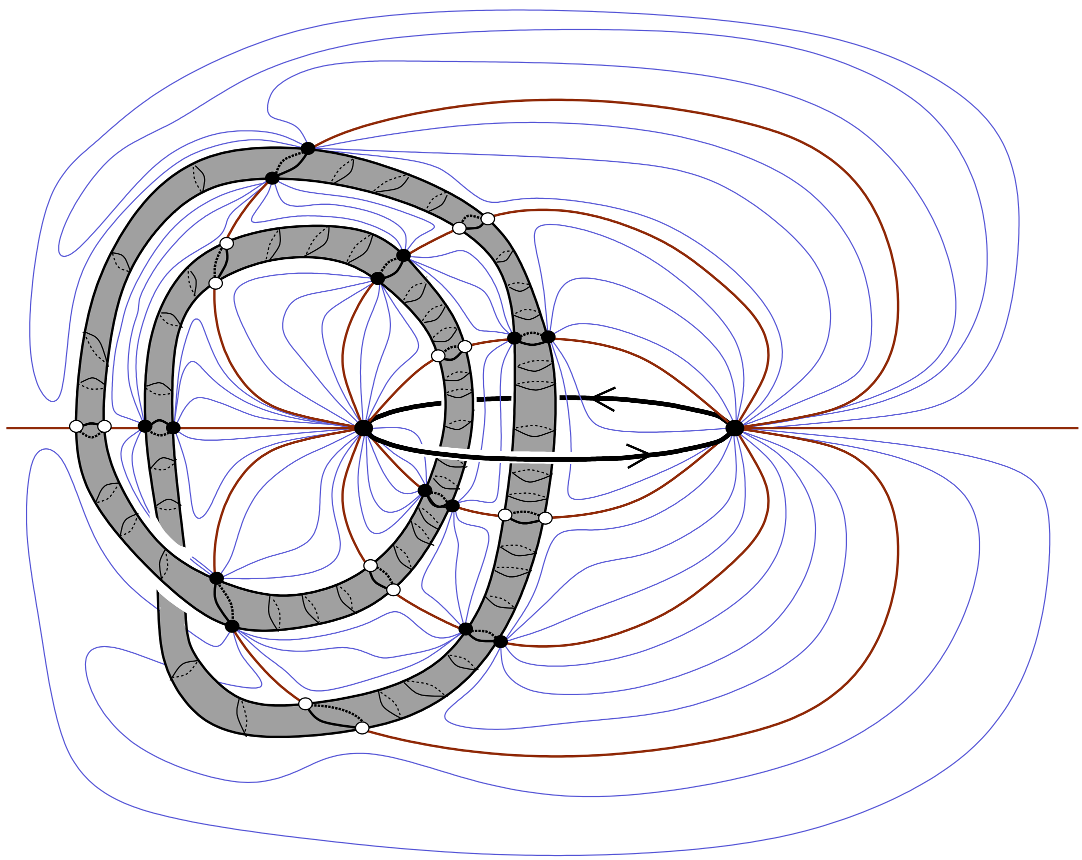
Holomorphic Foliations and Low-Dimensional Symplectic Field Theory



Chris Wendl

MIT

http://math.mit.edu/~wendlc/publications.html

Outline

I. Floer-type theories and holomorphic curves

II. Good holomorphic curves

III. Finite energy foliations

IV. Compactness for good holomorphic curves

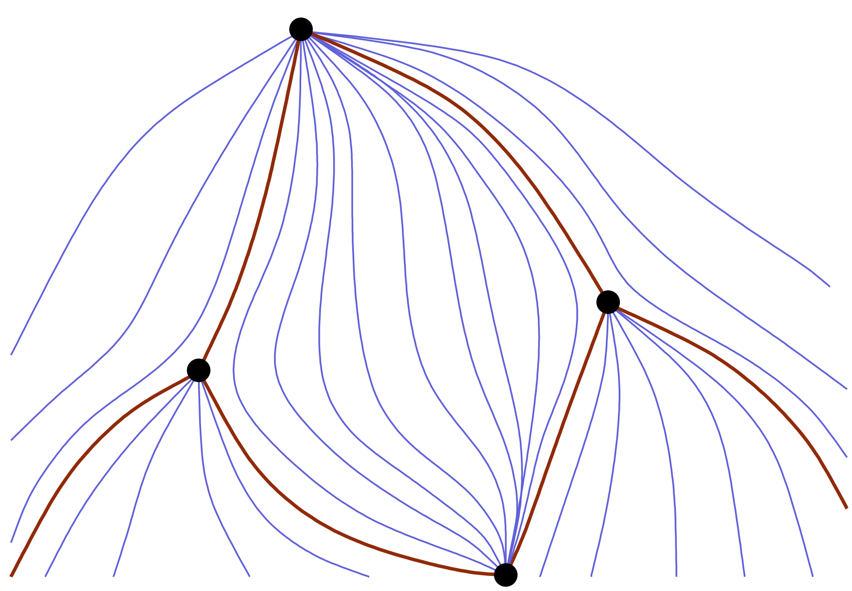
V. Foliations, concordance and SFT

I. Floer-type theories and holomorphic curves

Why Floer homology works:

