has a natural identification with M, though its virtual dimension is 0. Show that the vector spaces $\operatorname{coker} \mathbf{D}_u$ form the fibers of an obstruction bundle isomorphic to TM.

(b) Use the result of part (a) to complete the computation $\mathrm{GW}_{1,2,[S^2]}(\mathrm{pt}, \mathrm{pt}, [\mathrm{pt}]) = 2$ for $(S^2, d\,\mathrm{vol})$ that we sketched in lecture.
Hint: The relevant moduli space in this case is an S^2-parametrized family of nodal curves, each having a nontrivial spherical component that is Fredholm regular and a constant torus component with one nodal point. What will happen to this family of nodal curves after a generic inhomogeneous perturbation?

15. Week 15

Lecture 27 (7.02.2023): The Kontsevich-Manin axioms.

- Setting for the axioms: we assume (M, ω) is a closed symplectic $2n$-manifold and consider only the rational GW-invariants, so abbreviate

$$\mathrm{GW}_{m,A} := \mathrm{GW}_{0,m,A} : H^*(M)^{\times m} \times H_*(\overline{\mathcal{M}}_{0,m}) \to \mathbb{Q}$$

for $m \geqslant 3$ and $A \in H_2(M)$. Lifting all constraints on the forgetful map gives the rational m-point invariants

$$\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m) := \mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, [\overline{\mathcal{M}}_{0,m}]).$$

The axioms are meant to be valid for all symplectic manifolds, though we have only defined the invariants in the semipositive case (which was also the state of the art when [KM94] was written). Part of the idea is that one might imagine various different ways of extending the definition beyond the semipositive case, but whatever one defines should be required to satisfy the axioms and will, as a consequence, be uniquely determined in certain cases of interest.

- (E) *Effective* axiom: If $\omega(A) < 0$, then $\mathrm{GW}_{m,A} = 0$. This expresses the fact that holomorphic curves always have nonnegative energy.
- (S) *Symmetry* axiom: Under the interchange of two classes α_i, α_j in $\alpha_1, \ldots, \alpha_m \in H^*(M)$,

$$\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_m, \sigma_*\beta) = (-1)^{|\alpha_i||\alpha_j|}\, \mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_m, \beta),$$

  where $\sigma : \overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{M}}_{0,m}$ is the map defined by exchanging the ith and jth marked points. The sign change comes from the fact that oriented manifolds of the form $\bar{\alpha}_1 \times \ldots \times \bar{\alpha}_m$ change orientation whenever two odd-dimensional factors are exchanged.
- (G) *Grading* axiom: If $\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, \beta) \neq 0$, then

$$2n + 2c_1(A) = \sum_i |\alpha_i| - |\beta|.$$

  For the m-point invariants (when $|\beta| = \dim \mathcal{M}_{0,m} = 2(m-3)$), this becomes

$$2(n-3) + 2c_1(A) + 2m = \sum_i |\alpha_i|.$$

  Its meaning in each case is that the virtual dimension of the moduli space must match the codimension of the constraints.
- (H) *Homology* axiom (appears in a slightly different form as the *motivic* axiom in [KM94]): One can associate to each m, A a homology class

$$\sigma_{m,A} \in H_{2n+2c_1(A)+2m-6}(M^{\times m} \times \overline{\mathcal{M}}_{0,m}; \mathbb{Q})$$

  such that each $\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, \beta)$ is an evaluation of an appropriate cohomology class on $\sigma_{m,A}$, i.e.

$$\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, \beta) = \langle \mathrm{pr}_1^* \alpha_1 \cup \ldots \cup \mathrm{pr}_m^* \alpha_m \cup \mathrm{pr}_{m+1}^* \mathrm{PD}^{-1}(\beta), \sigma_{m,A} \rangle.$$

  In the semipositive setting, $\sigma_{m,A}$ is the singular homology class represented by the Gromov-Witten pseudocycle $(\mathrm{ev}, \Phi) : \mathcal{M}_{0,m}(J, K, A) \to M^{\times m} \times \overline{\mathcal{M}}_{0,m}$ under the correspondence in [Zin08]. One imagines it more generally as the pushforward under (ev, Φ) of the (virtual) fundamental class of $\overline{\mathcal{M}}_{0,m}(J, A)$, whenever one has a way of defining the latter.
- (FC) *Fundamental class* axiom: so named because the unit $1 \in H^0(M)$ is Poincaré dual to the fundamental class of M, so this involves invariants in which one of the marked points is not constrained under evaluation:

$$\mathrm{GW}_{m+1,A}(\alpha_1, \ldots, \alpha_m, 1, \beta) = \mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, \pi_*\beta),$$

  where $\pi : \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,m}$ is the map that forgets the last marked point and stabilizes (we have previously called this the “universal curve”).
  Proof: Given a submanifold $\bar{\beta} \subset \overline{\mathcal{M}}_{0,m+1}$ representing β, use the map $\pi|_{\bar{\beta}} : \bar{\beta} \to \overline{\mathcal{M}}_{0,m}$ to represent $\pi_*\beta \in H_*(\overline{\mathcal{M}}_{0,m})$ and compute the right hand side as an intersection number.
  Corollary: The $(m+1)$-point invariant $\mathrm{GW}_{m+1,A}(\alpha_1, \ldots, \alpha_m, 1)$ always vanishes!
  This follows for stupid dimensional reasons since $\dim \overline{\mathcal{M}}_{0,m} < \dim \overline{\mathcal{M}}_{0,m+1}$, so $\pi_*[\overline{\mathcal{M}}_{0,m+1}] = 0$. Another explanation is as follows: if $\mathrm{GW}_{m+1,A}(\alpha_1, \ldots, \alpha_m, 1)$ were nonzero, the grading axiom would imply $\text{vir-dim}\, \mathcal{M}_{m,0}(J, A) + 2 = \sum_{i=1}^m |\alpha_i|$, which makes the dimension of $\mathcal{M}_{m,0}(J, K, A)$ too small to satisfy the constraints imposed by $\alpha_1, \ldots, \alpha_m$.

