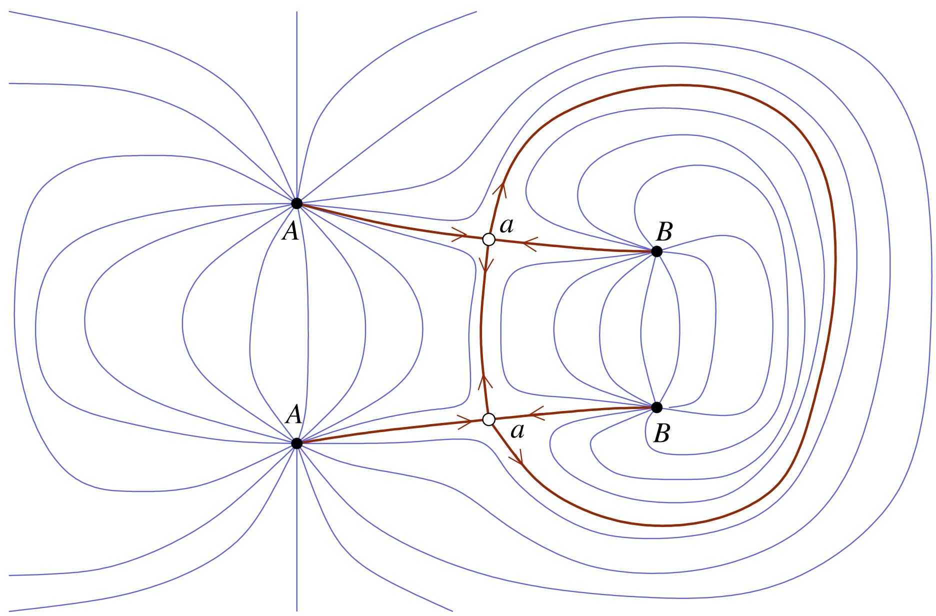
Dynamics, Holomorphic Curves and Foliations:

Using a PDE to solve an ODE problem



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http://math.mit.edu/~wendlc/publications.html

Outline

1. Hamiltonian dynamics. . . contact geometry
2. Pseudoholomorphic curves
3. Symplectizations and finite energy curves
4. Holomorphic foliations

References:

- Hofer, H. *Holomorphic curves and real three-dimensional dynamics.* *Geom. Funct. Anal.* **2000**, Special Volume, Part II, 674–704.

- Wendl, C. *Finite energy foliations on overtwisted contact manifolds.* Preprint SG/0611516.

PART 1: Hamiltonian Dynamics...
Contact Geometry

$x = (q_1, p_1, q_2, p_2) \in \mathbb{R}^4$

Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$

Equations of motion:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

In terms of a Hamiltonian vector field:

$$X_H : \mathbb{R}^4 \to \mathbb{R}^4, \qquad \dot{x} = X_H(x)$$

Hamiltonian flow: $\varphi_H^t : \mathbb{R}^4 \to \mathbb{R}^4$

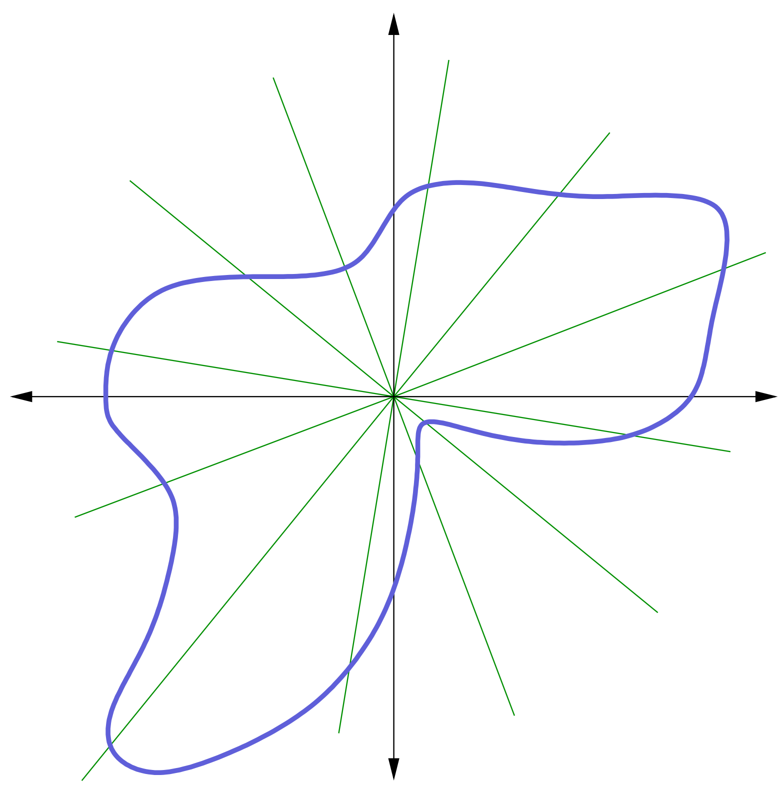
Fact: φ_H^t preserves level sets $H^{-1}(c)$

