h-principle seminar — TALK 6 — 28/2/2013 — A. JUHÁSZ

Legendrian & Lagrangian immersions

defn: $f : (V, \xi_V) \to (W, \xi_W)$ is isocontact if it induces ξ_V on V. ($\iff$ if $\ker \alpha_V = \xi_V$, $\ker \alpha_W = \xi_W$ then $f^* \alpha_W = \varphi \alpha_V$ for some function $\varphi > 0$)

$F : TV \to TW$ is isocontact if $\xi_V = F^{-1}(\xi_W)$ and $F|_{\xi_V} : \xi_V \to \xi_W$ is conformally symplectic wrt $CS(\xi_V)$ & $CS(\xi_W)$.

(recall: $\xi = \ker \alpha$ contact $\Rightarrow$ conformal sympl. str. $CS(\xi) := [d\alpha|_\xi]$. Well defined since $d(f\alpha)|_\xi = f\, d\alpha|_\xi$ for any function $f > 0$)

local h-principle for isocontact immersions

(A = polyhedron of codim > 0, $F_0 : \mathcal{O}p\, A \to \mathcal{R}$ homot. to genuine sol. F_1)

(V, ξ_V), (W, ξ_W), $A \subset V$ polyhedron of codim > 0, then all forms of the local h-principle hold for isocontact immersions $(\mathcal{O}p\, A, \xi_V|_{\mathcal{O}p\, A}) \to (W, \xi_V)$.

pf: $\mathcal{R}_{\text{isocont}}$ is locally integrable, microflexible & inv. wrt the capacious group $\text{Diff}_{\text{cont}}(V)$.

lemma: (V, ξ) ctct, $E \to V$ sympl. vec. bndl. Then $\exists$ a ctct str. $\tilde{\xi}$ on a nbhd U of the 0-section V s.t. $\tilde{\xi}|_V = \xi$, $\tilde{\xi}$ is tangent to the fibers of E viewed as a subbndl of $TE|_V$ & the given sympl. str. on these fibers belongs to the conformal class $CS(\tilde{\xi})$.

pf: $\xi = \ker \alpha$. Let $\eta \in \Omega^2(E)$ s.t. $d\eta = 0$ & $\eta|_{E_V}$ = given symp. str. & $\eta|_V = 0$. (note: maybe this requires a partition of unity on V?)

$\Rightarrow \exists\, \beta$ s.t. $\eta = d\beta$ & $\beta|_V = 0$.

(fiberwise Poincaré lemma)

Then $\beta + \pi^* \alpha$ defines $\tilde{\xi}$ on $\mathcal{O}p\, V$. $\boxtimes$

h-principle for isocontact immersions (Gromov)

If $\dim V < \dim W$, then all forms of h-princ hold for isocontact $(V, \xi_V) \to (W, \xi_W)$.

pf: microextension trick: $N :=$ ~~normal~~ bndl to $F(\xi_V)$, w.r.t. $CS(\xi_W)$

given $F : (TV, \xi_V) \rightarrowtail (TW, \xi_W)$.

N is symp. $\underset{(\text{lemma})}{\Longrightarrow}$ $\exists$ ξ_N on $\mathcal{O}p\, V \subset N$ s.t. ~~[illegible]~~

(V, ξ_V) is a ctct subbndl of $(\mathcal{O}p\, V, \xi_N)$, fibres N_v are tangent to $\xi_N|_V$ & are $\perp$ complements of ξ_V in ξ_N w.r.t. $CS(\xi_N)$.

Now any isocontact homomorphism $F : (TV, \xi_V) \to (TW, \xi_W)$ homotopically canonically extends to equidimensional isoctct hom.

$\hat{F} : (T\,\mathcal{O}p\, V, \xi_N) \to (TW, \xi_W)$.

In other direction: every isoctct immersion $(\mathcal{O}p\, V, \xi_N) \hookrightarrow (W, \xi_W)$ restricts to $(V, \xi_V) \hookrightarrow (W, \xi_W)$

$\boxtimes$

Similarly, every formal Legendrian immersion $F : TV^n \to TW^{2n+1}$ can be canonically extended to a formal equidimensional isocontact immersion

$$\hat{F} : T(\underset{\substack{\parallel \\ T^*V \times \mathbb{R}}}{J^1(V, \mathbb{R})}) \longrightarrow TW$$

$T^*V \times \mathbb{R}$ with its canonical ctct str. $\xi_V := \ker(dz - p\,dq)$

$\Rightarrow$ h-principle for Legendrian immersions (Gromov):

all forms of h-princ. hold for Legendrian immersions of $V^n \hookrightarrow (W^{2n+1}, \xi_W)$.

Lagrangian immersions: (W, ω) is exact if $\omega = d\alpha$.

$f : V \underset{\text{Lagrangian}}{\hookrightarrow} (W, \omega = d\alpha)$ is exact if $f^*\alpha$ is exact.

prop: $(W^{2n}, \omega = d\alpha)$, V^n, h-princ. holds for exact Lagr. immersions $V \hookrightarrow W$.

pf: $F : TV \rightarrowtail TW$ isotropic, then $\exists$ homotopically canonical lift to an isotropic monomorphism to

$\xi = \ker(dz - \alpha)$ on $W \times \mathbb{R}$. (Think: Lagrangian projection of a Legendrian!)

Conversely, any Legendrian imm. $V \hookrightarrow (W \times \mathbb{R}, \xi)$ projects to an exact Lagrangian immersion into $(W, d\alpha)$.

thm (Gromov): V^n, (W^{2n}, ω) all forms of h-princ. hold for Lagr. $V \hookrightarrow (W, \omega)$ as long as $[f^*\omega] = 0$ is incorporated in the defn of formal sols.

pf: $F : TV \xrightarrow{\text{isotropic}} TW$, $f = \mathrm{bs}\, F$ s.t. $[f^*\omega] = 0$.

By Hirsch-Smale, can assume f is a smooth immersion — can assume this change is C^0-small, so not hard to move F along s.t. still isotropic.

$N :=$ total space of normal bndl $\nu(f(V) \subseteq W)$

$\exists$ extension $\tilde{f} : N \hookrightarrow W$ s.t. $F = d\tilde{f} \circ G$, where

$G : TV \to TN$ Lagr. w.r.t. $\tilde{\omega} = \tilde{f}^*\omega$

$[f^*\omega] = 0 \Rightarrow \tilde{\omega}$ is exact, then apply previous prop. ∎