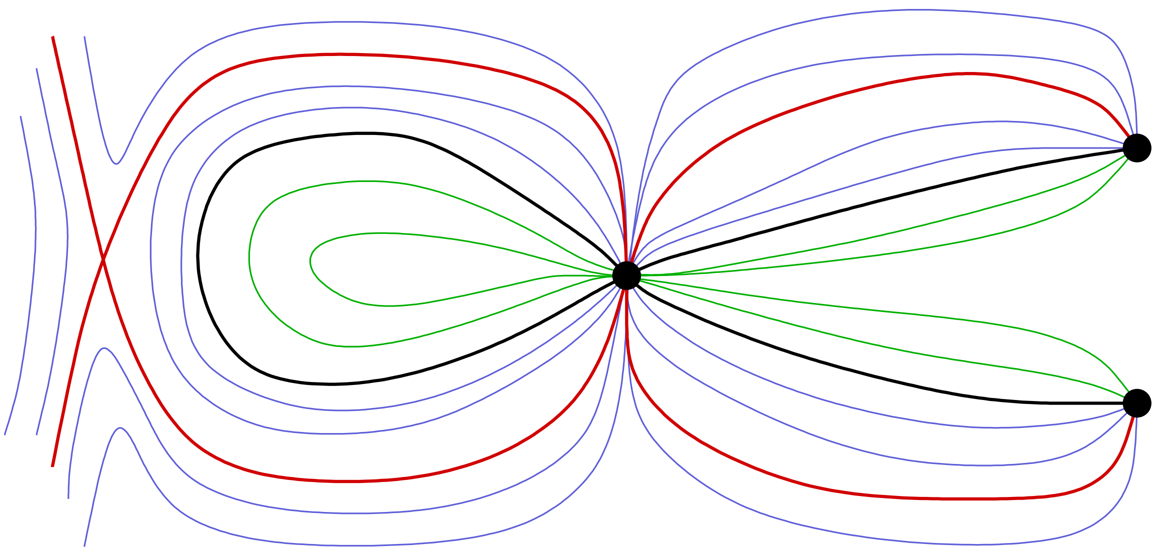
Automatic Transversality for Holomorphic Curves in Low Dimensions



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Outline

I. Motivation: transversality problems

II. Foliations and miracles of analysis

III. Automatic transversality

IV. Orbifolds of holomorphic curves

I. Motivation: transversality problems

Enumerative invariants in an ideal world (a recipe):

M = manifold, X = auxiliary data on M,

$$\Rightarrow \text{equation (PDE): } \boxed{F_X(u) = 0}$$

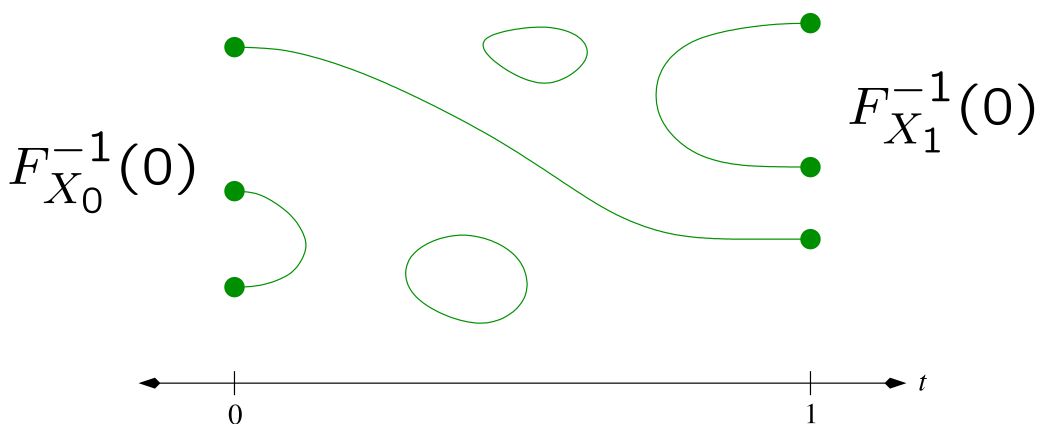
Define "invariant" $I(M, X) := \#F_X^{-1}(0)$, for generic X, then prove...

"Theorem": $I(M, X)$ doesn't depend on X.

"Proof": For generic homotopies $\{X_t\}_{t\in[0,1]}$,

$$\mathcal{M}_{[0,1]} := \{(t, u) \mid t \in [0, 1],\ F_{X_t}(u) = 0\}$$

is a compact smooth manifold with boundary.