Flow on Level Sets

For generic c, $M := H^{-1}(c)$ is a three-dimensional manifold.

Natural question: Does X_H have any periodic orbits on M? (If so, how many?)

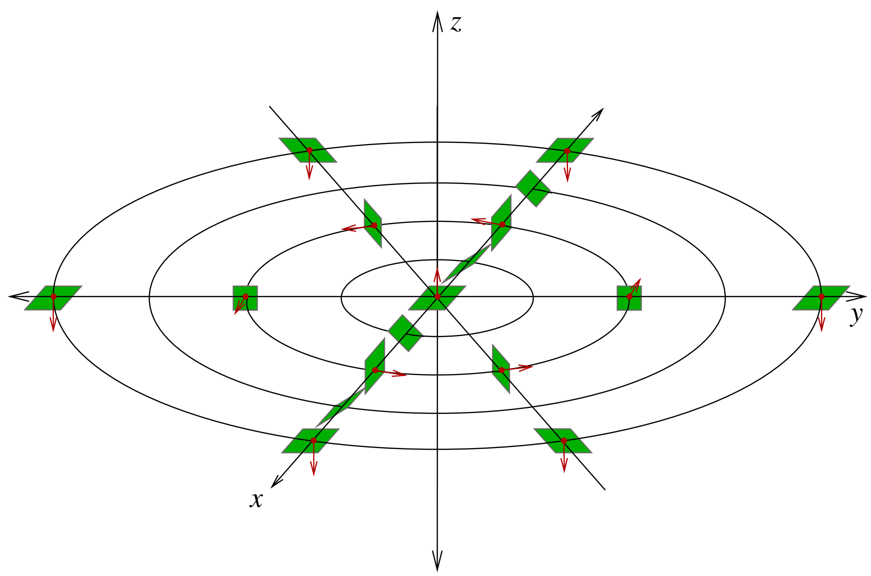
Theorem (P. Rabinowitz, 1978)**.** *: If M is a compact star-shaped hypersurface, then it has a periodic orbit.*



Contact Manifolds

$M =$ an oriented 3-dimensional manifold

A **contact structure** ξ on M is a choice of oriented 2-planes $\xi_x \subset T_xM$ at every point $x \in M$, such that ξ is totally nonintegrable.

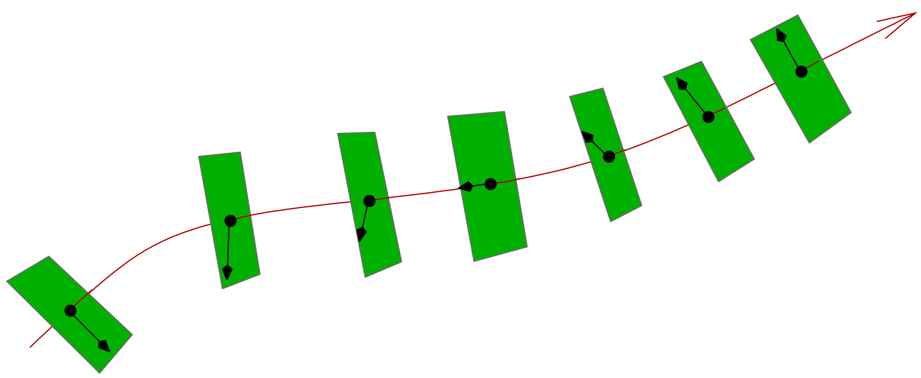


$(M, \xi) \cong (M', \xi')$ means there is a diffeomorphism $\psi : M \to M'$ such that $d\psi(\xi) = \xi'$.