Families of flow lines are compact up to breaking, that is:

$$\partial\{\text{flow lines}\} = \{\text{broken flow lines}\}$$



$\Longrightarrow$ can define invariant homology algebras by counting isolated flow lines.

Holomorphic curves as flow lines

$(M, \lambda) =$ contact manifold, $\xi = \ker \lambda$
$X_\lambda =$ Reeb vector field

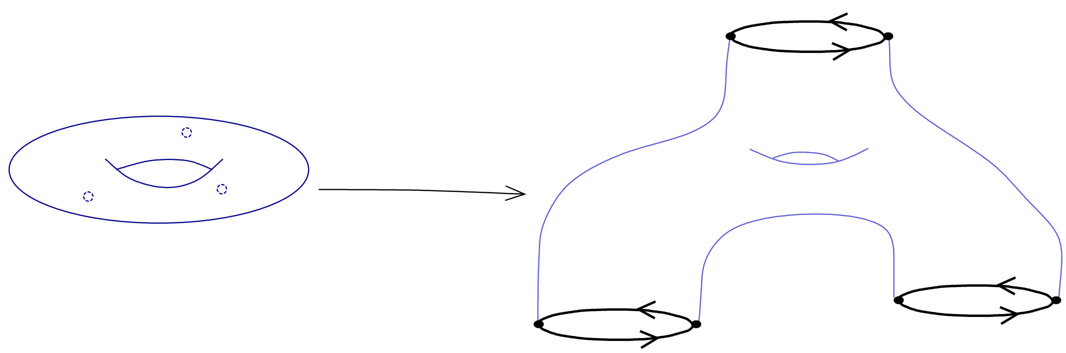
$\tilde{J} = \mathbb{R}$-invariant almost complex str. on $\mathbb{R} \times M$

$\dot{\Sigma} = \Sigma \setminus \Gamma$, punctured Riemann surface

We consider $\tilde{J}$-holomorphic maps

$$u : \dot{\Sigma} \to \mathbb{R} \times M$$

as flow lines between sets of closed Reeb orbits.

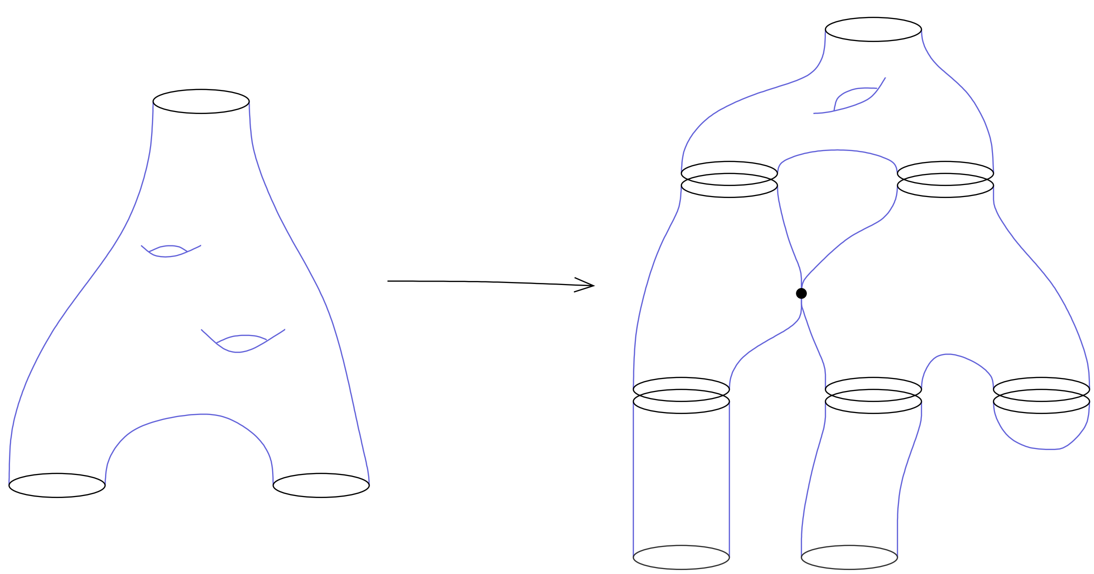


Also makes sense in symplectic cobordisms (W, ω, J).