For example: J–holomorphic curves

$(W, \omega) =$ symplectic manifold
$J =$ compatible almost complex structure
$(\Sigma, j) =$ Riemann surface

$\mathcal{M} := \{u : \Sigma \to W \mid Tu \circ j = J \circ Tu\}/\text{reparam.}$

Analysis: $\boxed{\mathcal{M} \cong \bar{\partial}_J^{-1}(0)/\text{symmetries}}$, where

$$\bar{\partial}_J : \mathcal{B} \to \mathcal{E} : (j, u) \mapsto Tu + J \circ Tu \circ j$$

is a smooth Fredholm section of a Banach space bundle.

We say $u : (\Sigma, j) \to (W, J)$ in $\mathcal{M}$ is **regular** if

$$D\bar{\partial}_J(j, u) : T_{(j,u)}\mathcal{B} \to \mathcal{E}_{(j,u)}$$

is surjective.

Then, implicit function theorem $\Rightarrow$
near u, $\bar{\partial}_J^{-1}(0)$ is a smooth manifold of dimension = Fredholm index of $D\bar{\partial}_J(j, u)$.

$$\boxed{\text{ind}(u) := \text{“}\dim \mathcal{M} \text{ near } u\text{”}}$$

An almost wonderful fact:

Theorem: For generic J, every simple curve $u \in \mathcal{M}$ is regular.

"*Simple*" = "*not multiply covered*":

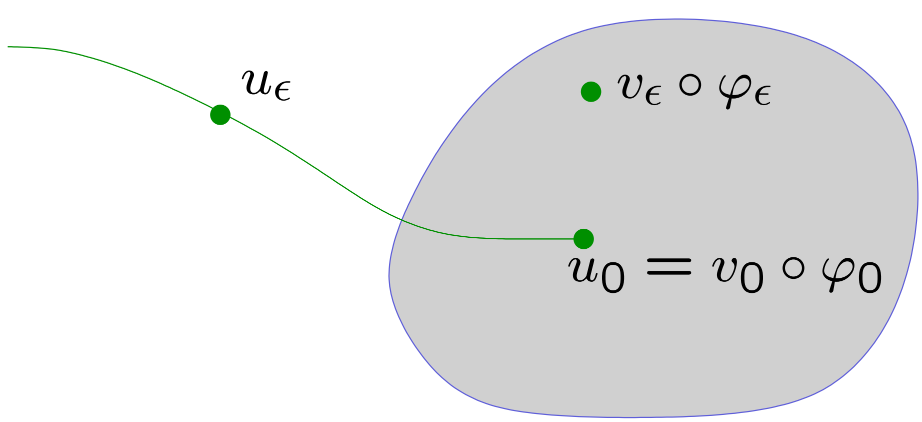
$$u \neq v \circ \varphi,$$

where $\varphi : (\Sigma, j) \to (\Sigma', j')$ branched cover, $\deg(\varphi) \geq 2$.

$\mathcal{M}$ is not generally smooth:
regularity fails at multiple covers.

How bad is this?

E.g. sometimes "$\dim \partial\mathcal{M} > \dim \mathcal{M}$":

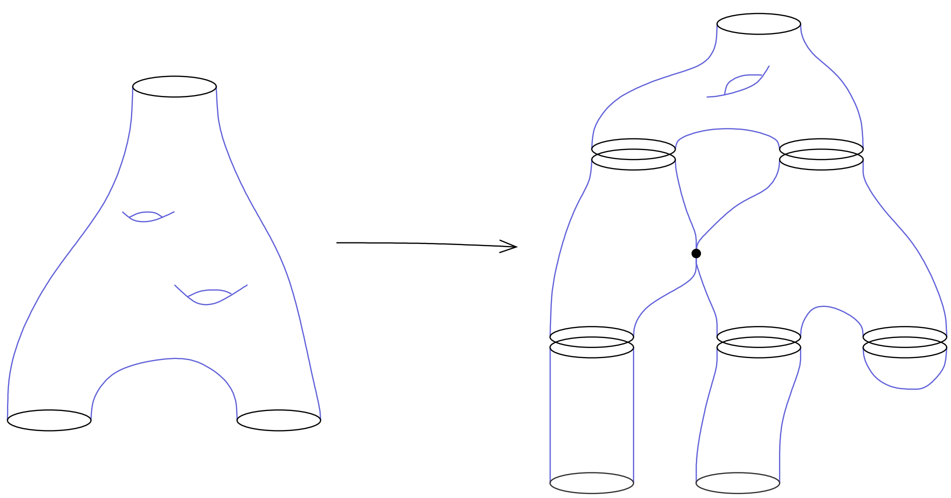


Possible transversality solutions:

1. Abstract perturbations: $\bar{\partial}_J(u) = \varepsilon$
   (destroys nice geometric properties,
   e.g. positivity of intersections)

2. Hope for a miracle
   (i.e. exploit geometrically nice properties)

Compactification:
$\mathcal{M} \subset \overline{\mathcal{M}} :=$ {nodal J–holomorphic buildings}



Goal: show that if $u \in \mathcal{M}$ is "nice", so is its connected component $\overline{\mathcal{M}}_u \subset \overline{\mathcal{M}}$

"Nice curves live in nice moduli spaces."

II. Foliations and miracles of analysis

IIa. Symplectizations

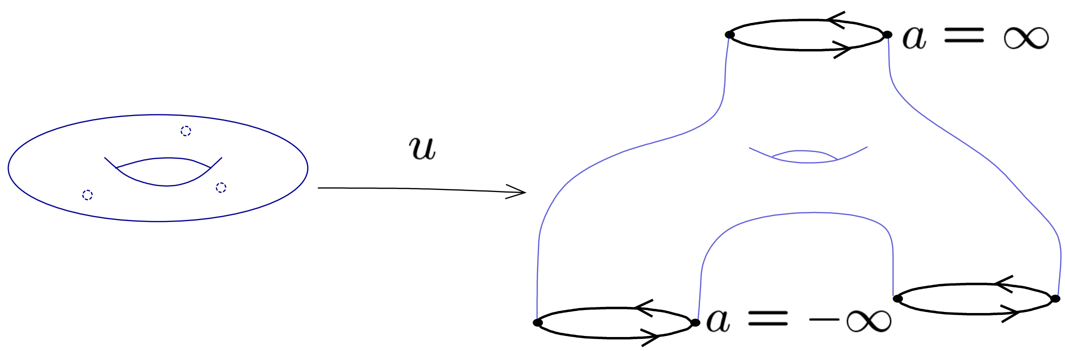
$(M, \lambda) =$ contact 3–manifold
$X_\lambda =$ Reeb vector field on M

On $W := \mathbb{R} \times M$, choose an $\mathbb{R}$–invariant almost complex structure $\tilde{J}$

Consider punctured $\tilde{J}$–holomorphic curves

$$\tilde{u} = (a, u) : \dot{\Sigma} \to \mathbb{R} \times M$$

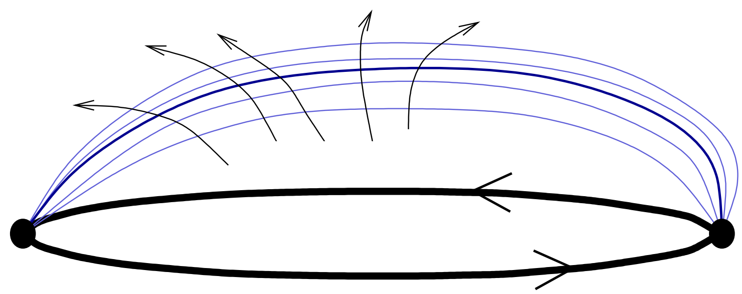
asymptotic to closed Reeb orbits.



We say $\tilde{u} = (a, u)$ is **nicely embedded** if $u : \dot{\Sigma} \to M$ is an embedding.