Then ψ is a **contactomorphism**.

Reeb Dynamics

Given (M, ξ), a **Reeb vector field** X is a vector field positively transverse to ξ such that the flow preserves ξ.



Example: $M = H^{-1}(c) \subset \mathbb{R}^4$ a star-shaped energy surface, then M has a natural contact structure and X_H is a Reeb vector field.

Conjecture (A. Weinstein '78). *Every Reeb vector field X on a compact contact manifold (M, ξ) admits a periodic orbit.*

Some Weinstein Conjecture History

- **C. Viterbo '87**: true for all contact hypersurfaces in $\mathbb{R}^4$

- **H. Hofer '93**: true for all contact structures on S^3, or any M with $\pi_2(M) \neq 0$, or any M if ξ is overtwisted

- **C. Taubes '06**: true for all contact 3-manifolds

But how many?

- **H. Hofer, K. Wysocki, E. Zehnder '03**: Generic compact star-shaped energy surfaces in R^4 admit either 2 or infinitely many periodic orbits!

Question: does any similar "2 or ∞" result hold for generic contact 3-manifolds?

PART 2: Pseudoholomorphic Curves

Suppose $u : \mathbb{C} \to \mathbb{C}$ is smooth.

Identify $\mathbb{C} = \mathbb{R}^2$, so $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$du(z) : T_z\mathbb{R}^2 \to T_{u(z)}\mathbb{R}^2$ is a 2-by-2 matrix

Then u is analytic (holomorphic) iff

$$\boxed{du(z) \circ i = i \circ du(z)}$$

for all z.

We define holomorphic maps $u : \mathbb{C}^n \to \mathbb{C}^m$ the same way using

$$i = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

W = an even-dimensional manifold

An almost complex structure J is a smooth family of linear maps $J(x) : T_xW \to T_xW$ with $J(x)^2 = -\mathrm{Id}$.

A map $u : (W, J) \to (W', J')$ is called pseudoholomorphic if

$$\boxed{du(x) \circ J(x) = J'(u(x)) \circ du(x)}$$

for all $x \in W$.

If $\dim \Sigma = 2$, (Σ, j) is a Riemann Surface, and $u : (\Sigma, j) \to (W, J)$ is a pseudoholomorphic (or J-holomorphic) curve.

embedded pseudoholomorphic curves $\cong$ surfaces in W with J-invariant tangent spaces

M. Gromov '85: These are useful in symplectic geometry.

This isn't complex analysis anymore. . .

In local coordinates (s, t) on Σ, $du \circ j = J \circ du$ becomes

$$\partial_s u + J(u)\partial_t u = 0,$$

a nonlinear first-order *elliptic* PDE.

Douglis, Nirenberg '55: The linearized operator $\bar{\partial} = \partial_s + i\partial_t : W^{1,p} \to L^p$ satisfies

$$\|v\|_{W^{1,p}} \leq C\|\bar{\partial}v\|_{L^p}$$

for all $v \in C_0^\infty(\mathbb{C}, \mathbb{C}^n)$.

Consequences: under certain assumptions, solution spaces are

- compact (bubbling off analysis)
- finite dimensional manifolds (Fredholm theory)

Local Structure of Solution Spaces

Suppose u_0 is a J-holomorphic curve.

Choose Banach spaces (or Banach manifolds) X and Y and a smooth map $F : X \to Y$ so that

- $u_0 \in X$
- a map u near u_0 is J-holomorphic iff $F(u) = 0$

The linearization $DF(u_0) : X \to Y$ is defined by

$$F(u_0 + h) = F(u_0) + DF(u_0)h + o(\|h\|)$$

Suppose $DF(u_0)$ is surjective and has kernel of dimension $N < \infty$. Then the implicit function theorem $\Rightarrow F^{-1}(0)$ is a smooth N-dimensional manifold.

$N =$ the Fredholm index of $DF(u_0)$.