Symplectic Field Theory

Count isolated (index 1) holomorphic curves in $\mathbb{R} \times M$ and isolated (index 0) curves in related symplectic cobordisms
$\Rightarrow$ invariants of (M, ξ).

Compactness theorem (BEHWZ):
Sequences of holomorphic curves in symplectic cobordisms converge to holomorphic buildings with nodes and multiple levels.

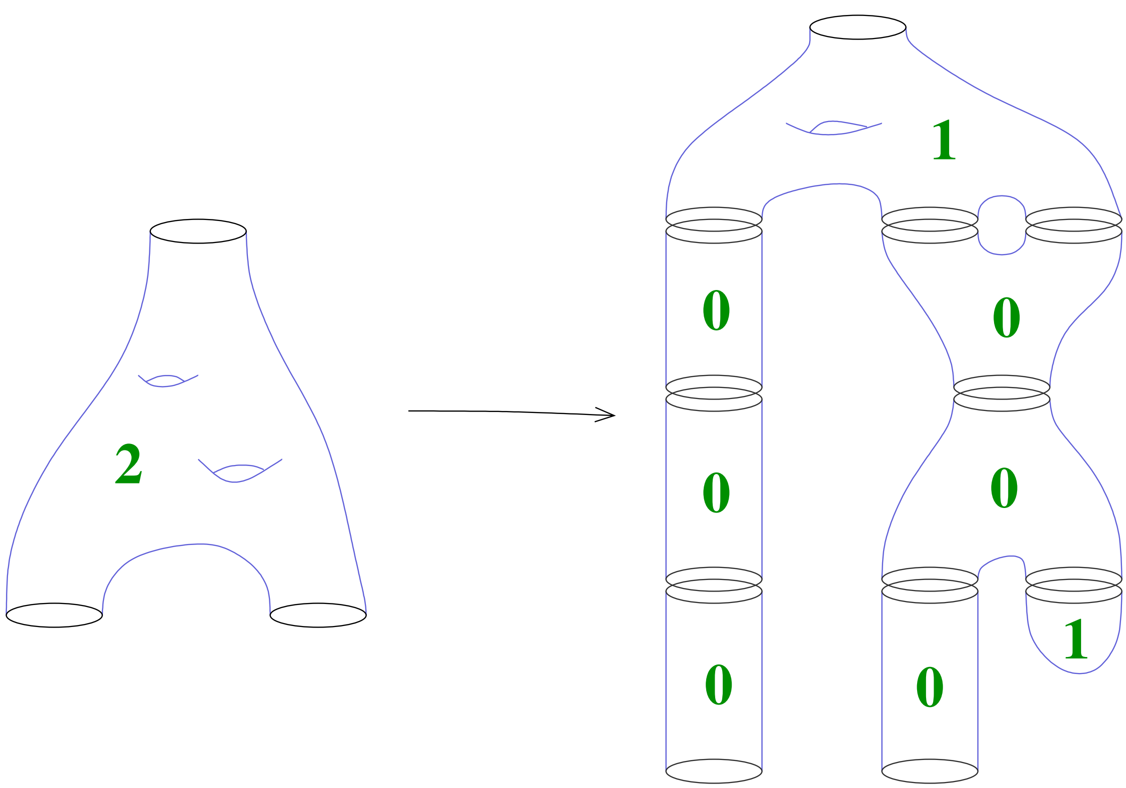


Trouble: transversality failes due to multiple covers. Need abstract perturbations.

Embedded Contact Homology

If $\dim M = 3$, restrict attention to a certain class of embedded holomorphic curves in $\mathbb{R}\times M$.

Compactness theorem (M. Hutchings): Sequences of admissible embedded index 2 curves in $\mathbb{R}\times M$ converge to broken curves with two admissible embedded index 1 levels plus index 0 covers of trivial cylinders.



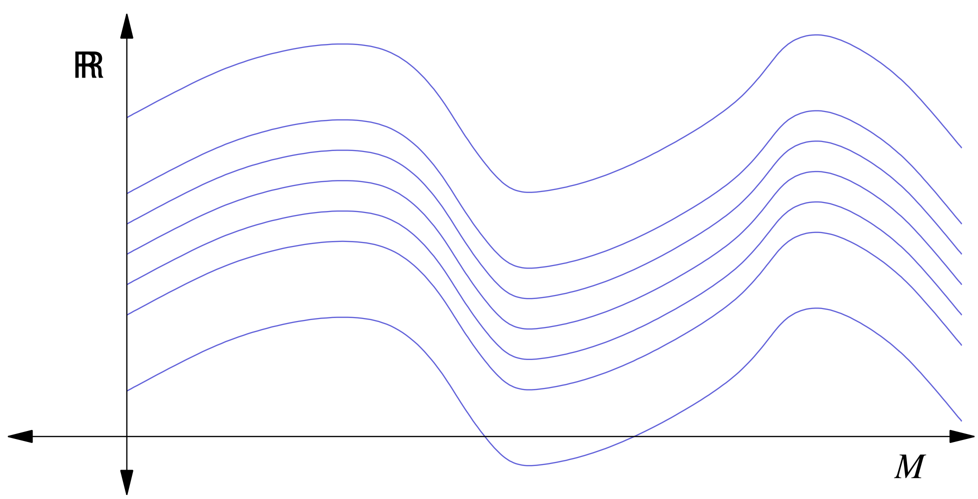
Trouble: still multiple covers.