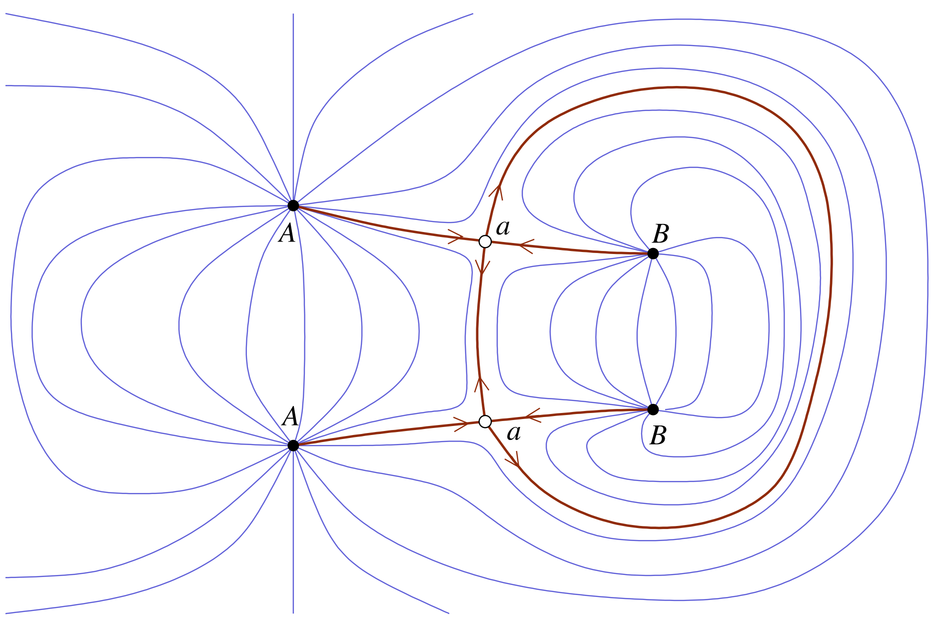
Nicely embedded $\Rightarrow$

- If $\text{ind}(\tilde{u}) = 2$, nearby curves foliate a neighborhood of $u(\dot{\Sigma}) \subset M$.



- If $\text{ind}(\tilde{u}) = 1$, $u(\dot{\Sigma}) \subset M$ appears isolated.

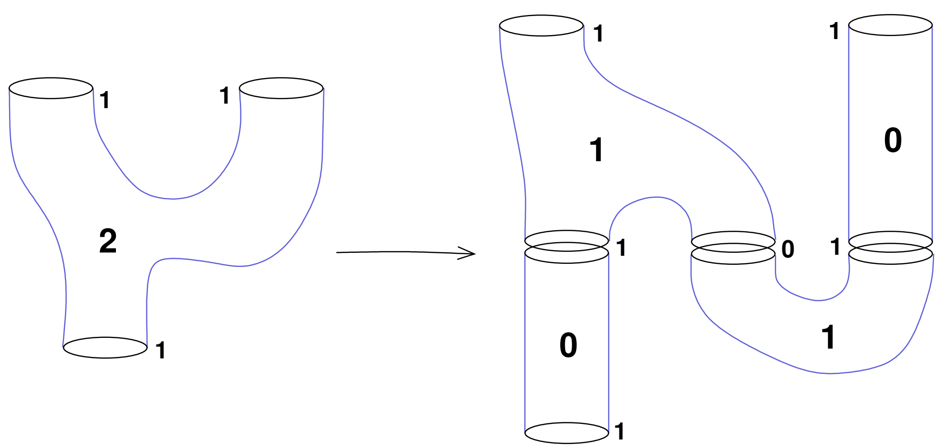
These can form *"finite energy foliations"*:



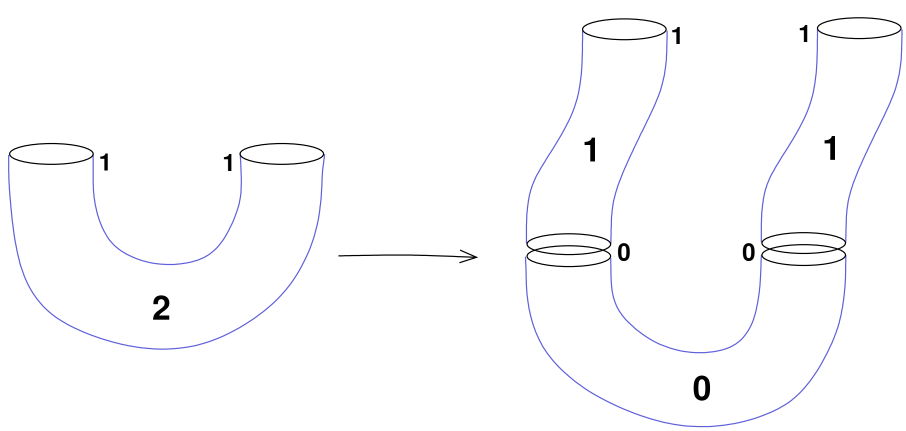
Theorem (arXiv:math/0703509)
If $\tilde{u}$ is nicely embedded, then all buildings in $\overline{\mathcal{M}}_{\tilde{u}}$ consist of nicely embedded curves and trivial cylinders over orbits.

Corollary: for generic $\tilde{J}$, all curves appearing in $\overline{\mathcal{M}}_{\tilde{u}}$ are regular

$\Rightarrow \overline{\mathcal{M}}_{\tilde{u}}$ is a compact manifold with boundary.

