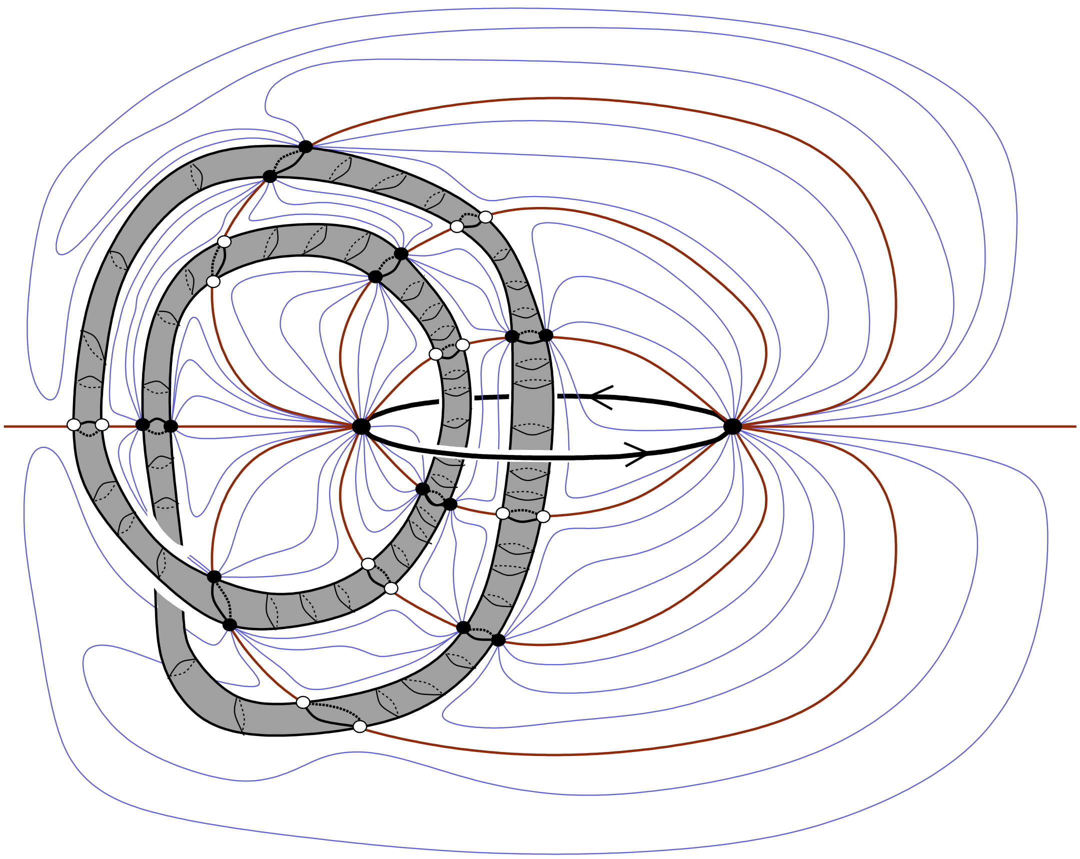
Holomorphic Foliations in Contact 3-Manifolds



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Outline

I. Background: contact geometry, holomorphic curves, Weinstein conjecture

II. The foliation program

III. Existence in the overtwisted case

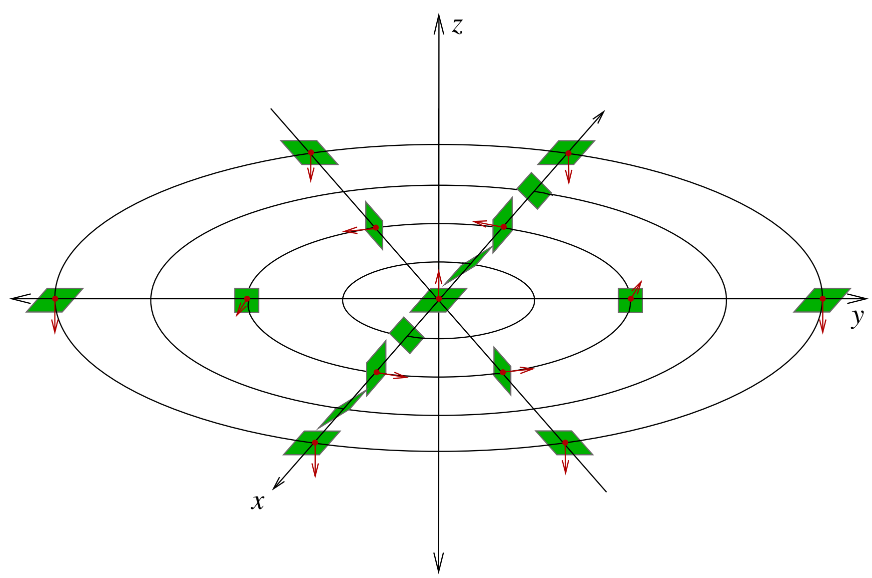
IV. Outlook

I. Background

- $M =$ closed oriented 3-manifold
- $\lambda =$ positive contact form, $\lambda \wedge d\lambda > 0$
- $\xi = \ker \lambda =$ contact structure
- $X_\lambda =$ Reeb vector field, defined by

$$d\lambda(X_\lambda, \cdot) = 0, \quad \lambda(X_\lambda) = 1$$

  Thus $TM = \mathbb{R}X_\lambda \oplus \xi$.



