h-principle seminar — TALK 5 — 21/2/2013 — A. JUHÁSZ

microflexibility

def: a diff. relation $R \subset X^{(r)}$ is (parametrically) locally integrable if given
- $h : I^k \to V$
- a family of sections $F_p : h(p) \to R$, $p \in I^k$
- a family of local holonomic extensions near ∂I^k

$$\tilde{F}_p : \mathcal{O}p\, h(p) \to R, \quad \tilde{F}_p(h(p)) = F_p(h(p)) \ \forall p \in \mathcal{O}p(\partial I^k)$$

$\Rightarrow$ $\exists$ family of local holon. extensions

$$\tilde{F}_p : \mathcal{O}p\, h(p) \to R, \quad \tilde{F}_p(h(p)) = F_p(h(p)) \ \forall p \in I^k$$

rks: (1) $\forall$ open relations are loc. integ.

(2) R_{iso} (isometric immersion of Riem. mfd $(V, g_V) \hookrightarrow (W, g_W)$) is not loc. integ.

(3) the closed relations R_{sag} $(V \hookrightarrow (W, \omega_W))$,

R_{leg} $(V \hookrightarrow (W, \xi_W))$

$R_{isocont}$ $(F : TV \to TW,\ \xi_V = F^{-1}(\xi_W))$

are all loc. integ.

(ex: Given a Lagrangian plane in a tangent space of a symplectic mfd, $\exists$ a local Lagrangian submfd tangent to it.)

prop (homotopy extension property for formal solutions, relative case)

$B \subseteq A \subseteq V$, A, B cpt, $F_A : \mathcal{O}p\, A \to R$ section

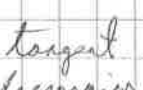
$F_B^\tau : \mathcal{O}p\, B \to R$, $\tau \in I = [0,1]$, a homot. of $F_B^0 := F_A|_{\mathcal{O}p\, B}$

Then F_B^τ extend to $F_A^\tau : \mathcal{O}p\, A \to R$ s.t. $F_A^0 = F_A$.

pf: Let $U = \mathcal{O}p\, B$, $B \subset U' \subset U$, cutoff fn $\delta : V \to \mathbb{R}_+$ w/ supp in U s.t. $= 1$ on U',

$$F^\tau(v) := \begin{cases} F(v) & \text{for } v \in V \setminus U \\ F_B^{\tau\delta(v)} & \text{for } v \in V \end{cases}$$ $\square$

Let $K^m = [-1,1]^m$, n fixed, $k < n$.

$\Theta_k := (K^n, K^k \cup \partial K^n)$ e.g.  $n=2$, $k=1$

$(A, B) \subset V$ is a Θ_k-pair if $(A, B) \underset{diff}{\cong} \Theta_k$.

defn: $R \subset X^{(r)}$ is parametrically k-microflexible if for every suff small open ball $U \subseteq V$ & families parametrized by $p \in I^m$ of
- O_k-pairs $(A_p, B_p) \subset U$,
- holon. sections $F_p^0 : Op\, A_p \to R$
- holon. homotopies $F_p^\tau : Op\, B_p \to R$, $\tau \in I$ of the sections F_p^0 over $Op\, B_p$ that are constant over $Op(\partial B_p)$ $\forall\, p \in I^m$ & const. over $Op\, B_p$ for $p \in Op(\partial I^m)$,

then $\exists\, \sigma > 0$ & a family of holonomic homotopies $F_p^\tau : Op\, A_p \to R$ for $\tau \in [0,\sigma]$, that extends the family of homotopies $F_p^\tau : Op\, B_p \to R$ for $\tau \in [0,\sigma]$, and are const. over $Op(\partial A_p)$ $\forall\, p \in I^m$ & const over $Op\, A_p$ $\forall\, p \in Op(\partial I^m)$.

$R \subset X^{(r)}$ is (micro)flexible if R is k-(micro)flexible $\forall\, k = 0, \dots, n-1$ where $n = \dim V$.

rk: (1) $\forall$ open relations, are microflexible.

(2) R_{Lag} is k-microflexible if $k \neq 1$.

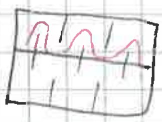
(3) R_{Lag} & $R_{isocont}$ are microflexible.

holonomic R-approximation thm

Spse $R \subset X^{(r)}$ locally integrable & microflexible, $A \subset V$ polyhedron of codim > 0, $F : Op\, A \to R$ a section

$\Rightarrow \forall\, \delta, \varepsilon > 0$ $\exists$ δ-small (in C^0-sense) diffeotopy $h^\tau : V \to V$, $\tau \in I$ & a holon. section $\tilde{F} : Op\, h^1(A) \to R$ s.t. $\forall\, v \in Op\, h^1(A)$, $\operatorname{dist}(\tilde{F}(v), F(v)) < \varepsilon$.

pf: Analogous to original holon. approx thm.
Over a cube, local integr. $\longrightarrow$ 1st step of induction
microflexibility $\Rightarrow$ version of the interpolation (Prop. 3.5.1) required in pf of inductive lemma 3.4.1



Then induction over skeleta of A.
Homot. extension lemma $\Rightarrow$ can extend the holonomic sol at each step as formal sol. to $Op\, A$. ∎

- Relative version: $B \subset A$, F holon. near B, $h|_{Op B} = Id$,
  $F|_{Op B} = \tilde{F}|_{Op B}$
- parametric version

cor (local h-principle): All forms of the local h-principle hold for loc. integr. & microflexible Diff(V)-invt. relations near any polyhedron $A \subset V$ of codim > 0.

pf: difference from before: constructing homot. btwn formal & genuine sols. inside R. Linear homot. doesn't necessarily lie in R. If R is a local nbhd retract (this holds in all our applications), compress lin. homot. into R by the retraction. ☐

cor (Gromov): V open mfd, $X \to V$ a natural bndl, every loc. integr. & microflexible Diff(V)-invt. $R \subset X^{(r)}$ satisfies parametric h-principle.

defn: $X \to V$ natural, $\mathcal{A} \subset Diff(V)$, R is $\mathcal{A}$-invt if $h_*(R) = R$ $\forall h \in \mathcal{A}$.

main examples: $\mathcal{A}$ = symplectic or contact diffeos.

defn: $\mathcal{A} \underset{\text{Lie subgroup}}{\subseteq} \{ d \in Diff(V) \mid supp(d) \text{ cpt} \}$

A Lie algebra $\mathfrak{g}$ of vector fields is capacious if

(CAP_2) $\forall$ tangent hyperplanes $\tau \subset T_xV$, $x \in V$, $\exists v \in \mathfrak{g}$ s.t. $v \pitchfork \tau$.

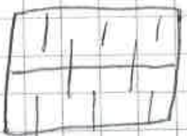
(CAP_1) $\forall v \in \mathfrak{g}$, $\forall A \subset V$ cpt, $\forall U \supset A$ $\exists \tilde{v}_{A,U} \in \mathfrak{g}$ s.t. $supp(\tilde{v}_{A,U}) \subset U$ & $\tilde{v}_{A,U}|_A = v|_A$.

+ CAP_1, CAP_2 ~ parametrically ~~for~~ $\forall$ cpt set of param.

ex: identity cpt. of group of compactly supp. contact diffeos. of a cont. mfd or Ham. diffeos. of a sympl. mfd

thm (local h-principle): $\mathcal{A} \subset Diff(V)$ capacious, $X \to V$ natural, R is an $\mathcal{A}$-invt locally integrable & microflexible relation, then all forms of local h-princ. hold near any subpolyhedron of codim > 0.

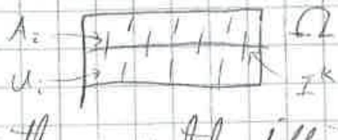
(rk: Follows from pf that suff to assume invariance under C^0-small diffeos.)

pf: Problem:  diffeotopy h^τ given by holon. R-approx. thm might not belong to $\mathcal{A}$. But

CAP$_2$ $\Rightarrow$ $\exists$ subdivision of A s.t. for every simplex Δ, $\exists\, v_\Delta \in g$ s.t. $v_\Delta \pitchfork \Delta$.

Near each Δ, choose a coord. system

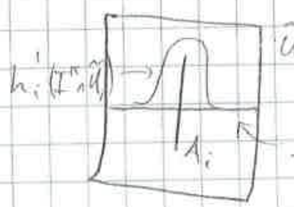
Pf of inductive lemma is same (use loc. integr. & microflex.) $\rightsquigarrow \tilde{F} : \Omega \to R$ section



claim: For $i = 1, \dots, N$, $\exists$ compactly supported diffeotopies

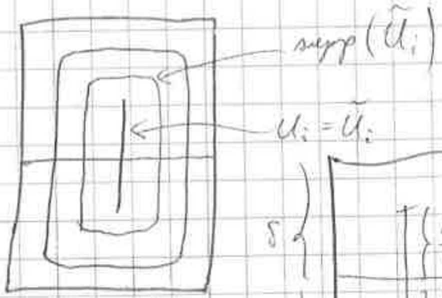
$h_i^\tau : \tilde{U}_i \to \tilde{U}_i$, $\tau \in [0,1]$ s.t.

- $h_i^\tau \in \mathcal{A}$
- $h_i^1(I^k \cap \tilde{U}_i) \cap A_i = \emptyset$



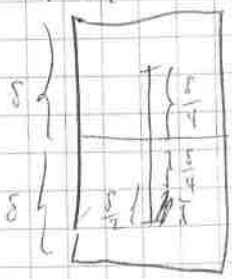
pf: By construction, $v = \frac{\partial}{\partial x_n} \in g$

CAP$_1$ $\Rightarrow$ $\exists\, \tilde{v}_i \in g$ s.t.



in time $\frac{\delta}{2}$, it slides $\hat{I}$ to $\tilde{I}$

$h_1^\tau = e^{\frac{\delta}{2}\tau \tilde{v}_1}$, $\tau \in I$ disjoins $I^k \cap \tilde{U}_1$ from A_1



$h_1^\tau, \dots, h_N^\tau$ fit together into a smooth isotopy $h^\tau \in \mathcal{A}$ defined on Op I^k, $h^1(I^k) \subset \Omega$ w/ $\hat{F}$ is def'd by pullback of $\tilde{F}$ along h^1 to a nbhd of I^k, continue as before. ☒