- (Z) *Zero* axiom (appears in [KM94] under the name *mapping to a point*):
For $A := 0 \in H_2(M)$,
$$\mathrm{GW}_{m,0}(\alpha_1, \dots, \alpha_m, [\mathrm{pt}]) = \langle \alpha_1 \cup \dots \cup \alpha_m, [M] \rangle,$$
and $\mathrm{GW}_{m,0}(\alpha_1, \dots, \alpha_m, \beta) = 0$ whenever $|\beta| > 0$.
Proof: The first statement is Exercise 14.6 and results from the fact that holomorphic spheres homologous to zero are constant (but also Fredholm regular!), after interpreting the right hand side as a signed count of intersections of submanifolds Poincaré dual to $\alpha_1, \dots, \alpha_m$ in general position. When $|\beta| > 0$, the grading axiom implies $\sum_i |\alpha_i| > 2n$, so for dimensional reasons, these submanifolds in general position will have no common intersections.
- (D) *Divisor* axiom: "divisor" is roughly the algebrogeometrisch word for "complex hypersurface", so this concerns constraints imposed by codimension 2 submanifolds: for $|\alpha| = 2$,
$$\mathrm{GW}_{m+1,A}(\alpha_1, \dots, \alpha_m, \alpha, \mathrm{PD}(\pi^* \mathrm{PD}^{-1}(\beta))) = \langle \alpha, A \rangle \, \mathrm{GW}_{m,A}(\alpha_1, \dots, \alpha_m, \beta),$$
where $\pi : \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,m}$ is again the map that forgets the last marked point.
Proof: If $\bar{\beta} \subset \overline{\mathcal{M}}_{0,m}$ represents β, then $\pi^{-1}(\bar{\beta}) \subset \overline{\mathcal{M}}_{0,m+1}$ is a submanifold representing $\mathrm{PD}(\pi^* \mathrm{PD}^{-1}(\beta))$. (Check this by intersecting it with other homology classes of complementary dimension.) The term $\langle \alpha, A \rangle$ is the intersection number of A with $\mathrm{PD}(\alpha)$, which gives a signed count of the number of places that an extra marked point can be put on any curve counted by $\mathrm{GW}_{m,A}(\alpha_1, \dots, \alpha_m, \beta)$ so that the α constraint is also satisfied.
- (Sp) *Splitting* axiom: This gives a relation (due to compactness and gluing) between counts of curves whose domains are close to degenerating in Delign-Mumford space and counts of pairs of curves that arise after a nodal degeneration. Choose a basis $e_0, \dots, e_N$ of $H^*(M; \mathbb{Q})$, write $g_{ab} := \langle e_a \cup e_b, [M] \rangle$ (defined to be zero whenever $|e_a| + |e_b| \neq 2n$), let g^{ab} denote the entries of the inverse matrix, and fix $m_0, m_1 \geqslant 2$ such that $m_0 + m_1 = m$. There is a natural map
$$\phi : \overline{\mathcal{M}}_{0,m_0+1} \times \overline{\mathcal{M}}_{0,m_1+1} \to \overline{\mathcal{M}}_{0,m}$$
defined by attaching two marked nodal Riemann surfaces Σ_1 and Σ_2 via a node formed from the last marked point in Σ_1 and the first marked point in Σ_2, then numbering the remaining marked points in the obvious order. The axiom then says that for any $\beta_0 \in H_*(\overline{\mathcal{M}}_{0,m_0+1})$ and $\beta_1 \in H_*(\overline{\mathcal{M}}_{0,m_1+1})$,
$$\begin{aligned}\mathrm{GW}_{m,A}(\alpha_1, \dots, \alpha_m, \phi_*(\beta_0 \times \beta_1)) = \sum_{A_1+A_2=A} \sum_{a,b=0}^{N} & \mathrm{GW}_{m_0+1,A_0}(\alpha_1, \dots, \alpha_{m_0}, e_b, \beta_0) \cdot g^{ab} \\ & \cdot \mathrm{GW}_{m_1+1,A_1}(e_a, \alpha_{m_0+1}, \dots, \alpha_m, \beta_1).\end{aligned}$$
- Exercise: Show by evaluating under a basis of cohomology classes that the homology class $\sum_{a,b} g^{ab} \, \mathrm{PD}(e_b) \times \mathrm{PD}(e_a) \in H_*(M \times M)$ is represented by the diagonal submanifold. Use this to interpret the right hand side of the formula in the splitting axiom as a count of nodal curves.
- Extension: define $\mathrm{GW}_{m,A}(\alpha_1, \dots, \alpha_m)$ for $m < 3$ and $A \neq 0$ so that the divisor axiom holds. One cannot insert any $\beta \in H_*(\overline{\mathcal{M}}_{0,m})$ in these cases because the Deligne-Mumford spaces $\overline{\mathcal{M}}_{0,0}$, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are not defined. One also cannot allow $A = 0$ because the holomorphic curves are then not stable, so Gromov's compactness theorem would not apply.
- (FC) extended: $\mathrm{GW}_{m,A}(\alpha_1, \dots, \alpha_m, 1) = 0$ also holds (for the same dimensional reasons) whenever $m \leqslant 2$ and $A \neq 0$.

Note: $\mathrm{GW}_{3,0}(\alpha_1, \alpha_2, 1)$ can be nontrivial due to the zero axiom. This exception occurs because $\mathcal{M}_{0,2}(J, 0)$ has the wrong dimension and, being a space of non-stable curves, there is no way to perturb it to a space with the right dimension.

- Theorem: All of the rational Gromov-Witten invariants are determined by the m-point invariants $\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m)$, i.e. the case $\beta = [\overline{\mathcal{M}}_{0,m}]$.
Proof: According to [Kee92], the ring $H^*(\overline{\mathcal{M}}_{0,m})$ is generated by degree 2 classes Poincaré dual to homology classes of the form $\phi_*([\overline{\mathcal{M}}_{0,m_0+1}], [\overline{\mathcal{M}}_{0,m_1+1}])$ for $m_0 + m_1 = m$, plus others obtained from these via permutation of marked points. It follows that it suffices to be able to compute $\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m, \beta)$ whenever β is an intersection of finitely many codimension 2 submanifolds of this type. The case $|\beta| = \max -2$ now follows from the case $|\beta| = \max$ via the splitting axiom, and after that, the splitting axiom determines the case $|\beta| = \max -4$ from these two cases, and so forth.
- First reconstruction theorem [KM94] (stated without proof): If $H^*(M)$ is generated as a ring by $H^2(M)$, then all $\mathrm{GW}_{m,A}$ are determined by the two-point invariants of the form[31]

$$\mathrm{GW}_{2,A}(\alpha_1, \alpha_2), \qquad A \neq 0 \text{ with } c_1(A) \leqslant n + 1.$$

- Example: In $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ with the generator $[L] \in H_2(\mathbb{CP}^n)$, $c_1([L]) = n + 1$, and this theorem then says that the computation $\mathrm{GW}_{2,[L]}(\mathrm{pt}, \mathrm{pt}) = 1$ from last week determines all other genus zero invariants on $\mathbb{CP}^n$.
(Note: Actually computing them is still not a trivial task; cf. the end of [MS12, Chapter 7].)

Lecture 28 (8.02.2023): The Gromov-Witten potential.