Conjecture (Weinstein). *For all (M, λ), the vector field X_λ has a periodic orbit.*

Symplectizations and holomorphic curves

$\mathbb{R} \times M$ = the *symplectization* of M

$$T(\mathbb{R} \times M) = (\mathbb{R} \oplus \mathbb{R}X_\lambda) \oplus \xi$$

The splitting yields a natural class of $\mathbb{R}$-invariant almost complex structures

$$\tilde{J} = i \oplus J : T(\mathbb{R} \times M) \to T(\mathbb{R} \times M).$$

We consider $\tilde{J}$-holomorphic maps

$$\tilde{u} = (a, u) : (\dot{\Sigma}, j) \to (\mathbb{R} \times M, \tilde{J})$$

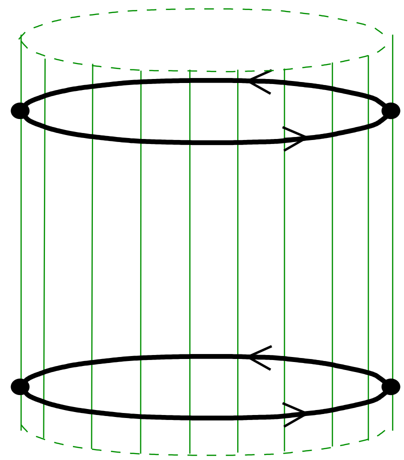
$$0 < E(\tilde{u}) < \infty$$

where $(\dot{\Sigma}, j)$ is a punctured Riemann surface.

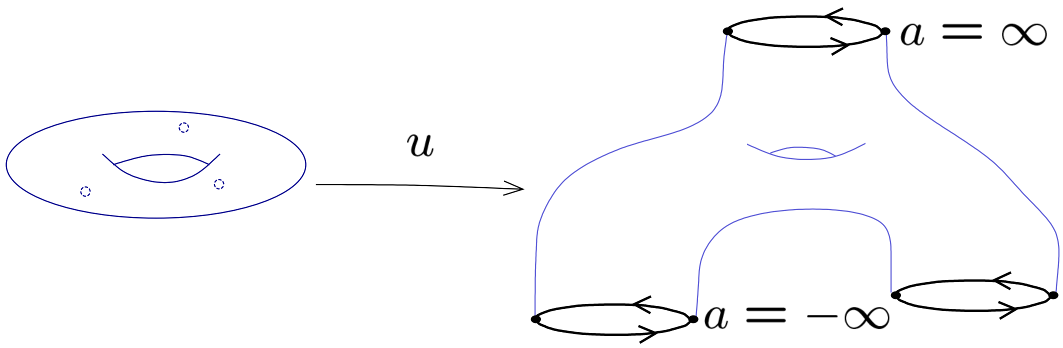
These are *finite energy surfaces*.

Asymptotics

- Simple example:
  Given a periodic orbit $P \subset M$, $\mathbb{R} \times P$ is an **orbit cylinder**.
  ($\tilde{J}$-holomorphic)



- All finite energy surfaces $\tilde{u} = (a, u)$ are *asymptotically cylindrical at the punctures*: $a \to \pm\infty$ and $u \to$ a periodic orbit.



$\therefore$ Existence of a holomorphic curve $\Rightarrow$ Weinstein conjecture! (Hofer '93)