II. Good Holomorphic Curves

Question:

Is there any such theory that counts only curves $u : \dot{\Sigma} \to \mathbb{R} \times M$ with embedded projections $\pi \circ u$ to M?

Observe: these curves are transverse to X_λ, and belong to families of nonintersecting curves.



Related question: when can we guarantee that two nearby curves u and v don't intersect in $\mathbb{R} \times M$?

What about the projections $\pi \circ u$ and $\pi \circ v$ in M?

Intersection theory with punctures (R. Siefring '05)

Consider $u : \dot{\Sigma} \to W$ (symp. cobordism).

Adjunction formula:

$$u \cdot u = 2\delta(u) + c_N(u) + \mathrm{cov}_\infty(u),$$

where

- $u \cdot u$ and $\delta(u)$ include "asymptotic intersections"
- $c_N(u) := c_1^\tau(u^*TW) - \chi(\dot{\Sigma}) + \ldots$
  the normal first Chern number of u
- $\Gamma_0 :=$ punctures with even CZ-index
- $\mathrm{cov}_\infty(u) \geq 0$, depends only on orbits.

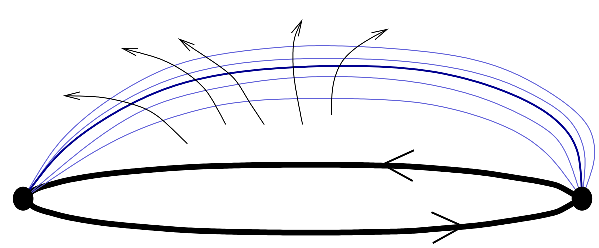
Index formula $\Rightarrow$

$$2c_N(u) = \mathrm{ind}(u) + 2g - 2 + \#\Gamma_0.$$

Implicit function theorem

For $u : \dot{\Sigma} \to W$ with $\mathrm{ind}(u) = 2$, call u **good** if $\delta(u) = 0$, $g = 0$, all orbits are elliptic and the asymptotic approach to each orbit is simply covered. Then for all compatible J *(no genericity required!)*,

- $u(\dot{\Sigma})$ *and its neighbors form a local 2-dimensional foliation of* W
- In $\mathbb{R}$-invariant case, $\pi \circ u : \dot{\Sigma} \to M$ *is embedded, and with its neighbors forms a local 1-dimensional foliation of* M, $\pitchfork X_\lambda$



For $\mathrm{ind}(u) = 1$, call u **good** if same as above except one orbit is hyperbolic.

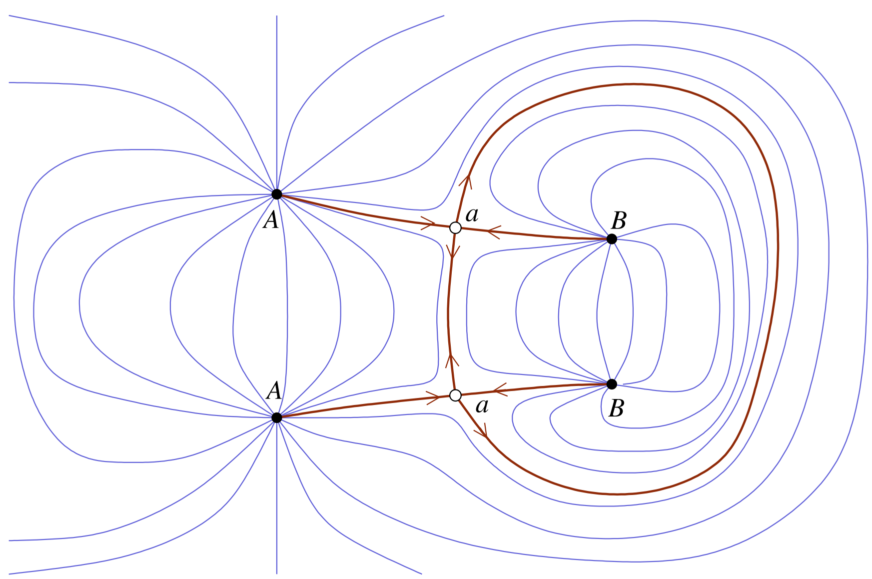
Then in $\mathbb{R}$-invariant case, $\pi \circ u : \dot{\Sigma} \to M$ is embedded and isolated.