- Goal: Repackage the information contained in the collection of all genus zero m-point invariants $\mathrm{GW}_{m,A}(\alpha_1, \ldots, \alpha_m)$ so as to minimize redundancy (arising e.g. from the divisor axiom).
- Fix the following choices on (M, ω). We take $H_2(M)$ to mean the quotient of $H_2(M; \mathbb{Z})$ by its torsion subgroup (note that $\omega(A)$ vanishes automatically when A is torsion), and define $H^*(M)$ always with rational coefficients. Choose a basis $A_1, \ldots, A_k$ of $H_2(M)$ over $\mathbb{Z}$ and a basis $e_0, \ldots, e_N$ of $H^*(M)$ over $\mathbb{Q}$ such that $e_0 = 1 \in H^0(M)$ and $e_1, \ldots, e_k \in H^2(M)$ form the dual basis to $A_1, \ldots, A_k$. We again write

$$g_{ab} := \langle e_a \cup e_b, [M] \rangle, \qquad a, b \in \{0, \ldots, N\},$$

and let g^{ab} denote the entries of the inverse matrix. Associate to each $t = (t_0, \ldots, t_N) \in \mathbb{Q}^{N+1}$ the cohomology class

$$\alpha_t := \sum_a t_a e_a \in H^*(M).$$

This symbol will have an additional formal meaning in the following, in which we regard $H^*(M)$ as a *supermanifold*, whose ring of functions (written as functions of the variables $t_0, \ldots, t_N$) thus has a natural splitting into spaces of *even* and *odd* functions. In particular, the variable t_a is defined to be even or odd in accordance with the degree of e_a, and the ring of functions on $H^*(V)$ is defined to satisfy the commutation relations

$$t_a t_b = (-1)^{|e_a| \cdot |e_b|} t_b t_a. \tag{15.1}$$

As a purely formal matter of notation, we will also write

$$e_a t_b := (-1)^{|e_a| \cdot |e_b|} t_b e_a. \tag{15.2}$$

[31] In [KM94], the condition on A is $c_1(A) \leqslant 2n + 1$ instead of $n + 1$, but I believe this to be a typo.

- Definition: The **group ring** of $H_2(M)$ over $\mathbb{Q}$ is the commutative ring $\mathbb{Q}[H_2(M)]$ generated by $\mathbb{Q}$ and all symbols of the form e^A for $A \in H_2(M)$, on which the product is defined by
$$e^A e^B := e^{A+B}.$$
- The **genus zero Gromov-Witten potential** Φ of (M, ω) is a formal power series consisting of a countable sum of monomials in the variables $t_0, \ldots, t_N$ multiplied by elements of $\mathbb{Q}[H_2(M)]$. It is defined by
$$\begin{aligned}\Phi &:= \sum_{A \in H_2(M)} \sum_{m=0}^{\infty} \frac{1}{m!} \mathrm{GW}_{m,A}(\alpha_t, \ldots, \alpha_t) e^A \\ &= \sum_{A,m} \frac{1}{m!} \sum_{a_1,\ldots,a_m=0}^{N} \epsilon(a)\, \mathrm{GW}_{m,A}(e_{a_1}, \ldots, e_{a_m}) e^A t_{a_1} \ldots t_{a_m},\end{aligned}$$
where the sign $\epsilon(a) = \pm 1$ of the tuple $a = (a_1, \ldots, a_m)$ is determined by the rule
$$t_{a_1} e_{a_1} \ldots t_{a_m} e_{a_m} = \epsilon(a) e_{a_1} \ldots e_{a_m} t_{a_1} \ldots t_{a_m}$$
in accordance with (15.2). The symmetry axiom and (15.1) guarantee that each term in this series is unchanged under permutations of $a_1, \ldots, a_m$. We can therefore rewrite it in terms of multi-indices $\gamma = (\gamma_0, \ldots, \gamma_N) \in \mathbb{N}_{\geqslant 0}^{N+1}$ as
$$\Phi = \sum_{A,\gamma} \frac{\epsilon(\gamma)}{\gamma!} \mathrm{GW}_{|\gamma|,A}(e^\gamma) e^A t^\gamma, \tag{15.3}$$
where $|\gamma| := \gamma_0 + \ldots + \gamma_N$, $\gamma! := \gamma_0! \ldots \gamma_N!$, $\epsilon(\gamma) := \epsilon(a) = \pm 1$ for
$$a = (a_1, \ldots, a_m) := \Big(\underbrace{0, \ldots, 0}_{\gamma_0}, \ldots, \underbrace{N, \ldots, N}_{\gamma_N}\Big),$$
and e^γ denotes the tuple $(e_{a_1}, \ldots, e_{a_m})$ for this same definition of $a_1, \ldots, a_m$. The summation is understood to omit terms with $m < 3$ and $A = 0$ since $\mathrm{GW}_{m,A}$ is not defined in these cases.
- Remark: For each individual multi-index γ, t^γ may appear in infinitely many terms of (15.3), but Gromov compactness implies that only finitely many of these can have $\omega(A)$ bounded above by any given constant. It follows that Φ is a power series in the variables $t_0, \ldots, t_N$ with coefficients in the **Novikov completion** Λ_ω of $\mathbb{Q}[H_2(M)]$, which is defined to consist of finite or countably infinite sums $\sum_i c_i e^{A_i}$ with $c_i \in \mathbb{Q}$ and $A_i \in H_2(M)$ such that, whenever the sum is infinite, $\lim_{i \to \infty} \omega(A_i) = +\infty$. This observation does not appear to follow directly from the Kontsevich-Manin axioms, but it certainly holds for any reasonable definition of the GW-invariants.
- Notational device: Associate to each of the basis elements $A_i \in H_2(M)$ a formal variable $q_i := e^{A_i} \in \mathbb{Q}[H_2(M)]$, so that for $d = (d_1, \ldots, d_k) \in \mathbb{Z}^k$, we can write
$$q^d := q_1^{d_1} \ldots q_k^{d_k} := e^{d_1 A_1} \ldots e^{d_k A_k} = e^{A_d} \quad \text{for} \quad A_d := \sum_{i=1}^{k} d_i A_i.$$
Now (15.3) becomes a formal Laurent series (remember that the d_i can be negative) in the variables $q_1, \ldots, q_k$ and $t_0, \ldots, t_N$ with rational coefficients, written as
$$\Phi(q,t) = \sum_{d,\gamma} \frac{\epsilon(\gamma)}{\gamma!} \mathrm{GW}_{|\gamma|,A_d}(e^\gamma) q^d t^\gamma.$$

- Example: $M := \mathbb{CP}^1 \cong S^2$, with any area form as ω. Taking $e_0 = 1 \in H^0(\mathbb{CP}^1)$, $e_1 = \mathrm{pt} := \mathrm{PD}^{-1}([\mathrm{pt}]) \in H^2(\mathbb{CP}^1)$ and $q := e^{[\mathbb{CP}^1]}$, we have

$$\Phi(q, t_0, t_1) = \frac{1}{2}t_0^2 t_1 + qe^{t_1} := \frac{1}{2}t_0^2 t_1 + q\sum_{m=0}^{\infty} \frac{1}{m!}t_1^m.$$