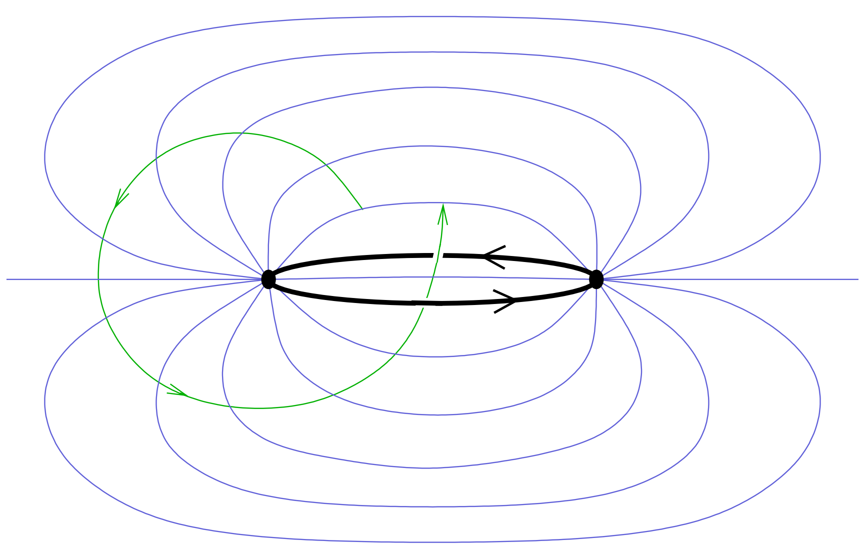
But let's not stop there...

Theorem (Hofer, Wysocki, Zehnder '03)**.** *For the standard contact structure on* S^3, *generic contact forms admit either* **2 or infinitely many** *periodic orbits.*

Idea of Proof:

"Holomorphic curves are everywhere!"

(And they're transverse to X_λ.)



II. The Foliation Program

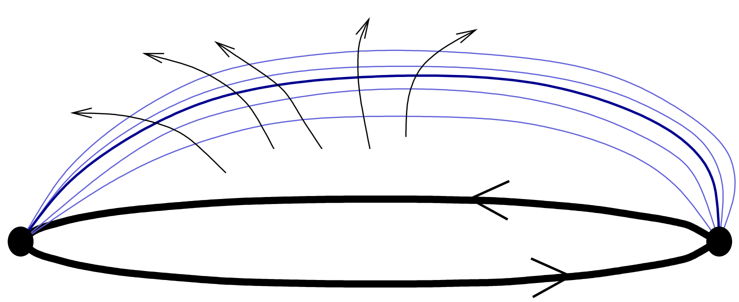
Thinking locally, consider

$$\tilde{u} = (a, u) : (\mathbb{C}, i) \to (\mathbb{R} \times M, \tilde{J})$$

embedded with Conley-Zehnder index 3.

Fredholm theory + intersection theory $\Rightarrow$

- $u : \mathbb{C} \to M$ is embedded, $u \pitchfork X_\lambda$
- $\tilde{u}(\mathbb{C}) \subset$ a local 2-dimensional foliation of $\mathbb{R} \times M$
- $u(\mathbb{C}) \subset$ a local 1-dimensional foliation of M, all leaves transverse to X_λ



Question: can we do this globally?

Definition. A <u>*finite energy foliation*</u> of M is a collection of embedded finite energy surfaces $\{\tilde{u}_\alpha = (a_\alpha, u_\alpha) : \dot{\Sigma}_\alpha \to \mathbb{R} \times M\}_{\alpha \in I}$ such that

1. $\tilde{u}_\alpha(\dot{\Sigma}_\alpha)$ foliate $\mathbb{R} \times M$
2. If (a, u) is a leaf, then so is $(a + c, u)$ for every constant $c \in \mathbb{R}$

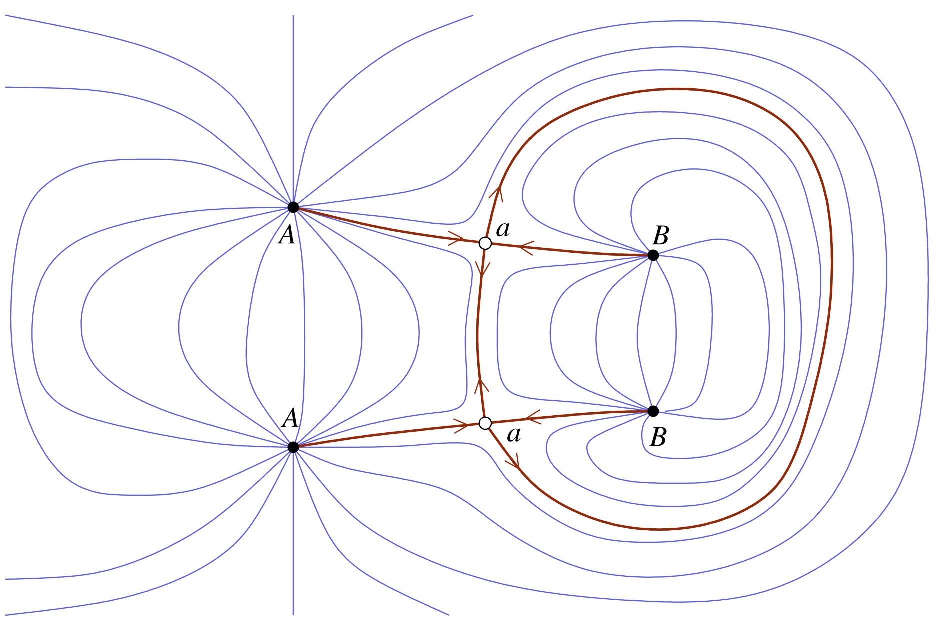
Consequences: (due to intersection theory)