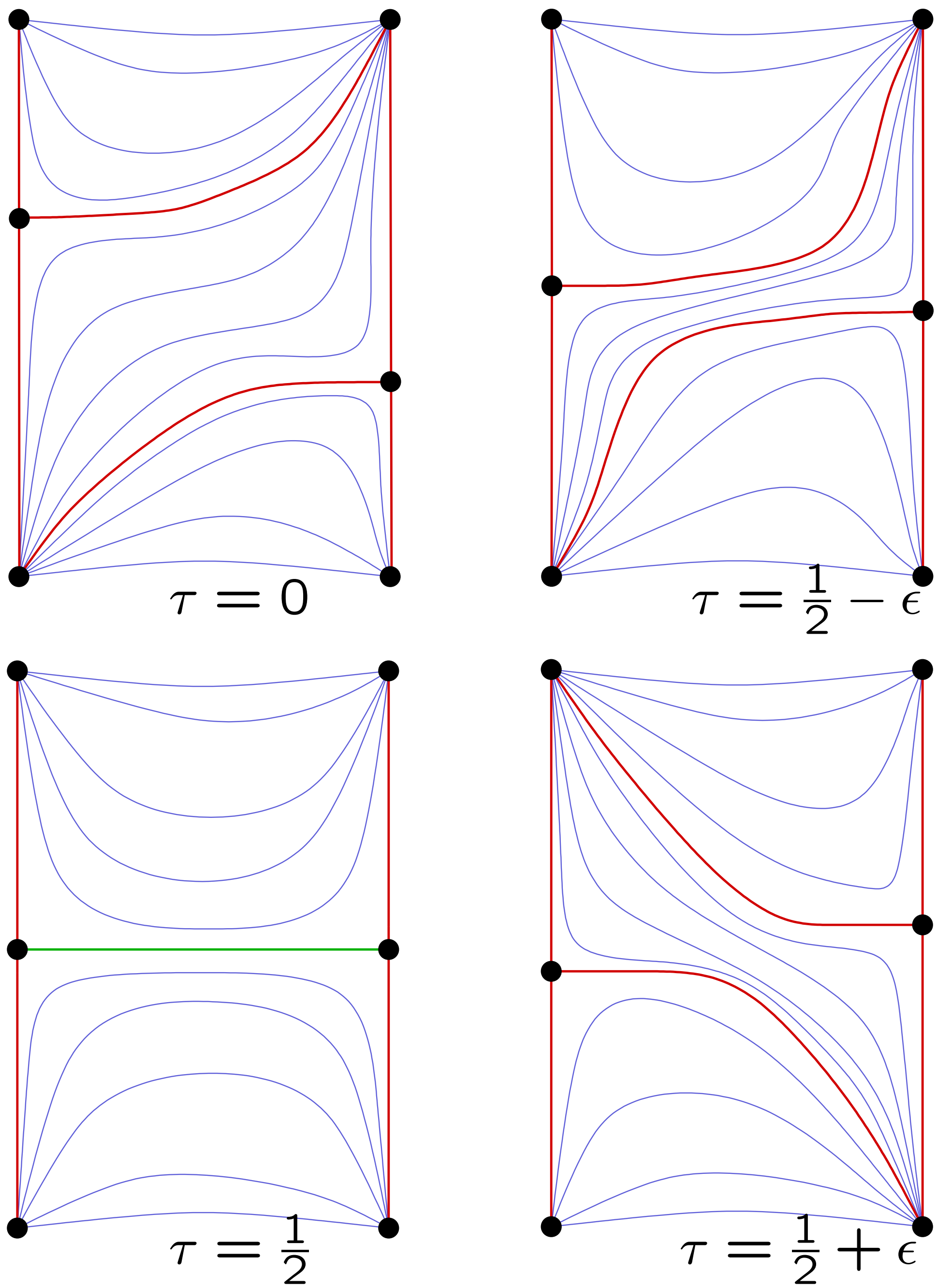


An example with non-generic J:



Application:

homotopies of finite energy foliations

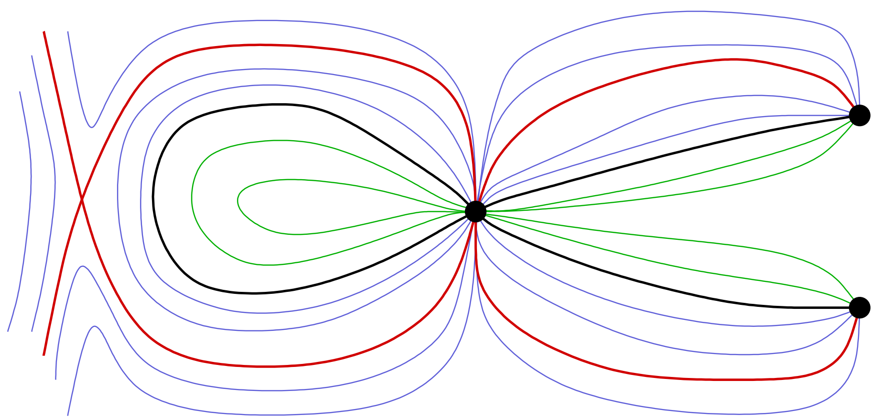


IIb. The closed case

(W, J) = closed almost complex 4–manifold,
(Σ, j) = closed Riemann surface

$u : (\Sigma, j) \to (W, J)$ nicely embedded $\Longleftrightarrow$
embedded, $\mathrm{ind}(u) = 2$ and $u \bullet u = 0$

(Can also generalize for immersed curves with fixed double points.)



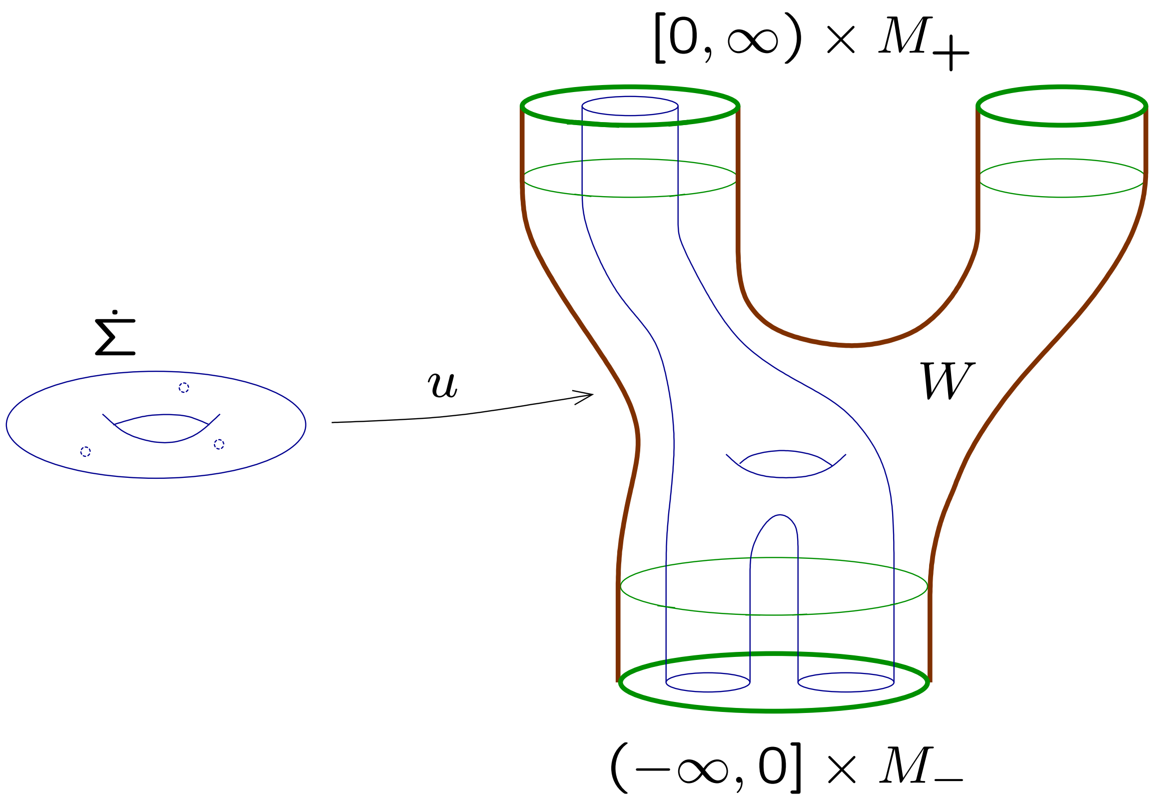
Theorem ($\Leftarrow$ *adjunction formula*):
u nicely embedded and J generic $\Rightarrow$
non-embedded curves in $\overline{\mathcal{M}}_u$ are nodal, with two embedded, transverse index 0 curves.

Corollary: regularity for generic J

$\Rightarrow$ *(by gluing)* $\overline{\mathcal{M}}_u$ is a closed manifold.