These foliations often extend globally...

III. Finite Energy Foliations

Definition. A *stable finite energy foliation* of M is a collection of good holomorphic curves which foliate $\mathbb{R} \times M$ and project to a foliation of M, outside some finite set of nondegenerate orbits.

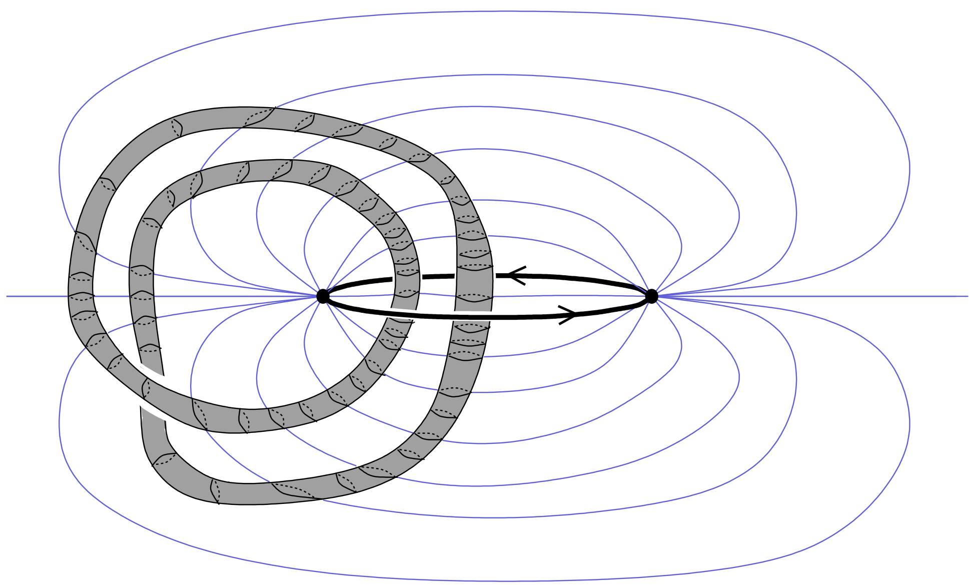
Index 0 $\Rightarrow$ trivial cylinder
Index 1 $\Rightarrow$ rigid surface
Index 2 $\Rightarrow$ 1-parameter family of leaves in M



Hofer, Wysocki, Zehnder '03: Foliations exist for generic contact forms on the tight three-sphere.

$\Rightarrow$ 2 or infinitely many periodic orbits!

W. '05: Foliations on all overtwisted (M, ξ) can be produced from open books on S^3 by transverse surgery.



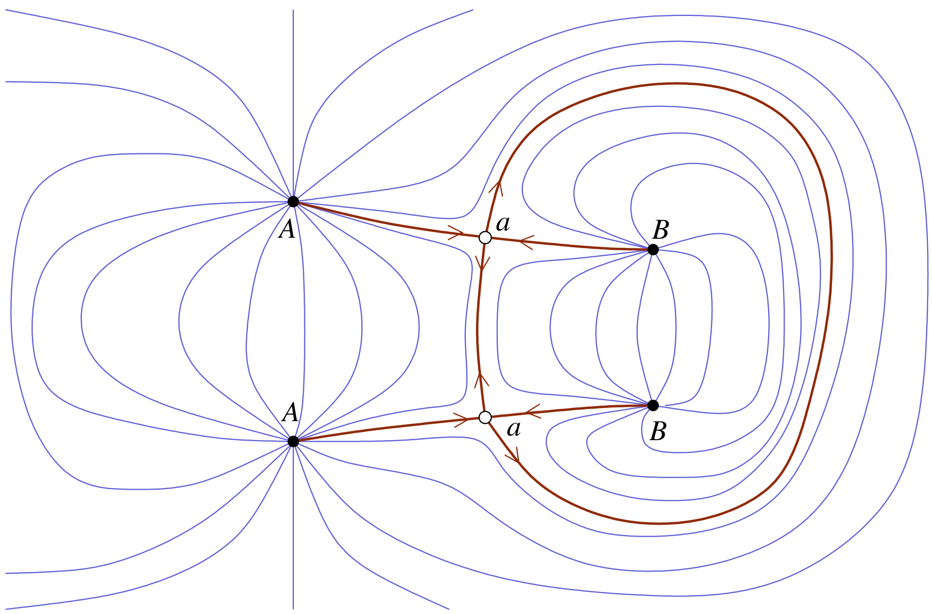
Abbas '04: Giroux's open book decompositions in the *planar* case can be made $\tilde{J}$-holomorphic.