1. If $\mathcal{P} \subset M$ is the union of all asymptotic orbits for leaves in the foliation, then every orbit cylinder $\mathbb{R} \times P$ for $P \subset \mathcal{P}$ is a leaf.
2. The maps $u_\alpha : \dot{\Sigma}_\alpha \to M$ are embedded and foliate $M \setminus \mathcal{P}$.

Foliations of interest here are

- **spherical**: each leaf has genus 0
- **stable**: each leaf has a *surjective* linearization with "the right Fredholm index"

Index 0 $\Rightarrow$ orbit cylinder
Index 1 $\Rightarrow$ rigid surface
Index 2 $\Rightarrow$ 1-parameter family of leaves in M



Some existence results

Theorem (Hofer, Wysocki, Zehnder '03). *Foliations exist for generic contact forms on the tight three-sphere.*

Corollary. *2 or* ∞.

Theorem (Abbas '04). *Giroux's open book decompositions in the* planar *case can be deformed into spherical finite energy foliations.*

Corollary (Abbas, Cieliebak, Hofer '04). *Weinstein conjecture for planar contact structures.*

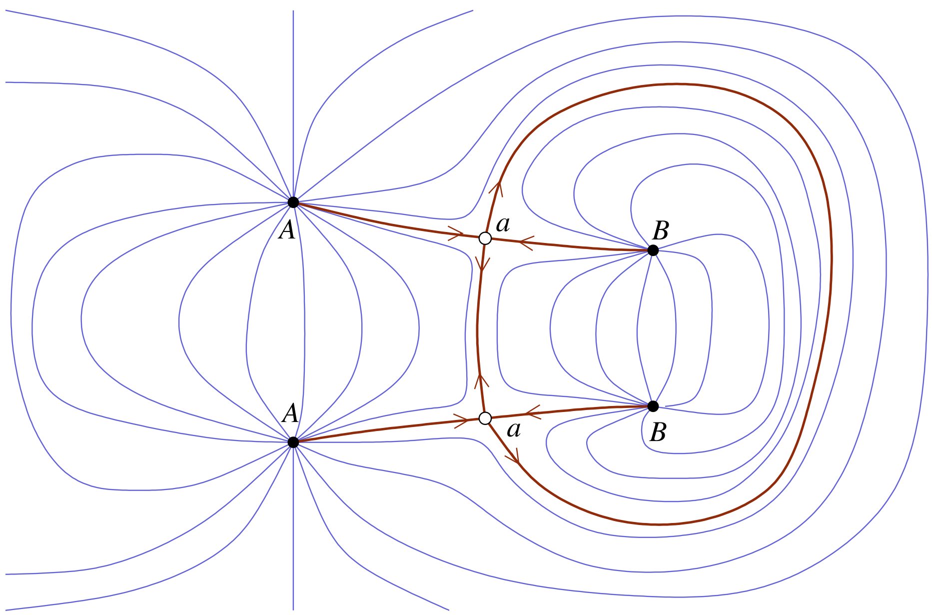
Theorem (W. '05). *All overtwisted contact manifolds admit foliations.*

Rallying cry:

"If holomorphic curves are everywhere, it's hard to kill them."

Floer-type invariants

Observation: Index 2 families in a stable foliation degenerate into broken index 1 leaves.



Given a foliation $\mathcal{F}$ of (M, λ), consider the contact homology algebra $HC_*(\mathcal{F})$ based on counting index 1 leaves with one positive puncture.

One could similarly define $H_*^{\mathsf{RSFT}}(\mathcal{F})$, $ECH_*(\mathcal{F})$ etc. . .

These should be invariants of. . . what?

Suppose:

- λ_+, λ_- are contact forms on (M, ξ)
- $\tilde{J}_\pm$ are $\mathbb{R}$-invariant almost complex structures on $\mathbb{R} \times M$ corresponding to $\lambda_\pm$
- $\mathcal{F}_\pm$ are finite energy foliations on $(\mathbb{R} \times M, \tilde{J}_\pm)$

Choose $\widehat{J}$ on $\mathbb{R} \times M$ to interpolate between $\tilde{J}_+$ near $\{+\infty\} \times M$ and $\tilde{J}_-$ near $\{-\infty\} \times M$.