  Explanation: By (E), only homology classes $A = d[\mathbb{CP}^1]$ with $d \geqslant 0$ contribute. The $d = 0$ contribution $\frac{1}{2}t_0^2 t_1$ comes from $\mathrm{GW}_{3,0}(e_0, e_0, e_1) = \langle e_0 \cup e_0 \cup e_1, [\mathbb{CP}^1]\rangle = 1$ by (Z). One uses (G) to show that all $d = 1$ contributions come from $\mathrm{GW}_{0,[\mathbb{CP}^1]} = 1$ (which counts only the identity map $S^2 \to \mathbb{CP}^1$) and its consequences via (D), which give $\mathrm{GW}_{m,[\mathbb{CP}^1]}(e_1, \ldots, e_1) = 1$ for every $m \geqslant 0$, producing the exponential series in t_1.
- Example: $(M, \omega) := (\mathbb{CP}^2, \omega_{\mathrm{FS}})$. Taking $q := e^{[L]}$ for the line $[L] \in H_2(\mathbb{CP}^2)$, $e_0 := 1 \in H^0(\mathbb{CP}^2)$, $e_1 := \mathrm{PD}^{-1}([L]) \in H^2(\mathbb{CP}^2)$ and $e_2 := e_1 \cup e_1 = \mathrm{pt} \in H^4(\mathbb{CP}^2)$, we have

$$\Phi(q, t_0, t_1, t_2) = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \sum_{d=1}^{\infty} \frac{N_d q^d}{(3d-1)!} e^{dt_1} t_2^{3d-1},$$

  where $N_d \in \mathbb{Z}$ is the count of rational curves of degree d through $3d - 1$ generic points, explicitly[32]

$$N_d := \mathrm{GW}_{3d-1,d[L]}(\mathrm{pt}, \ldots, \mathrm{pt}).$$

  Again (E) implies only classes $A = d[L]$ with $d \geqslant 0$ contribute, and the $d = 0$ contribution produces $\frac{1}{2}(t_0^2 t_2 + t_0 t_1^2)$ due to (Z), because $e_0 \cup e_0 \cup e_2 = e_0 \cup e_1 \cup e_1$ both evaluate to 1 on $[\mathbb{CP}^2]$. The exponential series comes from $\mathrm{GW}_{3d-1,d[L]}(\mathrm{pt}, \ldots, \mathrm{pt})$ and its consequences via (D), which give

$$\mathrm{GW}_{3d-1+r,d[L]}(\underbrace{e_1, \ldots, e_1}_{r}, e_2, \ldots, e_2) = d^r N_d$$

  for every integer $r \geqslant 0$. All other possible contributions are excluded via (G).
- Idea: Certain axioms of Kontsevich-Manin translate into partial differential equations satisfied by Φ as a function of $t_0, \ldots, t_N$ and $q_1, \ldots, q_k$, which can give nontrivial information toward computations of Φ. Note that when working with variables that are only *graded* commutative, the definition of the partial derivative with respect to an odd variable is slightly non-obvious: the basic property we require is that[33]

$$\partial_{t_a}(t_a F) = F$$

  whenever $F(q, t)$ does not contain the variable t_a. This might not be the same as $\partial_{t_a}(F t_a)$ since $t_a F$ and $F t_a$ might not be equal; in particular, for $a \neq b$, one deduces the relation

$$\partial_{t_a}\partial_{t_b} = (-1)^{|e_a||e_b|}\partial_{t_b}\partial_{t_a}.$$

  This plus $\partial_{t_a} t_a = 1$, $\partial_{t_a}(1) = 0$ and the obvious definition of ∂_{t_a} when t_a is an even variable suffice to determine the operator ∂_{t_a} on all power series, and one can show that it satisfies a graded Leibniz rule. The definition of ∂_{q_a} is straightforward because the variables q_a are even.
- Theorem: The Gromov-Witten potential $\Phi(q, t)$ satisfies the following relations:

[32]Our overarching goal is of course to compute the numbers N_d. Writing down Φ in terms of N_d is the first half of the argument toward that end.

[33]Note that if $t_a F \neq 0$ and t_a is an odd variable, then $F(q, t)$ *cannot* contain t_a since $t_a^2 = 0$.

(1) The **string equation**:

$$\partial_{t_0}\Phi(q,t) = \frac{1}{2}\sum_{a,b} g_{ab}t_b t_a.$$

(2) The **divisor equation**: for each $a = 1,\ldots,k$ (corresponding to the basis elements of $H^2(M)$ and $H_2(M)$),

$$\partial_{t_a}\Phi(q,t) = \partial_{t_a}\Phi(0,t) + q_a\partial_{q_a}\Phi(q,t).$$

(3) The **WDVV equation**: for each $i,j,k,\ell \in \{0,\ldots,N\}$,

$$\sum_{a,b}\left(\partial_{t_i}\partial_{t_j}\partial_{t_b}\Phi\right) g^{ba}\left(\partial_{t_a}\partial_{t_k}\partial_{t_\ell}\Phi\right) = (-1)^{|e_i|(|e_j|+|e_k|)}\sum_{a,b}\left(\partial_{t_j}\partial_{t_k}\partial_{t_b}\Phi\right) g^{ba}\left(\partial_{t_a}\partial_{t_i}\partial_{t_\ell}\Phi\right).$$

- Comments:
  (1) (FC) implies that very few terms in Φ can contain t_0, so the string equation is just telling us (via (Z)) what they are.
  (2) Since $|e_a| = 2$ in the divisor equation, $\partial_{t_a}\Phi$ is meant to detect terms that contain this particular degree 2 class, in which case the divisor axiom relates them to other terms that do not contain t_a.
  (3) The two sides of the WDVV equation are the same except for a cyclic permutation of the indices i, j, k (and a sign change associated with that cyclic permutation). These sums of quadratic products should remind you of the count of nodal curves in the splitting axiom, and we will justify the equation next time by relating both sides to two counts of smooth curves that are close to a nodal degeneration, in which certain permutations of the contraints obviously do not change the count.

Suggested reading. Our presentation of the Kontsevich-Manin axioms and the Gromov-Witten potential follows [MS12, §7.5 and §11.2] fairly closely. The original paper [KM94] also makes for interesting reading, though you need to be aware that Kontsevich and Manin regard what we call $\mathrm{GW}_{g,m,A}$ as a linear map $I_{g,m,A} : H^*(M)^{\otimes m} \to H^*(\overline{\mathcal{M}}_{g,m})$; our definition is obtained from theirs by setting $\mathrm{GW}_{g,m,A}(\alpha_1,\ldots,\alpha_m,\beta) := \langle I_{g,m,A}(\alpha_1\otimes\ldots\otimes\alpha_m),\beta\rangle$. Kontsevich and Manin also include some axioms for higher-genus invariants (including the result of Exercise 14.7(a)), though they do not do much with them.

In case you are curious what the actual definition of a *supermanifold* is, [Var04] is very good. The notion of supersymmetry first appeared in physics during the 1970's because it offered some hope for making string theory connect with reality,[34] but starting from Witten's interpretation [Wit82] of Morse theory as a supersymmetric quantum field theory, it has taken on a mathematical life of its own. These days, many algebraic objects that come with natural $\mathbb{Z}_2$-gradings can usefully be described in terms of supermanifolds.

Exercises (for the Übung on 15.02.2023).