IIc. The general (cobordism) case

$(W, J) =$ 4–manifold with cylindrical ends
$(\dot{\Sigma}, j) =$ punctured Riemann surface



Conjecture: u nicely embedded $\Rightarrow$
$\overline{\mathcal{M}}_u$ is a smooth object (in some sense)

Partial result (arXiv:0802.3842):
u nicely embedded and J generic $\Rightarrow$
$\mathcal{M}_u$ is a smooth **orbifold**, with isolated singularities that consist of **unbranched** multiple covers over embedded index 0 curves.

This partially implies the previous two results (multiple covers cannot arise):

1. *Symplectization:* $\mathbb{R}$–invariance $\Rightarrow$
   $\nexists$ embedded index 0 curves

2. *Closed:* nicely embedded curves have genus 0,
   $\nexists$ unbranched covers $\varphi : S^2 \to S^2$

In general, multiple covers can appear, but only the harmless type!
(We will show: *unbranched* $\Rightarrow$ *regular*)

Example:

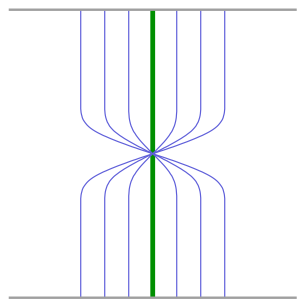
$W := (S^2 \times S^2) \setminus \{(0,0), (1,1), (\infty, \infty)\}$
(three negative S^3–ends)
$\dot{\Sigma} := S^2 \setminus \{0, 1, -1, \infty\}$
For $\zeta \in \mathbb{C}$ approaching 0, consider

$$u_\zeta : \dot{\Sigma} \to W : z \mapsto \left(z^3 \frac{z + \zeta}{\zeta z + 1}, z^2 \right).$$

Why orbifolds?

A lower-dimensional example:

$\mathcal{M}$:= smooth 1−parameter family of (unparametrized) closed orbits



Regularity $\Rightarrow$
{parametrized orbits} $\cong$ smooth surface
(Möbius strip)

$\Rightarrow \mathcal{M} \cong \text{surface}/S^1$.

Middle orbit has stabilizer $\mathbb{Z}_2$ under S^1−action,
$\Rightarrow \mathcal{M} \cong$ open subset of $\mathbb{R}/\mathbb{Z}_2$.

symmetry $\Leftrightarrow$ orbifold singularities

For holomorphic curves:

$$\mathcal{M} \cong \bar{\partial}_J^{-1}(0)/\text{symmetries}$$

u regular $\Rightarrow$ $\bar{\partial}_J^{-1}(0)$ is a manifold near u.

Stabilizer of u is

$$\text{Aut}(u) := \{\varphi : (\Sigma, j) \xrightarrow{\sim} (\Sigma, j) \mid u = u \circ \varphi\}.$$

This can be nontrivial if u is multiply covered.

$\therefore$ Regularity $\Rightarrow$

$$\boxed{\begin{array}{c} \text{nbhd}(u) \subset \mathcal{M} \\ \cong \\ \text{open subset} \subset \mathbb{R}^{\text{ind}(u)} / \, \text{Aut}(u). \end{array}}$$

Task: prove regularity for all curves in $\mathcal{M}_u$, *including the multiple covers*.

III. Automatic transversality

In dimension four, the following holds for closed curves and *all* (not just generic) J:

Theorem (Hofer-Lizan-Sikorav):
If $u : \Sigma \to W^4$ is immersed and $c_1(u^*TW) > 0$, then u is regular.

Claim: this applies to nicely embedded curves.

Define the *normal Chern number*:

$$\boxed{c_N(u) := c_1(u^*TW) - \chi(\Sigma)}$$

Then adjunction $\Rightarrow u \bullet u = 2\delta(u) + c_N(u)$,
$\Rightarrow$ nicely embedded curves have $c_N(u) = 0$.

$$c_1(u^*TW) > 0 \Longleftrightarrow \boxed{\mathrm{ind}(u) > c_N(u)}$$

$\therefore$ When $u_j \to u = v \circ \varphi$, regularity follows if u is immersed. Indeed, we will show:
(1) v is embedded,
(2) φ is unbranched.

Generalizing Hofer-Lizan-Sikorav:

IIIa: Punctured curves

The following argument generalizes nicely. For simplicity, assume Teichmüller space is trivial.

If u is immersed, $u^*TW = T\Sigma \oplus N_u$, $\Rightarrow$

$$D\bar{\partial}_J(u) = \begin{pmatrix} D_u^T & \cdot \\ 0 & D_u^N \end{pmatrix}$$

$D_u^T \cong$ natural CR-operator on $T\Sigma$, onto.