(Tentative) Definition. A *directed concordance* $\mathcal{F}_+ \to \mathcal{F}_-$ is a foliation $\widehat{\mathcal{F}}$ of $\mathbb{R} \times M$ by $\widehat{J}$-holomorphic cuves that converges to the $\mathbb{R}$-invariant foliations $\mathcal{F}_\pm$ near $\{\pm\infty\} \times M$.

A *concordance* between $\mathcal{F}_+$ and $\mathcal{F}_-$ is a pair of directed concordances $\mathcal{F}_+ \to \mathcal{F}_-$ and $\mathcal{F}_- \to \mathcal{F}_+$ which are "inverses".

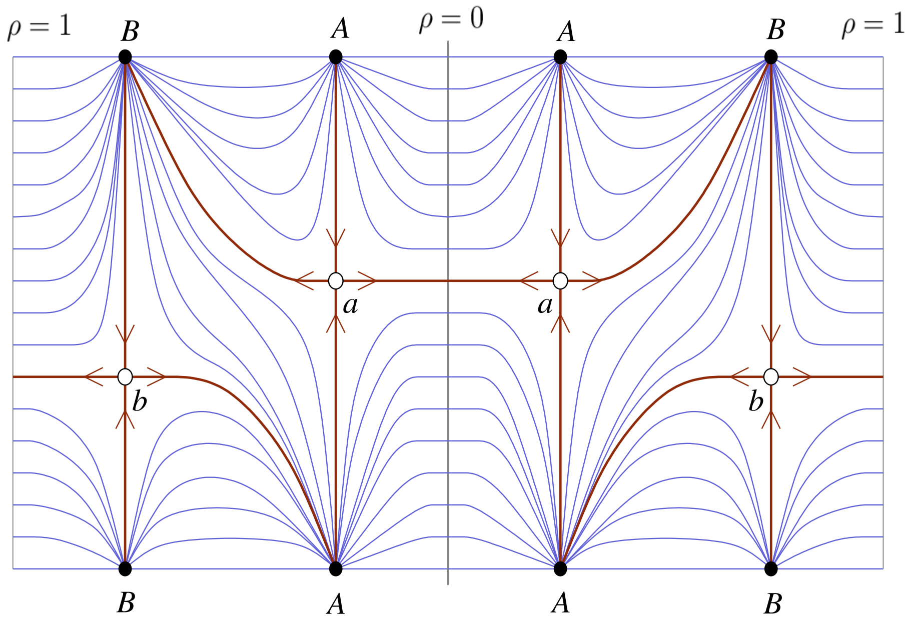
Conjecture. *Concordance defines an equivalence relation for foliations on* (M, ξ).

Conjecture. *Floer-type algebras such as* $HC_*(\mathcal{F})$ *are concordance invariants.*

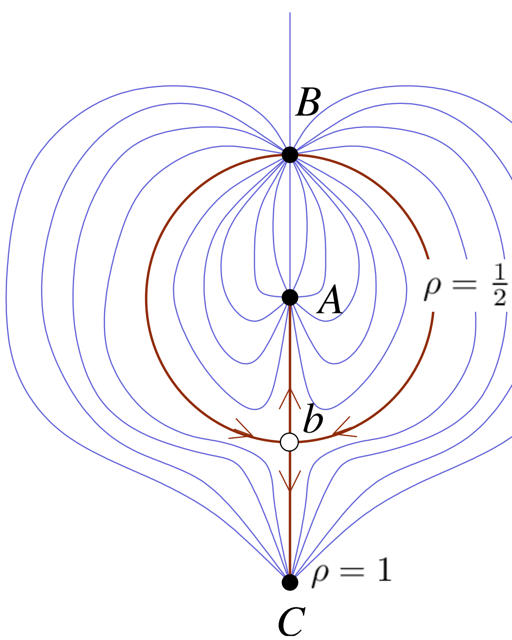
Question: given (M, ξ), what is the set of foliations up to concordance?

How is this related to the topology of M or contact topology of (M, ξ)?

Two foliations of an overtwisted $S^1 \times S^2$



$HC_*(\mathcal{F}_1) = 0$



$HC_*(\mathcal{F}_2) \neq 0$

Conjecture. *There is no concordance between $\mathcal{F}_1$ and $\mathcal{F}_2$.*

This should be related to the fact that $\pi_1(S^1 \times S^2) \neq 0$.