Exercise 15.1. Assume (M,ω) is a closed symplecic Calabi-Yau 3-fold, meaning $\dim M = 2n := 6$ and $c_1(TM) = 0$. In this case, every moduli space of J-holomorphic curves without marked points has virtual dimension zero, and we can therefore associate to every $A \neq 0 \in H_2(M)$ a number

$$N_A := \mathrm{GW}_{0,0,A} \in \mathbb{Q},$$

[34]The original version of string theory turned out to produce a self-consistent theory only if spacetime is assumed to be 26-dimensional, which was generally regarded as a problem. Putting supersymmetry into the picture reduced 26 to 10, and this was considered an improvement since 10 is closer to 4.

interpreted as a count of finitely many solutions to an inhomogeneous nonlinear Cauchy-Riemann equation for maps $u : S^2 \to M$ with $[u] = A$.[35]

(a) Prove that if $\mathrm{GW}_{0,m,A}(\alpha_1, \dots, \alpha_m) \neq 0$, then the cohomology classes $\alpha_1, \dots, \alpha_m$ must all have degree 2.
Hint: Show via the grading axiom that if this is not true, then at least one of the α_i has degree greater than 2, meaning that one of the marked point constraints in the definition of $\mathrm{GW}_{0,m,A}(\alpha_1, \dots, \alpha_m)$ involves a submanifold of codimension greater than 2. Argue that generically, no solution will intersect that submanifold.

(b) The argument I have in mind for part (a) requires some knowledge of how the GW-invariants are defined, so it does not appear to follow from the axioms alone. Show however that if $H^1(M) = 0$, then the same conclusion can be deduced purely from the axioms.
Remark: If you can also do this without assuming $H^1(M) = 0$, then more power to you!

(c) Using a basis $A_1, \dots, A_k \in H_2(M)$ with corresponding formal variables $q_a := e^{A_a}$ and dual basis $e_1, \dots, e_k \in H^2(M)$ with corresponding formal variables $t_1, \dots, t_k$, show that the Gromov-Witten potential of (M, ω) satisfies

$$\Phi(q, t) = \Phi(0, t) + \sum_{d \neq 0 \in \mathbb{Z}^k} N_d q^d e^{\langle d, t \rangle},$$

where we abbreviate $N_d := N_{d_1 A_1 + \dots + d_k A_k}$ and $\langle d, t \rangle := d_1 t_1 + \dots + d_k t_k$.

(d) Would you expect the splitting axiom to provide any useful information about the numbers N_A in this situation?

Exercise 15.2. We have mentioned a few times that the m-point invariant $\mathrm{GW}_{g,m,A}$ can be defined for $2g + m < 3$ and $A \neq 0$ in a unique way so that the divisor axiom is satisfied: concretely, this means choosing any $k \in \mathbb{N}$ such that $2g + m + k \geqslant 3$, along with classes $\beta_1, \dots, \beta_k \in H^2(M)$ that satisfy $\langle \beta_i, A \rangle \neq 0$, and defining

$$\mathrm{GW}_{g,m,A}(\alpha_1, \dots, \alpha_m) := \frac{\mathrm{GW}_{g,m+k,A}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_k)}{\langle \beta_1, A \rangle \cdot \ldots \cdot \langle \beta_k, A \rangle} \in \mathbb{Q}$$

for each $\alpha_1, \dots, \alpha_m \in H^*(M)$. Show that the result is independent of the choice of k and $\beta_1, \dots, \beta_k$.
Hint: All you need is the knowledge that the divisor axiom holds for $2g + m \geqslant 3$.

Exercise 15.3. Let's take a closer look at the differential equations satisfied by the Gromov-Witten potential.

(a) Show that if any of the indices i, j, k, ℓ in the WDVV equation are 0, then the equation follows from the string equation.
(b) Verify explicitly that the Gromov-Witten potential of $\mathbb{CP}^1$ satisfies the string, divisor and WDVV equations.
(c) Check the string and divisor equations explicitly for the Gromov-Witten potential of $\mathbb{CP}^2$.

16. Week 16

Lecture 29 (14.02.2023): WDVV equations and the Kontsevich recursion formula.

- Proof of the string equation (via fundamental class and zero axioms)

[35] Recall that in order to define the inhomogeneous perturbation needed for transversality, one needs to add three marked points and impose incidence conditions on them via degree 2 cohomology classes $\alpha_1, \alpha_2, \alpha_3$, so that the divisor axiom gives the actual definition of N_A as $\frac{\mathrm{GW}_{0,3,A}(\alpha_1,\alpha_2,\alpha_3)}{\langle \alpha_1, A\rangle\langle \alpha_2, A\rangle\langle \alpha_3, A\rangle}$. This reveals why N_A might not be an integer, although $\mathrm{GW}_{0,3,A}(\alpha_1, \alpha_2, \alpha_3)$ is one.

- Proof of the divisor equation:
  Acting on Φ with $q_a \partial_{q_a}$ multiplies any term containing $q_a^{d_a}$ by d_a. For any multi-index γ not involving the variable t_a, acting on Φ with ∂_{t_a} and subtracting off the terms with no q replaces $\frac{1}{\gamma! k!} \mathrm{GW}_{|\gamma|+k,A_d}(e^\gamma, e_a, \dots, e_a) t^\gamma t_a^k$ for $k \geqslant 0$ and $d \neq 0$ with

$$\frac{1}{\gamma!(k-1)!} \mathrm{GW}_{|\gamma|+k,A_d}(e^\gamma, \underbrace{e_a, \dots, e_a}_{k}) t^\gamma t_a^{k-1} \overset{\text{(D)}}{=} \frac{1}{\gamma!(k-1)!} \langle e_a, A_d \rangle \mathrm{GW}_{|\gamma|+k-1,A_d}(e^\gamma, \underbrace{e_a, \dots, e_a}_{k-1}) t^\gamma t_a^{k-1}.$$

  Summing over all γ and k, the effect is the same since $\langle e_a, A_d \rangle = d_a$.
- Proof of the WDVV equation:
  Ignoring the signs for simplicity, we have

$$\partial_{t_i} \partial_{t_j} \partial_{t_b} \Phi = \sum_{d,\gamma} \frac{1}{\gamma!} \mathrm{GW}_{|\gamma|+3,A_d}(e^\gamma, e_i, e_j, e_b) q^d t^\gamma.$$

  Inserting this on the left hand side of the WDVV equation produces a summation over two multi-indices γ, γ' and two homology classes d_0, d_1 of terms that contain

$$\sum_{a,b} \mathrm{GW}_{|\gamma|+3,A_{d_0}}(e^\gamma, e_i, e_j, e_b) g^{ba} \mathrm{GW}_{|\gamma'|+3,A_{d_1}}(e_a, e_k, e_\ell, e^{\gamma'}) q^{d_0+d_1} t^{\gamma+\gamma'}.$$