$\therefore$ Sufficient to prove D_u^N is onto.
By Riemann-Roch,

$$c_1(N_u) < 0 \Rightarrow D_u^N \text{ is injective}$$
$$\text{ind}(D_u^N) > c_1(N_u) \Rightarrow D_u^N \text{ is surjective.}$$

Key point: can generalize $c_N(u)$ for punctured curves so that it counts zeros of sections in $\ker D_u^N$ if u is immersed.

IIIb: Non-immersed curves

Ivashkovich-Shevchishin observed:

$\mathbb{C}$–linear part of $D\bar{\partial}_J(u) \Rightarrow$
holomorphic structure on u^*TW such that

$$du \in \Gamma(\mathrm{Hom}_{\mathbb{C}}(T\Sigma, u^*TW))$$

is a holomorphic section. Therefore:

(1) Critical points have positive orders.
(2) There is still a splitting

$$\boxed{u^*TW = T_u \oplus N_u}$$

such that $T_u = \mathrm{im}(du)$ on $\Sigma \setminus \mathrm{Crit}(u)$.

Counting $\mathrm{Crit}(u)$ algebraically,

$$c_1(T_u) = \chi(\Sigma) + \#\,\mathrm{Crit}(u)$$
$$c_1(N_u) = c_N(u) - \#\,\mathrm{Crit}(u).$$

Lemma: D_u^T is again surjective, with index increased by $2\,[\#\,\mathrm{Crit}(u)]$. This implies:

$$\boxed{\mathrm{coker}\, D\bar{\partial}_J(u) \cong \mathrm{coker}\, D_u^N}$$

D_u^N is surjective if $\mathrm{ind}(D_u^N) > c_1(N_u)$, $\Longleftrightarrow$ $\mathrm{ind}(u) - 2\,[\#\,\mathrm{Crit}(u)] > c_N(u) - \#\,\mathrm{Crit}(u)$.

Also valid for punctured curves, implying:

Theorem (generalized automatic $\pitchfork$):
If $u : \dot{\Sigma} \to W^4$ satisfies

$$\boxed{\mathrm{ind}(u) > c_N(u) + \#\,\mathrm{Crit}(u),}$$

then u is regular.

Remark 1: This is most useful for genus 0 curves, because by the index formula,

$$2c_N(u) = \mathrm{ind}(u) - 2 + 2g + \#\Gamma_0,$$

where $\Gamma_0 := \{z \in \Gamma \mid \mu_{\mathrm{CZ}}(z) \text{ is even}\}$.

Remark 2: It's a nice result, but we won't use it directly. It will be more useful to note that even when u isn't regular,

$$\boxed{\dim\ker D\bar{\partial}_J(u) = \dim\ker D_u^N + 2[\#\,\mathrm{Crit}(u)].}$$

IV. Orbifolds of holomorphic curves

For simplicity, consider the *closed case*:
$u_j : \Sigma \to W$ embedded, with $u_j \bullet u_j = 0$,

$$u_j \to u = v \circ \varphi$$

a branched cover of degree $k \geq 2$, with $v : \Sigma' \to W$ simple.

It remains to prove two claims:

Claim 1: For generic J, v is embedded.

$$0 = u \bullet u = k^2(v \bullet v)$$

$\Rightarrow 0 = 2\delta(v) + c_N(v)$. Then since

$$2c_N(v) = \text{ind}(v) - 2 + 2g' \geq -2$$

for generic J, $c_N(v) \geq -1 \Rightarrow \delta(v) = 0$.

This generalizes to the punctured case using the intersection theory of R. Siefring.
Notably: $\boxed{u \bullet u \geq k^2(v \bullet v)}$ *in general.*

Claim 2: Crit$(\varphi) = \emptyset$.

Suppose not. Then $\#\,$Crit$(u) = \#\,$Crit$(\varphi) \Rightarrow$

$$c_1(N_u) = c_N(u) - \#\,\text{Crit}(u) < 0,$$

thus D_u^N is injective, and

$$\dim \ker D\bar{\partial}_J(u) = 2\,[\#\,\text{Crit}(\varphi)].$$

But the space

$$\{u' = v \circ \varphi' \mid \varphi' = \text{a branched cover near } \varphi\}$$

is in $\mathcal{M}_u$ and has exactly this dimension!

$\therefore$ Implicit function theorem $\Rightarrow$
All $u' \in \mathcal{M}_u$ near u are branched covers.

Contradiction!