PART 3: Symplectizations and Finite Energy Curves

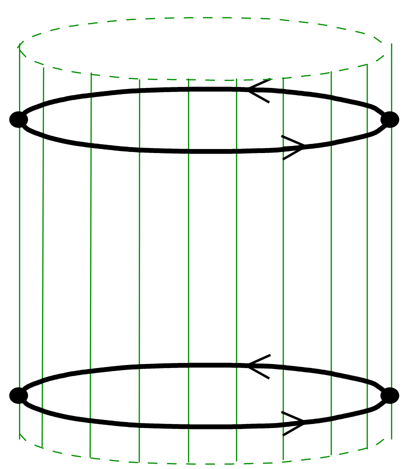
Choose 3-dimensional (M, ξ), with Reeb X

Let $W = \mathbb{R} \times M$, the symplectization of M

$$T(\mathbb{R} \times M) = (\mathbb{R} \oplus \mathbb{R}X) \oplus \xi$$

Define J : $\mathbb{R} \to \mathbb{R}X$, $\xi \to \xi$.

If γ is a closed Reeb orbit, $\mathbb{R} \times \gamma \subset \mathbb{R} \times M$ is a J-holomorphic cylinder! (orbit cylinder)



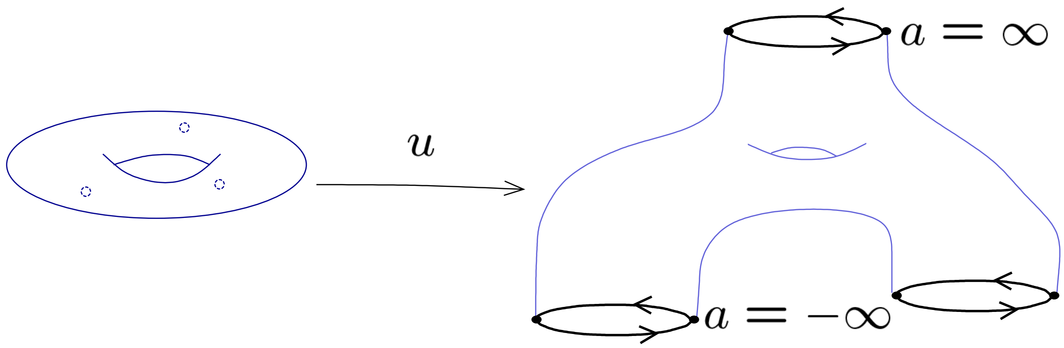
Cylinder $\cong$ 2-punctured sphere:
$\mathbb{R} \times S^1 \cong \mathbb{C} \setminus \{0\} \cong S^2 \setminus \{0, \infty\}$

Consider J-holomorphic curves

$$\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$$

where $\dot{\Sigma}$ is a closed Riemann surface with finitely many punctures.

All such maps with finite energy are asymptotically cylindrical at the punctures.

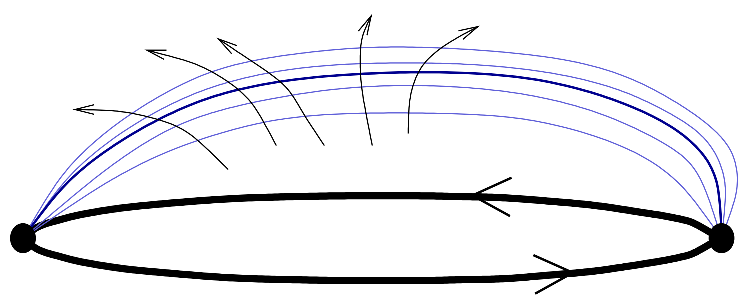


Existence of $\tilde{u} \Rightarrow$ Weinstein!

Bonus: if $u : \dot{\Sigma} \to M$ is embedded, it's transverse to X.

Index 2 planes

$\tilde{u} = (a, u) : \mathbb{C} = S^2 \setminus \{\infty\} \to \mathbb{R} \times M$