  If we fix γ and γ' but sum over all terms for which $d_0 + d_1$ takes a fixed value d, the splitting axiom identifies the product with a GW-invariant counting curves homologous to A_d and constrained by $e^\gamma, e_i, e_j, e_k, e_\ell, e^{\gamma'}$, plus a condition on the domains. By the symmetry axiom, this is unchanged under a cyclic permutation of i, j, k.
- The example of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$: writing $\Phi_1 = \partial_{t_1}\Phi$, $\Phi_{01} = \partial_{t_0}\partial_{t_1}\Phi$ and so forth, our previous computation of Φ shows that $\Phi_{012} = \Phi_{022} = 0$ and $\Phi_{011} = 1$, so plugging $i = j = 1$ and $k = \ell = 2$ into the WDVV equation gives

$$\Phi_{111}\Phi_{122} + \Phi_{222} = (\Phi_{112})^2.$$

  Writing out these four derivatives as summations over $d \in \mathbb{N}$ and then writing the quadratic products as double summations over $k, \ell \in \mathbb{N}$ produces sums of various coefficients times $q^{k+\ell} e^{(k+\ell)t_1} t_2^{3(k+\ell)-4}$, except for the term Φ_{222}, which is a summation over $d \in \mathbb{N}$ of coefficients multiplied by $q^d e^{dt_1} t_2^{3d-4}$. Matching coefficients on both sides then leads to the **Kontsevich recursion formula**

$$N_d = \sum_{k,\ell \in \mathbb{N},\ k+\ell=d} \left(k^2 \ell^2 \binom{3d-4}{3k-2} - k^3 \ell \binom{3d-4}{3k-1} \right) N_k N_\ell,$$

  which determines all N_d starting from $N_1 = 1$.
- Theorem: Given any $d \in \mathbb{N}$ and distinct points $p_1, \dots, p_{3d-1} \in \mathbb{CP}^2$, for generic $J \in \mathcal{J}_\tau(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, there are exactly N_d curves u in $\mathcal{M}_{0,3d-1}(J, d[L])$ satisfying $\mathrm{ev}(u) = (p_1, \dots, p_{3d-1})$, all of them simple and immersed. (In other words, the rational GW-invariants of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ really are "enumerative".)[36]
  Proof sketch: By Exercise 14.1, $\mathrm{GW}_{3d-1,d[L]}(\mathrm{pt}, \dots, \mathrm{pt})$ is a signed count of simple curves in $\mathrm{ev}^{-1}(p_1, \dots, p_{3d-1}) \subset \mathcal{M}_{0,3d-1}(J, d[L])$, so it suffices to prove that all of these count with the same sign.

[36]Algebraic geometers sometimes complain that the GW-invariants are in general only "virtually" enumerative, in that they answer a slightly different enumerative question—one about the count of solutions to an equation that is inhomogeneously perturbed for the sake of transversality—than the one that is natural to ask in complex algebraic geometry. The content of this theorem is that for the particular example of $\mathbb{CP}^2$, the situation is much better than that.

Lemma 1 (proved via transversality of "jet evaluation maps"): Generically, the set of non-immersed simple curves is contained in a codimension 2 subset of the moduli space. In particular, for a generic family $\{J_s\}_{s\in[0,1]}$, we may assume that all curves in the 1-dimensional parametric moduli space

$$\mathcal{M} := \left\{(s,u) \;\middle|\; s \in [0,1],\; u \in \mathrm{ev}^{-1}(p_1,\ldots,p_{3d-1}) \subset \mathcal{M}_{0,3d-1}(J_s,d[L]) \text{ simple}\right\}$$

are immersed.
Lemma 2 (Hofer-Lizan-Sikorav [HLS97]): For all $J \in \mathcal{J}(M)$ on a 4-manifold M, immersed J-holomorphic spheres $u : S^2 \to M$ with $c_1([u]) > 0$ are always Fredholm regular.
Reason: The splitting $u^*TM \cong TS^2 \oplus N_u$ decomposes the linearized Cauchy-Riemann operator $\mathbf{D}_u$ in block form as

$$\mathbf{D}_u \cong \begin{pmatrix} \mathbf{D}_{S^2} & \ldots \\ 0 & \mathbf{D}_u^N \end{pmatrix},$$

where $\mathbf{D}_u^N$ is a Cauchy-Riemann type operator on the normal bundle $N_u \to S^2$. Since $\mathbf{D}_{S^2}$ is already known to be surjective, $\mathbf{D}_u$ is surjective if and only if $\mathbf{D}_u^N$ is, and the latter holds due to the similarity principle if $c_1(N_u) = c_1([u]) - \chi(S^2) \geqslant -1$. This works because N_u is a line bundle, so it depends crucially on the assumption that M is 4-dimensional.
For the parametric moduli space $\mathcal{M}$, Lemmas 1 and 2 together imply that the projection $\mathcal{M} \to [0,1] : (s,u) \mapsto s$ is a local diffeomorphism, so the moduli spaces for J_0 and J_1 do not just have the same *signed* count of elements, but are actually diffeomorphic. This shows that the actual number of curves is the same for all generic J; to show that this number really is N_d, one can also apply the Hofer-Lizan-Sikorav result to deduce from the determinant line bundle that all curves in these moduli spaces count positively.

- The following additional fun fact about $\mathbb{CP}^2$ was also mentioned but (by a vote of the majority) not proved:
  Theorem: Suppose (M,ω) is a closed symplectic 4-manifold containing a symplectically embedded sphere $S^2 \cong S \subset (M,\omega)$ such that $[S] \bullet [S] = 1$ but no symplectically embedded spheres $S^2 \cong E \subset (M,\omega)$ with $[E] \bullet [E] = -1$. Then (M,ω) is symplectomorphic to $(CP^2, c\omega_{\mathrm{FS}})$ for some $c > 0$, via a symplectomorphism identifies S with a complex line.
  Comment: This result is due to Gromov and McDuff [Gro85, McD90], and is discussed with full details in [Wen18]. The proof uses the same moduli space that underlies the computation of $N_1 = \mathrm{GW}_{2,[L]}(\mathrm{pt},\mathrm{pt})$, plus positivity of intersections; it is closely related to the fact that any two distinct points in $\mathbb{CP}^2$ are connected by a unique J-holomorphic line for every J.

Lecture 30 (15.02.2023): Quantum cohomology.

- Idea: Interpret the Gromov-Witten potential as a function on $H^*(M)$ and derive geometric/algebraic structure on $H^*(M)$ from the WDVV equation
- Toy model: Assume $(V, g = \langle\ ,\ \rangle)$ is a pseudo-Riemannian manifold with an *affine flat structure*, i.e. an atlas of charts in which the components $g_{ij} := \langle \partial_i, \partial_j \rangle$ are constant and all transition maps are affine.[37] The Levi-Cività connection ∇ is then flat, and all parallel local vector fields commute with each other. All other symmetric connections ∇' on V are of the form

$$\nabla'_X Y = \nabla_X Y + X \circ Y$$

[37]Being a vector space, $H^*(M)$ naturally has an affine flat structure, at least if one uses coefficients in $\mathbb{R}$ instead of $\mathbb{Q}$. For this informal discussion, we will pretend that all vector spaces can be regarded as manifolds, regardless of the ground field.

for a symmetric fiberwise-bilinear pairing $TV \oplus TV \xrightarrow{\circ} TV$, which can always be defined via the relation

$$\langle X \circ Y, Z \rangle = A(X, Y, Z)$$

for some covariant rank 3 tensor field A that is symmetric in X and Y.