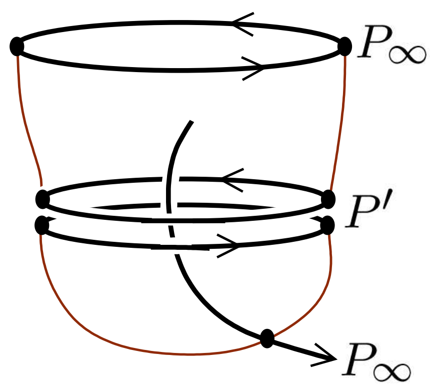
III. Existence in the overtwisted case
(or, *how to actually prove some things. . .*)

Warmup: a simple compactness argument

Suppose (S^3, λ_0) admits a holomorphic open book decomposition $\mathcal{F}_0 = \{\tilde{u}_\tau : \mathbb{C} \to \mathbb{R} \times S^3\}_{\tau \in S^1}$, asymptotic to $P_\infty \subset S^3$.

Choose a homotopy $\{\lambda_r\}_{r\in[0,1]}$ such that X_{λ_r} remains transverse to $\mathcal{F}_0$.

Suppose IFT $\Rightarrow$ a smooth family of foliations $\mathcal{F}_r$ for $r \in [0,1)$. Then this extends to $r = 1$.
Linking prevents bubbling off!



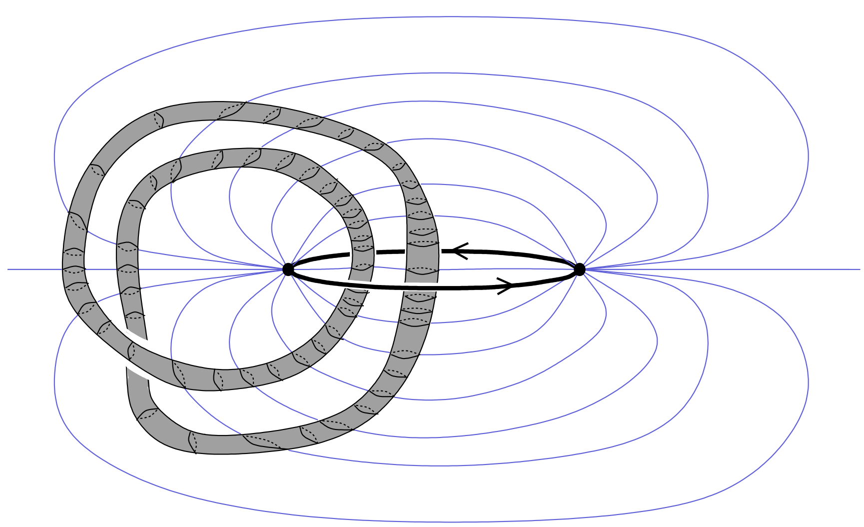
$\text{lk}(P_\infty, P') \neq 0$

Such phenomena are uniquely 3-dimensional:
compactness arises from topological constraints.

Idea:

1. (S^3, ξ_0) has a stable holomorphic open book decomposition asymptotic to a Hopf circle.

2. Modify this under surgery and Lutz twists along a transverse link.
   Bennequin ('83) $\Rightarrow$ we can also assume *link* $\pitchfork$ *foliation*.



Martinet + Lutz ('71) + Eliashberg ('89) $\Rightarrow$ *this gives every overtwisted contact structure on every closed 3-manifold.*

How can a foliation survive surgery?

A Mixed Boundary Value Problem:

$L \subset M$ a torus tangent to X_λ

Given a function $G : L \to \mathbb{R}$ and $\sigma \in \mathbb{R}$, let

$$\tilde{L}^\sigma = \{(G(x) + \sigma, x) \in \mathbb{R} \times L\} \subset \mathbb{R} \times M$$

This is a family of totally real submanifolds.

Fix G and consider embedded holomorphic punctured disks

$$\tilde{v} = (b, v) : \mathbb{D} \setminus \{0\} \to \mathbb{R} \times M$$

with finite energy and $\tilde{v}(\partial\mathbb{D}) \subset \tilde{L}^\sigma$ for *arbitrary* $\sigma \in \mathbb{R}$.

Can generalize to arbitrary surfaces $\widehat{\Sigma}$ with boundary and interior punctures.

Cutting out disks

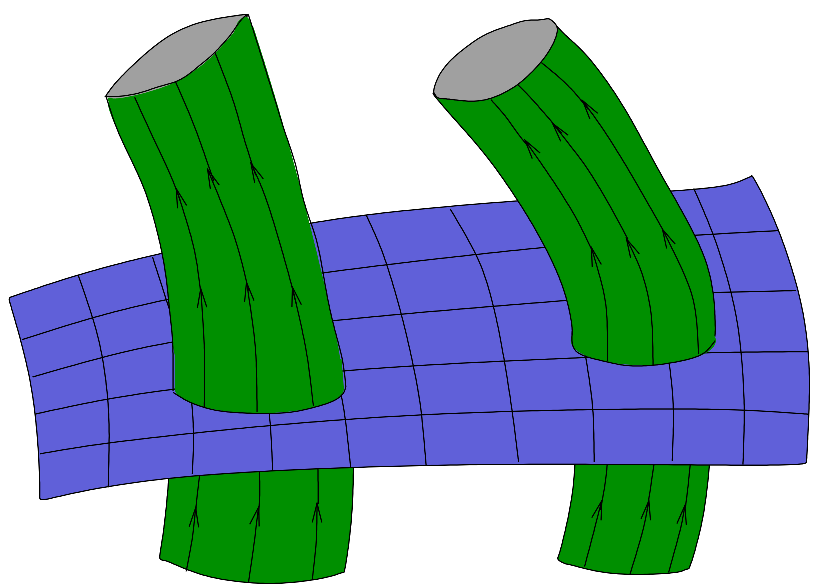
$$\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times M$$