Implicit function theorem

$\Rightarrow$ smooth 2-parameter family in $\mathbb{R} \times M$

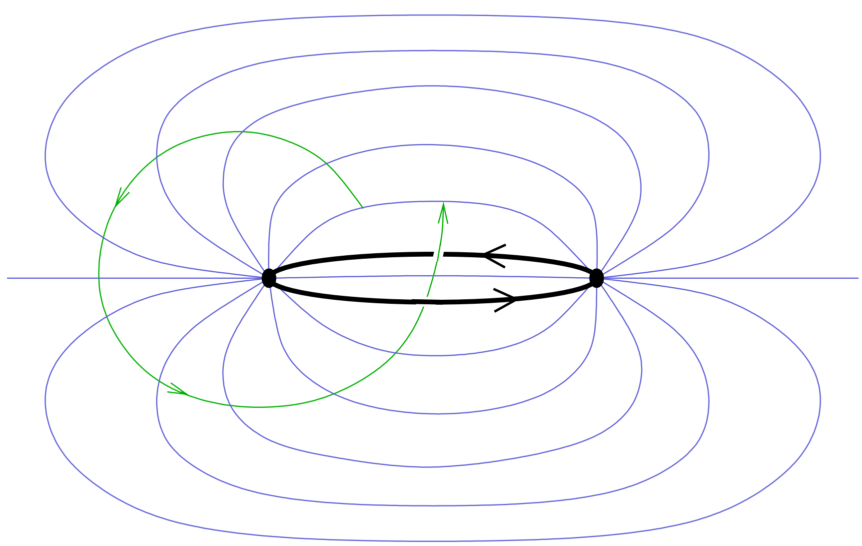
$\Rightarrow$ projects to 1-parameter family $\pitchfork X$ in M.

PART 4: Holomorphic Foliations

Fredholm theory $\Rightarrow$ holomorphic curves appear in families

Compactness $\Rightarrow$ families don't go on forever

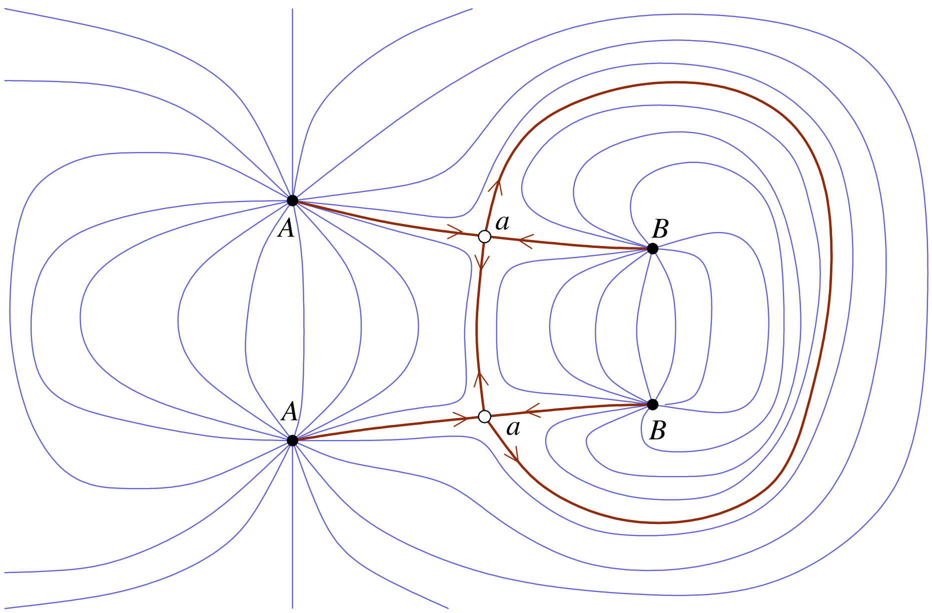
Open book decomposition of S^3 (i.e. $\mathbb{R}^3 \bigcup \{\infty\}$):



Return map is area-preserving.

Consequence: 2 or ∞.

Theorem (Hofer, Wysocki, Zehnder '03). *Generic star-shaped energy surfaces in* $\mathbb{R}^4$ *admit finite energy foliations.*