- Observation: ∇' is compatible with $g \iff \langle X \circ Y, Z \rangle + \langle Y, X \circ Z \rangle = 0$ for all $X, Y, Z \iff A(X, Y, Z)$ is antisymmetric in Y and Z. But this could only happen if $A = 0$, since the Levi-Cività connection is unique. With this in mind, we will instead assume A is *symmetric* in Y and Z, i.e. A is fully symmetric.
- Observation: If A is symmetric, then the perturbed connections $\nabla^\lambda_X Y := \nabla_X Y + \lambda X \circ Y$ are also flat for all $\lambda \in \mathbb{R}$ if and only if the following two conditions are satisfied:
  (1) The product $\circ$ on each fiber of TV is associative;
  (2) $\nabla_X(Y \circ Z) - \nabla_Y(X \circ Z) = 0$ for all X, Y, Z that are parallel with respect to ∇.
  Proof: Easy computation.
- Proposition: The second condition above is satisfied if and only if every point in V has a neighborhood on which the tensor A can be written as[38]

$$A(X, Y, Z) = \mathcal{L}_X \mathcal{L}_Y \mathcal{L}_Z \Phi$$

  for all parallel local vector fields X, Y, Z and some "potential" function Φ. Given this, the product $\circ$ is then associative if and only if Φ satisfies the WDVV equation in local flat coordinates.
  Proof: In local flat coordinates, using the Einstein summation convention, we can write $(X \circ Y)^i = A^i{}_{jk} X^j Y^k$ where $A^i{}_{jk} := g^{i\ell} A_{\ell jk}$ and $A_{ijk} := A(\partial_i, \partial_j, \partial_k)$. The condition $\nabla_X(Y \circ Z) - \nabla_Y(X \circ Z) = 0$ is then equivalent to $\partial_i A_{jk\ell} - \partial_k A_{ji\ell} = 0$, and combining this with the fact that A_{ijk} is symmetric under permutations of i, j, k leads via three successive applications of the Poincaré lemma to the existence of a function Φ satisfying $\partial_i \partial_j \partial_k \Phi = A_{ijk}$. Associativity of $\circ$ is now a quadratic relation on the components A_{ijk} that becomes the WDVV equation (in a simpler version without the annoying signs) when we substitute $A_{ijk} = \partial_i \partial_j \partial_k \Phi$.
- Definition: V together with its affine flat structure and a globally defined potential function Φ as described above is called a **Frobenius manifold**. This is only the "classical" version—one can also generalize the whole discussion allowing V to be a *supermanifold*, in which Φ will depend locally on a mixture of even and odd variables, with the function ring defined so that odd variables anticommute with each other. This necessitates adding quite a lot of signs to the discussion above, but otherwise changes very little. If you don't want to think about signs, just assume your symplectic manifold has no cohomology in odd degrees.
- Simplifying assumption (SA) on (M, ω): the Gromov-Witten potential $\Phi(q, t)$ is a *convergent* power series with respect to $t_0, \ldots, t_N$, having coefficients in the Novikov ring

$$\Lambda_\omega = \left\{ \sum_{d \in \mathbb{Z}^k} c_d q^d \;\middle|\; c_d \in \mathbb{Q} \text{ and } \{d \mid c_d \neq 0 \text{ and } \omega(A_d) \leqslant C\} \text{ is a finite set for every } C \in \mathbb{R} \right\},$$

  and can thus be interpreted literally as a function $\Phi : H^*(M) \to \Lambda_\omega$. This is true in many interesting examples such as $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, though the most important definition below will not actually depend on it.
- Definition: Equip $H^*(M; \Lambda_\omega) \cong H^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega$ with the Poincaré duality pairing

$$(\alpha, \beta) := \langle \alpha \cup \beta, [M] \rangle \in \Lambda_\omega,$$

[38] Here $\mathcal{L}_X$ denotes the Lie derivative with respect to a vector field, so applying it to a function f just gives the differential $\mathcal{L}_X f = df(X)$. In the lecture I wrote ∇_X instead of $\mathcal{L}_X$, but on closer inspection this was not a good choice of notation, as the derivative we're using here does not depend on the connection.

which allows us to think of $H^*(M;\Lambda_\omega)$ informally as a pseudo-Riemannian manifold with an affine flat structure. Given (SA), the **big quantum cohomology** of (M,ω) is a family of Λ_ω-bilinear products $*_t$ on $H^*(M;\Lambda_\omega)$ determined for each $t \in H^*(M) := H^*(M;\mathbb{Q})$ via the relation[39]

$$(e_a *_t e_b, e_c) = \partial_{t_c}\partial_{t_b}\partial_{t_a}\Phi(t) \in \Lambda_\omega$$

for our chosen basis elements $e_0,\dots,e_N \in H^*(M)$. For each $t \in H^*(M)$, this product is graded commutative due to the commutation relations satisfied by the operators ∂_{t_a} and ∂_{t_b}, and the previous discussion of Frobenius manifolds was meant to convince you that it is associative as a result of the fact that Φ satisfies the WDVV equation, i.e. it is a consequence of the splitting axiom for the Gromov-Witten invariants. Informally, we can think of $*_t$ as a product on the tangent space to $H^*(M;\Lambda_\omega)$ at t.

- Definition: The **small quantum cup product** on $H^*(M;\Lambda_\omega)$ is $* := *_0$. This is well defined without (SA) since evaluating a power series at $t = 0$ does not require the series to converge. This is also a graded commutative and associative product, and the ring $H^*(M;\Lambda_\omega)$ with this product is often called the **quantum cohomology** of (M,ω) and denoted by $QH^*(M,\omega;\Lambda_\omega)$, or simply $QH^*(M,\omega)$. It is uniquely characterized by the formula

$$(\alpha * \beta, \gamma) = \sum_d \mathrm{GW}_{3,A_d}(\alpha,\beta,\gamma)q^d \qquad \text{for all } \alpha,\beta,\gamma \in H^*(M), \tag{16.1}$$

or equivalently,

$$\alpha * \beta = \sum_{c=0}^{N}\left(\sum_d \mathrm{GW}_{3,A_d}(\alpha,\beta,e_c)q^d\right)e^{(c)},$$

where the expression in parentheses belongs to Λ_ω and we let $e^{(0)},\dots,e^{(N)}$ denote the basis of $H^*(M)$ satisfying $(e^{(a)},e_b) = \delta^a_b$ for all a,b.
- Notice: The $d = 0$ term in (16.1) is $(\alpha \cup \beta, \gamma) = \langle \alpha \cup \beta \cup \gamma, [M]\rangle$, which is the count of constant holomorphic spheres passing through generic submanifolds Poincaré dual to α,β,γ and thus reduces to a count of intersections between those submanifolds. The $d \neq 0$ terms are called **quantum corrections**, and are obtained morally by counting nonconstant J-holomorphic spheres through those same three submanifolds.
- Proposition (easy):
  1. $\alpha * \beta$ is $\alpha \cup \beta$ plus terms that depend on q^d for $d \neq 0$, so in particular, the quantum cup product reduces to the classical cup product in the "limit as $q \to 0$".
  2. $QH^*(M,\omega;\Lambda_\omega)$ has a unit: $1 * \alpha = \alpha * 1 = \alpha$.
  3. In some situations (e.g. in $\mathbb{CP}^2$), the formula above for $\alpha * \beta$ contains only finitely many terms, in which case the coefficient ring Λ_ω can be replaced by something simpler such as the group ring $\mathbb{Q}[H_2(M)]$. In this case, $*$ respects a grading defined on $QH^*(M,\omega;\mathbb{Q}[H_2(M)])$ such that each of the variables $q_i = e^{A_i} \in \mathbb{Q}[H_2(M)]$ corresponding to the chosen basis elements $A_i \in H_2(M)$ is defined to have degree $|q_i| := 2c_1(A_i)$, hence $|q^d| = 2c_1(A_d)$ for every $d \in \mathbb{Z}^k$. This is a consequence of the grading axiom for the GW-invariants.
- Computation of $QH^*(\mathbb{CP}^2,\omega_{\mathrm{FS}})$: Using the usual basis $e_0 = 1 \in H^0(\mathbb{CP}^2)$, $e_1 = \mathrm{PD}^{-1}([L]) \in H^2(\mathbb{CP}^2)$ and $e_2 = \mathrm{pt} \in H^4(\mathbb{CP}^2)$, the classical cohomology ring of $\mathbb{CP}^2$ is generated by

[39]My reversal of the order of the differential operators on the right hand side of this relation is a half-hearted attempt to get all the signs right, but honestly, everything I say about signs in this lecture should be taken with a grain of salt.

$p := e_1$, which satisfies $p^2 := e_1 \cup e_1 = e_2$ but $p^3 = 0$, so

$$H^*(\mathbb{CP}^2; \mathbb{Q}) \cong \mathbb{Q}[p]/\langle p^3 \rangle.$$

The Novikov ring introduces one additional generator $q = e^{[L]} \in \mathbb{Q}[H_2(\mathbb{CP}^2)]$. The quantum product depends only on the cubic terms in the Gromov-Witten potential, which are

$$\Phi(q,t) = \frac{1}{2}(t_0^2 t_2 + t_0 t_1^2) + \frac{1}{2} q t_1 t_2^2 + \text{non-cubic terms}.$$

Here the first two terms come from GW-invariants with $d = 0$, so they will reproduce the classical cup product. The third term contains the computation $\mathrm{GW}_{3,[L]}(e_1, e_2, e_2) = \mathrm{GW}_{2,[L]}(\mathrm{pt}, \mathrm{pt}) =: N_1 = 1$. Since the only term quadratic in t_1 is classical, we have

$$p^2 := p * p = e_1 * e_1 = e_1 \cup e_1 = e_2.$$

However, the term $t_1 t_2^2$ produces a quantum correction in $e_1 * e_2$, giving

$$p^3 = e_1 * e_2 = \partial_{t_1} \partial_{t_2}^2 \frac{1}{2} q t_1 t_2^2 e^{(2)} = q e_0 = q.$$

All other products are determined from these via associativity, thus

$$QH^*(\mathbb{CP}^2, \omega_{\mathrm{FS}}) \cong \mathbb{Q}[p,q]/\langle p^3 - q \rangle.$$

Notice that the coefficient ring here can be reduced to $\mathbb{Q}[H_2(\mathbb{CP}^2)]$, and since $c_1([L]) = 3$, $QH^*(\mathbb{CP}^2, \omega_{\mathrm{FS}}; \mathbb{Q}[H_2(\mathbb{CP}^2)])$ then has a natural grading in which $|p| = 2$ and $|q| = 6$, so it contains a 1-dimensional subspace in every even degree.

Suggested reading. Chapter 11 of [MS12] contains a good general discussion of the Gromov-Witten potential and the PDEs that it satisfies. It also contains far more than you probably want to read (at least on a first pass) about the possible choices of coefficients for $QH^*(M, \omega)$, and a very interesting but necessarily incomplete discussion of some computations that are much deeper and less trivial than the one we carried out for $\mathbb{CP}^2$.

If you want to learn more about Frobenius manifolds and the "big" quantum cohomology, I recommend the book by Manin [Man99]. (For a quick summary of the main points, [KM94, §4] is also not bad.) You may find it more digestible if you first learn from [Var04] what ringed spaces and supermanifolds are.

If you find pseudo-mathematical speculation by visionary physicists fascinating, then I also recommend taking a look at [Vaf92, §4], which was written shortly after Witten's topological sigma-model paper [Wit88b] and has sometimes been cited as the first place where the construction of quantum cohomology was ever suggested.[40] From Vafa's perspective, the classical cohomology ring of a Kähler manifold M embeds naturally into the operator algebra of a fermionic string theory whose underlying Hilbert space is the space of semi-infinite differential forms on the loop space of M. Quantum cohomology was then predicted based on the properties of this operator algebra in quantized string theory! The theory later took on its present mathematical form due mainly to the work of Ruan-Tian [RT95], Kontsevich-Manin [KM94] and McDuff-Salamon [MS94], though Vafa's paper already contains a heuristic description of something that you will easily recognize as the small quantum cup product.

Since the course is now over, I'll take this opportunity to tie up one other loose end: Gerard asked at some point whether the GW-invariants can actually be used for the obvious symplectic

[40]Witten's paper [Wit91], which appeared around the same time, also contains (around page 274) a brief sketch of the main idea, in addition to several other ideas (e.g. gravitational descendants) that soon became central to the subject of Gromov-Witten theory. I do not happen to know why Witten's and Vafa's papers did not cite each other.

application, namely to distinguish symplectic manifolds that are diffeomorphic but not symplectically deformation equivalent. In fact, there are several results of this type in the early papers on this subject by Ruan [Rua94, Rua96], and these were presented as one of the original selling points of the rigorous construction on semipositive symplectic manifolds.

Exercises (just for fun).

Exercise 16.1. Suppose (M, ω) is a Calabi-Yau 3-fold, and after choosing a basis $A_1, \ldots, A_k$ of $H_2(M)$, write $N_d := N_{A_d} := \mathrm{GW}_{0,0,A_d} \in \mathbb{Q}$ for $d = (d_1, \ldots, d_k) \in \mathbb{Z}_k$ and $A_d := d_1 A_1 + \ldots + d_k A_k$ as in Exercise 15.1. Since $c_1(A_d) = 0$ for all d, the small quantum cup product $*$ on $H^*(M; \Lambda_\omega)$ preserves the obvious grading in which all elements of Λ_ω are assigned degree 0. Show that $\alpha * \beta$ then differs from $\alpha \cup \beta$ only when $\alpha, \beta \in H^*(M)$ both have degree 2, and for this case, prove the formula

$$\alpha * \beta = \alpha \cup \beta + \sum_{d \neq 0} N_d \langle \alpha, A_d \rangle \langle \beta, A_d \rangle q^d \, \mathrm{PD}^{-1}(A_d) \in H^4(M; \Lambda_\omega).$$

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