Corollary (Abbas, Cieliebak, Hofer '04). *Weinstein conjecture for planar contact structures.*

Rallying cry:

"If holomorphic curves are everywhere, it's hard to kill them."

IV. Compactness for Good Holomorphic Curves



Conjecture:
Sequences of good index 2 curves in $\mathbb{R} \times M$ converge to broken holomorphic curves with two good index 1 levels (and no other levels).

More generally, good curves in symplectic cobordisms may produce nodal and/or multiply covered limits. . . but with severe restrictions.

Partial compactness results

Theorem 1. Suppose $u_j : \dot{\Sigma} \to W$ are good index 2 curves and converge to a multiple cover $u = v \circ \varphi$. Then u is immersed, and v is embedded with index 0.

Moreover, all curves near u are embedded, and fit together with v in a foliation.

Idea of proof:

- Intersection theory $\Rightarrow$ v embedded index 0.
- Immersed $\Rightarrow$ regular:
  Linearized CR-operator $\mathbf{L}_u$ acts on sections of νu. $c_N(u) = 0$
  $\Rightarrow \dim \ker \mathbf{L}_u \leq 2$.
- Not immersed $\Rightarrow$ contradiction:
  If φ has $C > 0$ critical points, similar arguments show $\dim \ker \mathbf{L}_u \leq 2C$.
  $\Rightarrow$ all u' near u are of form $v \circ \varphi'$.

Observe: no trouble with transversality!

Corollary: no multiple covers in $\mathbb{R}$-invariant case, or when W is closed.

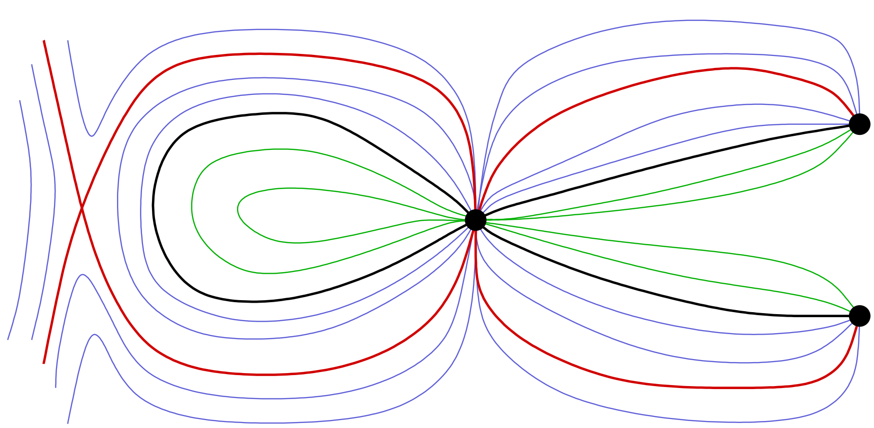
Theorem 2: Suppose W is closed and contains an immersed symplectic sphere $u : S^2 \to W$ with transverse self-intersections and

$$u \cdot u - 2\delta(u) \geq 2\#(\text{noninjective points}).$$

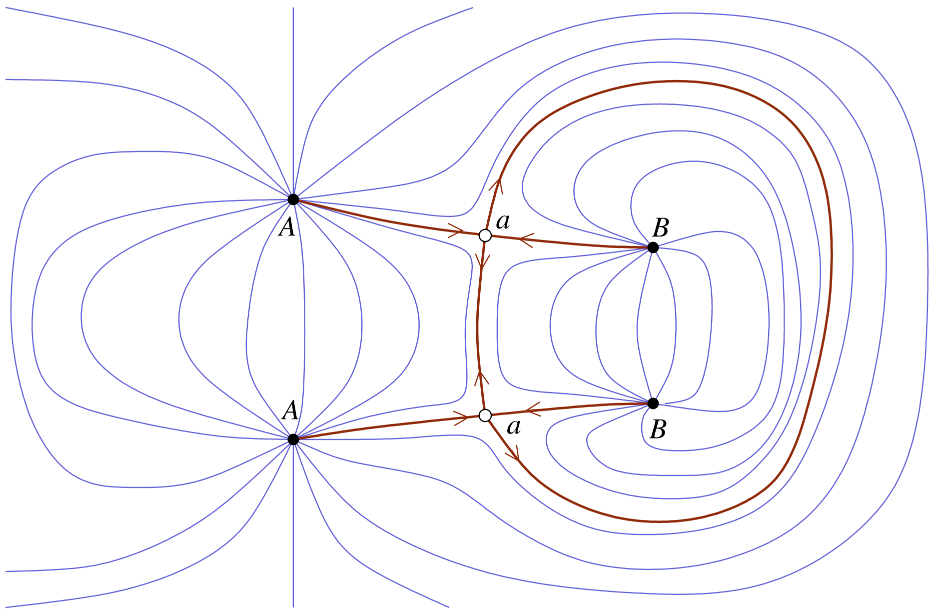
Then W admits a symplectic Lefschetz pencil with u as a fiber.

Idea of proof:

Choose J so that u is J-holomorphic, and fix marked point constaints so that $\text{ind}(u) = 2$. Compactification includes only good curves.



V. Foliations, Concordance and SFT



Given a finite energy foliation $\mathcal{F}$ of (M, λ), define a contact homology algebra $HC_*(\mathcal{F})$ generated by orbits in $\mathcal{F}$, with

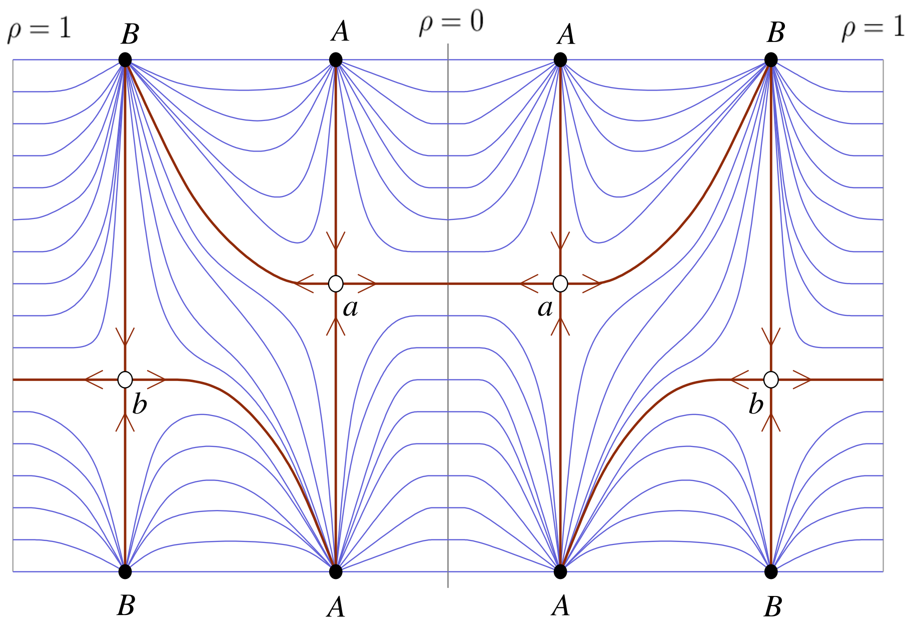
$$\partial\alpha = \sum_{\beta} \#\Big(\mathcal{M}(\alpha; \beta_1, \ldots, \beta_k)/\mathbb{R}\Big)\beta_1 \ldots \beta_k.$$

$\mathcal{M}(\alpha; \beta_1, \ldots, \beta_k) :=$ moduli space of rigid leaves in $\mathcal{F}$ with one positive puncture at α and negative punctures at $\beta_1, \ldots, \beta_k$.

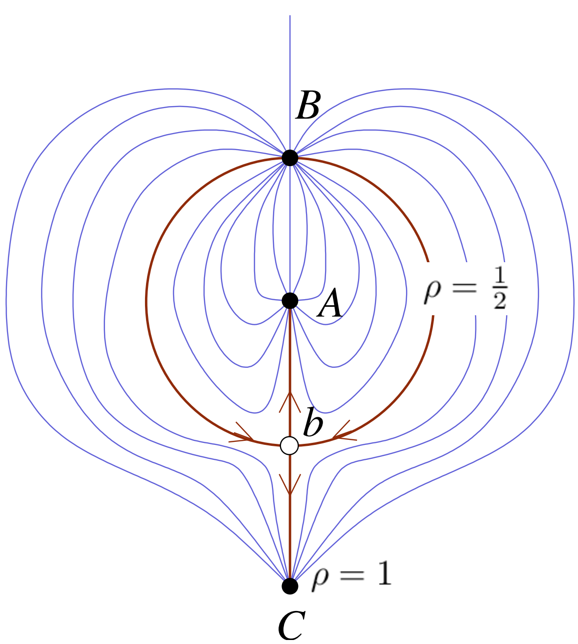
This should be functorial under concordance $\mathcal{F}_+ \to \mathcal{F}_-$: a symplectic cobordism $(\mathbb{R} \times M, \hat{J})$ with holomorphic foliation $\mathcal{F}$ that approaches $\mathcal{F}_\pm$ near $\{\pm\infty\} \times M$.

Question: given (M, ξ), *what is the set of foliations up to equivalence by concordance?*

Example: two (conjecturally) non-equivalent foliations on an overtwisted $S^1 \times S^2$

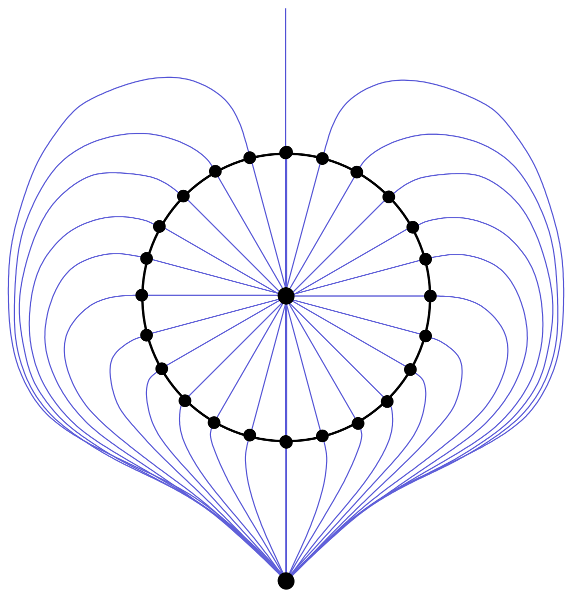


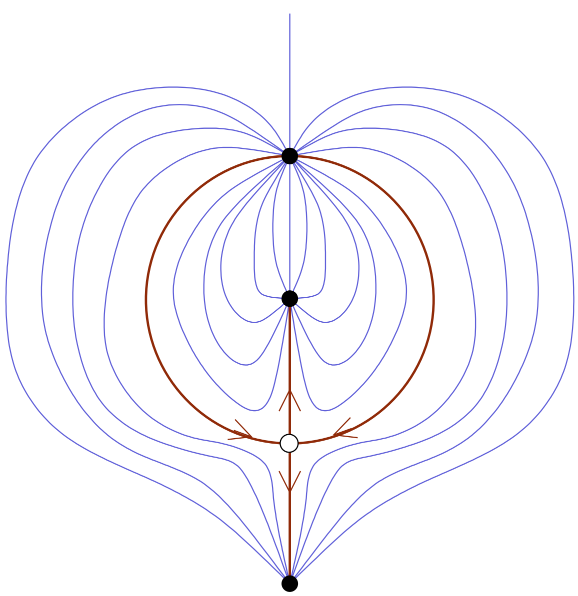
$HC_*(\mathcal{F}_1) = 0$

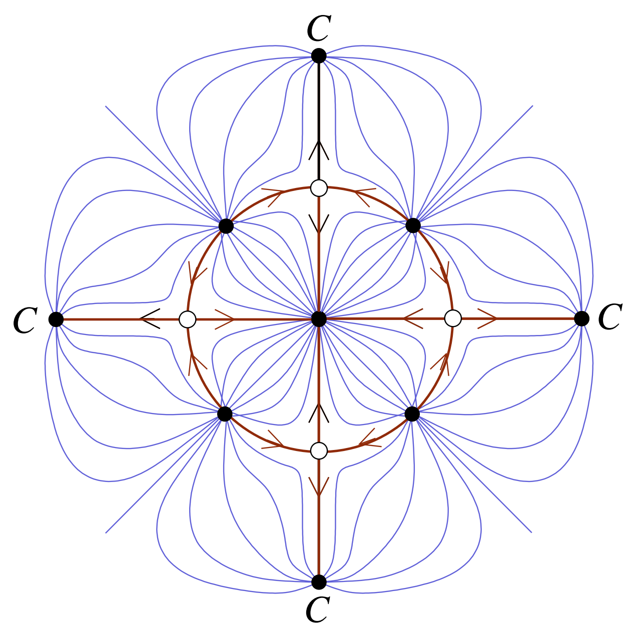


$HC_*(\mathcal{F}_2) \neq 0$

Morse-Bott foliations and perturbations







$$HC_*(\mathcal{F}_2) = HC_*(\mathcal{F}_3)$$

Conjecture. *All nondegenerate perturbations of a given Morse-Bott foliation are concordant.*

Example: a stupid Morse-Bott foliation.

For (S^3, λ_0), every Hopf circle is an orbit, so take $\mathcal{F} = \{\mathbb{R} \times P\}_{P=\text{Hopf}}$. This is the *only* stable foliation of (S^3, λ_0).

Conjecture. *This is the only stable foliation of (S^3, ξ_0) up to concordance.*