yields

$$\tilde{v} : \widehat{\Sigma} = \dot{\Sigma} \setminus \{\text{disks}\} \to \mathbb{R} \times M$$

with same Fredholm index:

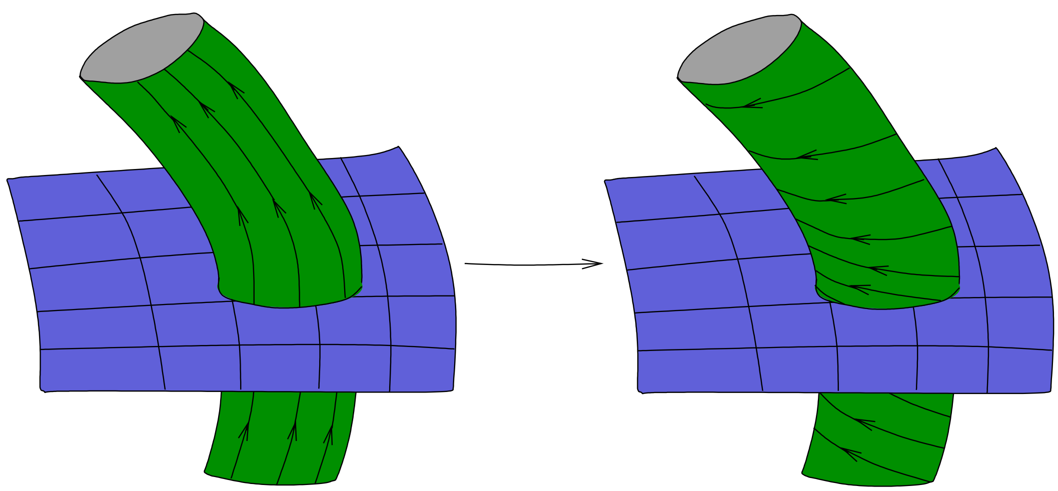
$$\mathrm{Ind}(\tilde{v}) = \mathrm{Ind}(\tilde{u})$$



Twisting

Our new foliation $\mathcal{F}$ lives outside L, so it survives (discontinuous!) changes inside L.

Now choose a homotopy $\{\lambda_r\}_{r\in[0,1]}$ near L so that orbits on L become meridians.



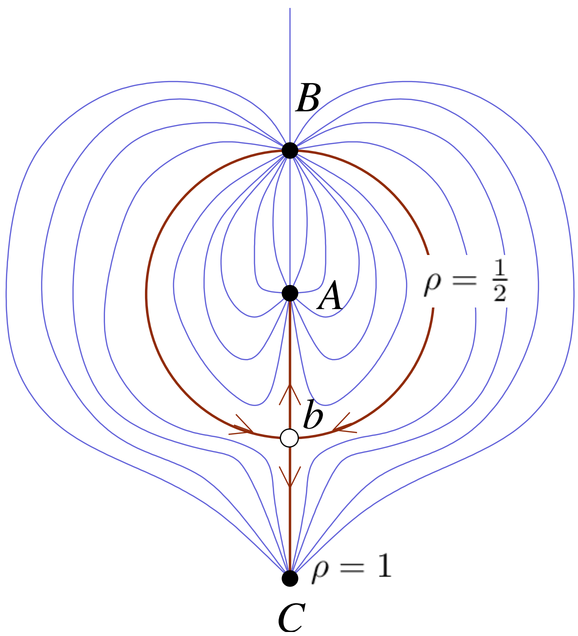
Two things are needed to finish the proof:

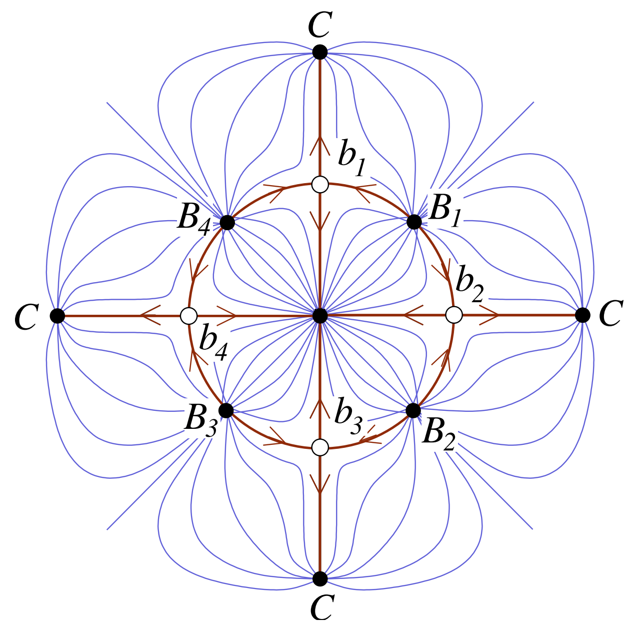
1. a **compactness result**:
   for $\tilde{v}_r$ as $r \to r_\infty < 1$
2. a **noncompactness result**:
   for $\tilde{v}_r$ as $r \to 1$: *boundary* $\to$ *punctures*.

Both follow from topological constraints. $\square$

IV. Outlook

Nondegenerate perturbations





$$HC_*(\mathcal{F}_2) = HC_*(\mathcal{F}_3)$$

Conjecture. *Every foliation of stable Morse-Bott type can be perturbed to a stable foliation for a C^∞-close nondegenerate contact form.*

All such perturbed foliations are concordant.

Example: a stupid, yet strangely illuminating Morse-Bott foliation.

For (S^3, λ_0), every Hopf circle is an orbit, so take $\mathcal{F} = \{\mathbb{R} \times P\}_{P=\mathsf{Hopf}}$. This is the *only* stable foliation of (S^3, λ_0).

Conjecture. *All stable foliations of the tight 3-sphere are concordant.*

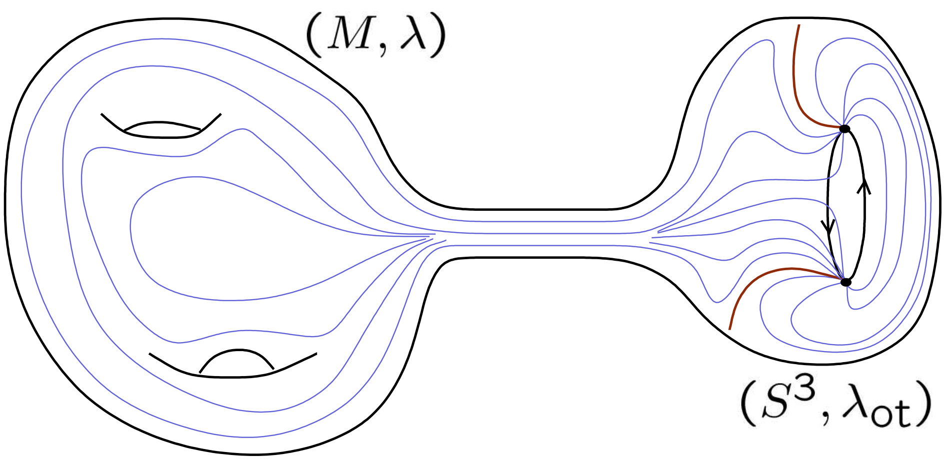
This should follow from the above remarks and. . .

Conjecture (Hofer, Fish, Siefring, W.)**.** *If* (M, λ) *admits a holomorphic foliation, then so does* $(M, f\lambda)$ *for generic functions* f*, and the two are concordant.*

This is hard, but might follow from a homotopy argument.

Foliations for tight contact structures?

Hofer's idea:
given (M, λ), take an overtwisted $(S^3, \lambda_{\text{ot}})$, find a foliation on $(M \# S^3, \lambda \# \lambda_{\text{ot}})$ and stretch.



This should prove the Weinstein conjecture in dimension 3.

But it won't be that simple...

Foliations can't always exist

$(S, g) =$ a closed surface with curvature -1
$(S^1TS, \lambda) =$ the unit tangent bundle with its natural contact form. Then

$$\text{Reeb flow on } S^1TS \cong \text{geodesic flow on } S$$
$$\text{Conley-Zehnder index} = \text{Morse index}$$

So $\mu_{CZ}(P) = 0$ for all orbits $P \subset S^1TS$.

In this situation, generic curves $\tilde{u} : \dot{\Sigma} \to \mathbb{R} \times S^1TS$ of index $\text{Ind}(\tilde{u}) = B$ should intersect their neighbors B times.

Question: what is a *singular finite energy foliation*, and do they always exist?