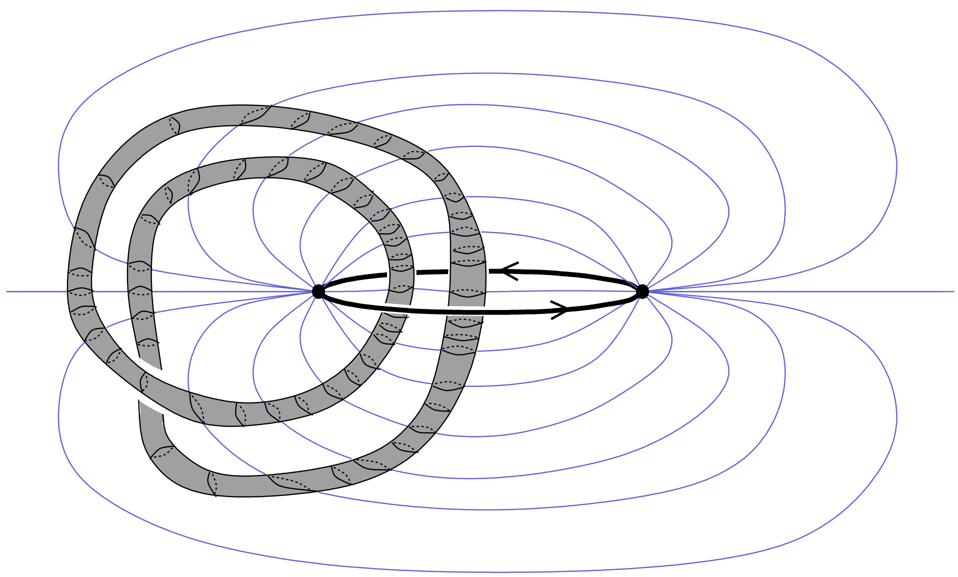
index 0: orbit cylinders

index 1: rigid surfaces

index 2: 1-parameter families in M

Theorem (W. '05). *Every overtwisted contact manifold has a Reeb vector field that admits a finite energy foliation.*

Construct by Dehn surgery on S^3:

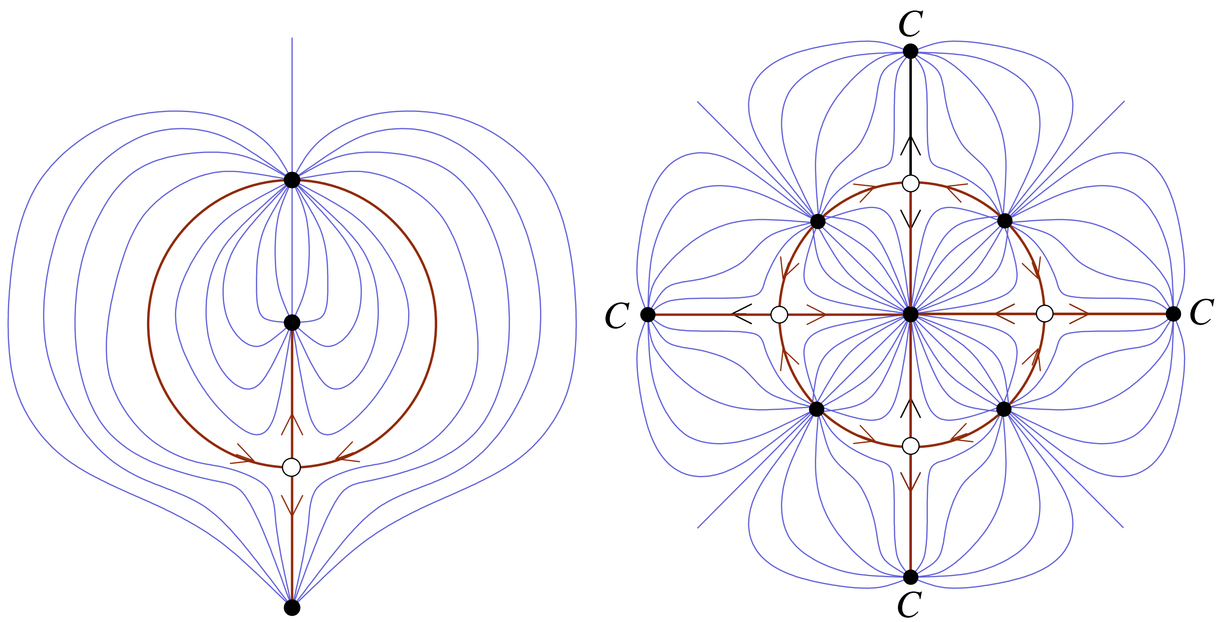


Conjecture. *This is true for generic Reeb vector fields on overtwisted contact manifolds.*

Rallying cry:

"If holomorphic curves are everywhere, it's hard to kill them."

Homotopy of foliations: $S^1 \times S^2$



These are homotopic to each other.
But (conjecturally) not to this one: